Chapter 1

Section 1.2

1. (b) First order. y_1: $2\sin x\cos x \neq 9\sin 2x$ so No. y_2: $2(3\sin x)(3\cos x)$ does $= 9\sin 2x$ (because $2\sin x\cos x = \sin 2x$) so Yes. y_3: $2(e^x)(e^x) \neq 9\sin 2x$ so No.

(h) $y_1' + 2xy_1 - 1 = -2xAe^{-x^2}\int_0^x e^{t^2}dt + Ae^{-x^2}e^{x^2} + 2Axe^{-x^2}\int_0^x e^{t^2}dt - 1 = 0$ only if $A=1$. Thus, in general, No.
$y_2' + 2xy_2 - 1 = -2xe^{-x^2}\int_a^x e^{t^2}dt + e^{-x^2}e^{x^2} + 2xe^{-x^2}\int_a^x e^{t^2}dt - 1$ does $=0$ for all choices of a, so Yes.

3. Evaluating u_{xx}, u_{yy}, u_{zz}, we obtain
$u_{xx} + u_{yy} + u_{zz} = (c^2 - a^2 - b^2)\sin ax \sin by \sinh cz = 0$ if $c^2 = a^2 + b^2$ (or if $a=b=c=0$ so that $\sin ax \sin by \sinh cz = 0$, but this is a subcase of $c^2 = a^2 + b^2$).

5. (b) $y' + 3y^2 = \lambda e^{\lambda x} + 3e^{2\lambda x} = e^{\lambda x}(\lambda + 3e^{\lambda x})$. The $e^{\lambda x}$ factor is not 0 for any x, let alone for all x. And for the second factor to be 0 for all x requires that $e^{\lambda x}$ is a constant and that, in turn, requires that $\lambda = 0$. But if $\lambda = 0$ then $\lambda + 3e^{\lambda x} = 0 + 3 \neq 0$. Thus, no such solutions.

(c) $y'' - 3y' + 2y = (\lambda^2 - 3\lambda + 2)e^{\lambda x} = 0$ if $\lambda^2 - 3\lambda + 2 = 0$, i.e., if $\lambda = 1$ or 2. Thus, e^x and e^{2x} are solutions.

6. (b) $y'' - y - x^2 = (-2 + A\sinh x + B\cosh x) - (-x^2 - 2 + A\sinh x + B\cosh x) - x^2$ does $= 0$.
$y(0) = -2 = -2 + B$ and $y'(0) = 0 = A$ give $A = B = 0$, so $y(x) = -x^2 - 2$.

7. (b) Nonlinear due to the yy' term
(d) Nonlinear due to the $\exp(y)$ term
(g) Nonlinear due to the yy''' term. All others linear.

8. $y'' \approx C$, since $y'^2 \ll 1$.

Section 1.3

3. (a) Since $\Delta W = w\Delta x = \mu\Delta s$, we see that $w = \mu\, ds/dx = \mu\sqrt{1+y'^2}$. Integrating (11a) and (11b),

$$\left.\begin{aligned} T\cos\theta &= A \\ T\sin\theta &= \mu\int^x \sqrt{1+y'^2}\,dx + B \end{aligned}\right\} \text{so } \frac{T\sin\theta}{T\cos\theta} = \tan\theta = y' = \frac{\mu}{A}\int^x \sqrt{1+y'^2}\,dx + \frac{B}{A}$$

and d/dx gives $y'' = C\sqrt{1+y'^2}$.

Chapter 2

Section 2.2

2. (b) $y' + 4y = 8$, so (21) gives $y(x) = e^{-\int 4dx}\left(\int e^{\int 4dx} 8dx + C\right) = e^{-4x}\left(\frac{8e^{4x}}{4} + C\right) = 2 + Ce^{-4x}$. Or, by integrating factor method, consider $\sigma y' + 4\sigma y = \sigma 8$. For $\sigma y' + 4\sigma y$ to $\overset{?}{=} (\sigma y)' = \sigma y' + \sigma' y$ we need $\sigma' = 4\sigma$ so, from (7), $\sigma(x) = e^{4x}$. Thus, $(e^{4x}y)' = 8e^{4x}$, so $e^{4x}y = \int^x 8e^{4x}dx + C$ or

$y(x) = 2 + Ce^{-4x}$ again.

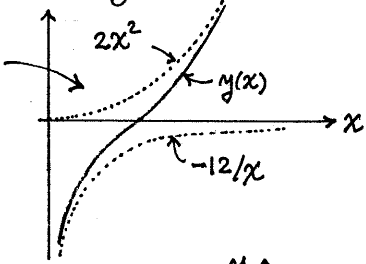
(e) $y(x) = e^{-\int -\tan x\,dx}\left(\int e^{\int -\tan x\,dx} 6dx + C\right) = e^{\int \frac{\sin x}{\cos x} dx}\left(\int e^{-\int \frac{\sin x}{\cos x} dx} 6dx + C\right)$
$= e^{-\int d(\cos x)/\cos x}\left(\int e^{\int d(\cos x)/\cos x} 6dx + C\right) = e^{-\ln|\cos x|}\left(\int e^{\ln|\cos x|} 6dx + C\right)$
$= \frac{1}{|\cos x|}\left(\int 6|\cos x|\,dx + C\right)$. Recall that the $\tan x$ in the ODE is defined only on $\ldots, -3\pi/2 < x < -\pi/2, -\pi/2 < x < \pi/2, \pi/2 < x < 3\pi/2, \ldots$ etc. On $-\pi/2 < x < \pi/2$, for ex., $\cos x > 0$ so $|\cos x| = \cos x$ and $y(x) = \frac{1}{\cos x}(6\sin x + C)$. On $\pi/2 < x < 3\pi/2$, for ex., $\cos x < 0$ so $|\cos x| = -\cos x$ and $y(x) = \frac{1}{-\cos x}\left(\int -6\cos x\,dx + C\right)$
$= \frac{1}{\cos x}(6\sin x - C)$, and so on, so on <u>any</u> of the stated x intervals the solution is $y(x) = \frac{1}{\cos x}(6\sin x + \text{“}A\text{”})$ where A is an arbitrary constant.

(f) $y(x) = e^{-\int 2dx/x}\left(\int e^{\int 2dx/x} x^2 dx + C\right) = e^{-2\ln|x|}\left(\int e^{2\ln|x|} x^2 dx + C\right)$
$= \frac{1}{|x|^2}\left(\int |x|^2 x^2 dx + C\right) = \frac{1}{x^2}\left(\int x^4 dx + C\right) = \frac{x^3}{5} + \frac{C}{x^2}$ for $0 < x < \infty$ or

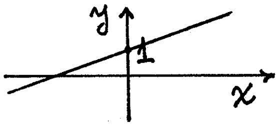
for $-\infty < x < 0$.

3. (b) $y_h(x) = Ae^{-4x}$ so seek $y(x) = A(x)e^{-4x}$. Then $(A'e^{-4x} - \cancel{4Ae^{-4x}}) + \cancel{4Ae^{-4x}} = 8$
gives $A' = 8e^{4x}$, $A(x) = \int 8e^{4x} dx + C = 2e^{4x} + C$, so
$y(x) = (2e^{4x} + C)e^{-4x} = 2 + Ce^{-4x}$, as in 2(b).

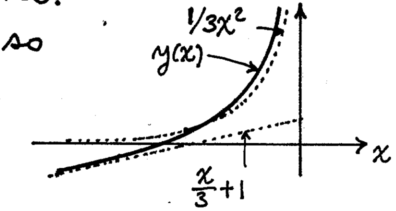
5. (b) $y(x) = 2x^2 + C/x$. $y(1) = 2 = 2 + C$ gives $C = 0$ so $y(x) = 2x^2$ on $-\infty < x < \infty$.
(c) $y(x) = 2x^2 + C/x$. $y(2) = 2 = 8 + C/2$
gives $C = -12$ so $y(x) = 2x^2 - \frac{12}{x}$ on $0 < x < \infty$.



6. (21) gives general solution $y(x) = \frac{x}{3} + 1 + \frac{C}{x^2}$
(b) $y(0) = 1 = 0 + 1 + 0$ if we choose $C = 0$.
Thus, $y(x) = \frac{x}{3} + 1$. (NOTE: If, instead, $y(0) = y_0$ where $y_0 \neq 1$, then there is <u>no</u> solution.) That solution holds on $-\infty < x < \infty$.

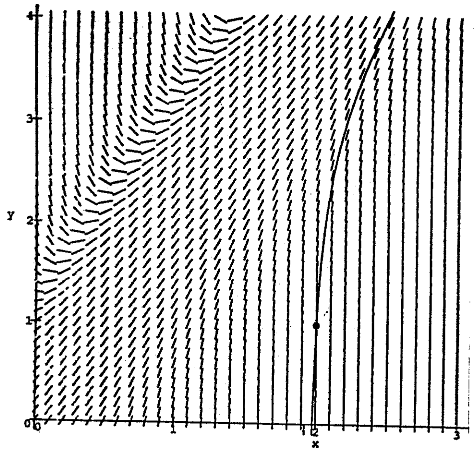


(c) $y(-1) = 1 = -\frac{1}{3} + 1 + C$ gives $C = \frac{1}{3}$, so
$y(x) = \frac{x}{3} + 1 + \frac{1}{3x^2}$ on $-\infty < x < 0$.



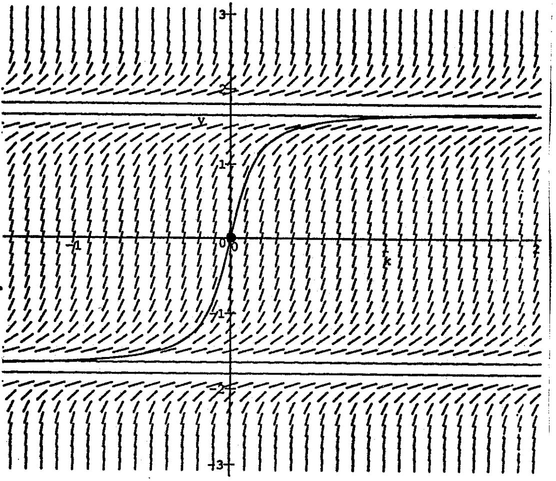
7. (b) Consider $x = x(y)$ rather than $y = y(x)$. Then $\frac{dx}{dy} = 6x + y^2$ or
$\frac{dx}{dy} - 6x = y^2$, $x(y) = e^{-\int -6dy}\left(\int e^{\int -6dy} y^2 dy + C\right)$
$= -\frac{1}{6}y^2 - \frac{1}{18}y - \frac{1}{108} + Ce^{6y}$

8. (a) Shown at the right is only the $0 < x < 3$, $0 < y < 4$ part of the display, using the command
phaseportrait (2+(2*x-y)^3, [x,y], x=-4..4, {[2,1]}, y=-4..4, grid=[40,40], stepsize=0.01, arrows=LINE);
NOTE: The default grid is [20,20] and is too coarse so we use the Grid option **grid=[40,40]**. Also, the stepsize needs to be reduced sufficiently to get

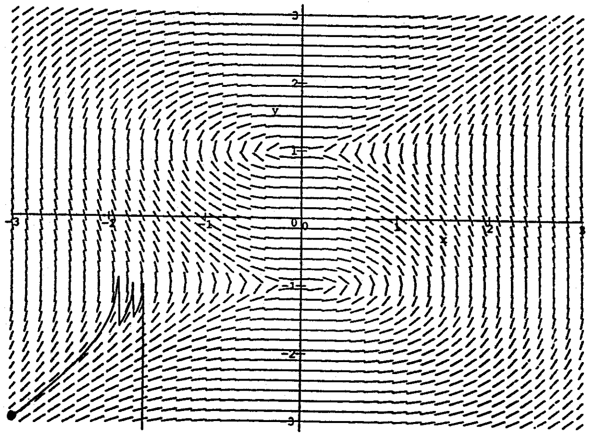


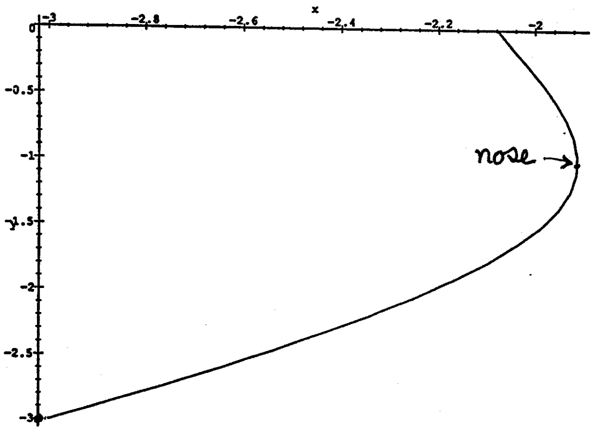
render the solution curve through [2,1] smooth so we used the additional option **stepsize** = 0.01. (For further discussion of the phaseportrait command see the Index.) Looking at the lineal element field (and peeking at the ODE) reveals the simple integral curve $y=2x$. The integral curve through [3,0], for instance, is almost vertical and bends to the right, eventually approaching $y=2x$.

(c) `phaseportrait((3-y^2)^2, [x,y], x=-2..2, {[0,0]}, grid = [40,40], stepsize = 0.04, y=-3..3, arrows = LINE);` gives the phaseportrait shown at the right. We observe the integral curves $y=+\sqrt{3}$ and $y=-\sqrt{3}$.



(e) `phaseportrait(x^2/(y^2-1), [x,y], x=-3..3, {[-3,-3]}, y=-3..3, grid = [40,40], stepsize = 0.01, arrows = LINE);` gives the portrait shown below left. To resolve the mysterious zigzags we reduced the stepsize to 0.01 but the zigzags persisted. It looks like the problem is that the integral curve rises from [-3,-3] reaches a vertical tangent at $y=-1$ (as can also be seen from the ODE) and then bends to the left, in which case a single-valued differentiable solution $y(x)$ would exist only up to the point of vertical tangency, the "nose" of the curve. NOTE: If we use separation of variables (not discussed until Sec. 2.4), we obtain, in implicit form, the solution $y^3-3y=x^3+9$. Next, the commands `with(plots):` and `implicitplot(y^3-3*y = x^3+9, x=-3..0, y=-3..0, numpoints=500);` gives the integral curve plot shown below right, which plot substantiates the foregoing reasoning.

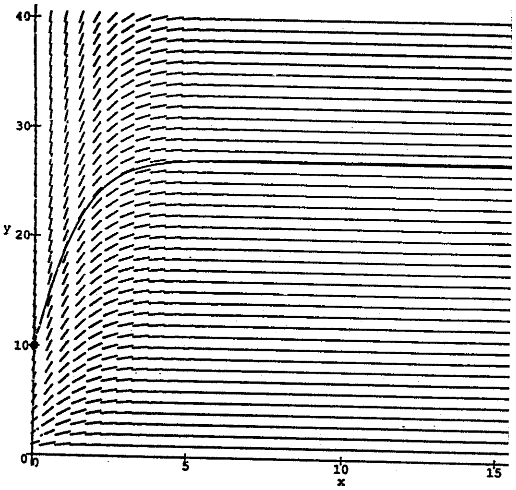




(g) $y' = e^{-x}y$. `phaseportrait(exp(-x)*y, [x,y], x=0..20, {[0,10]}, grid = [40,40], y=0..40, stepsize = 0.04, arrows = LINE);` gives the portrait shown on next page.

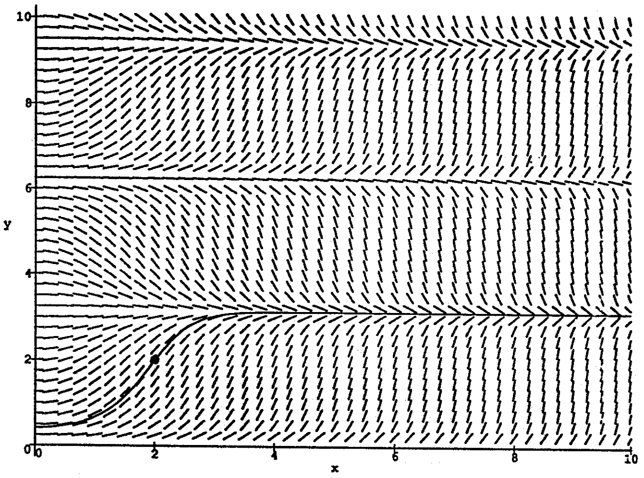
(h) `phaseportrait(x*sin(y), [x,y], x=0..10, {[2,2]}, y=0..10, grid = [40,40], stepsize = 0.04, arrows = LINE);` gives the portrait shown on next page.

(g) continued:



An exact integral curve is $y=0$.

(h) continued:



Exact integral curves are $y = n\pi$ $(n=0,\pm1,\pm2,\ldots)$

9. (b) $y'+py=qy^n$, $v=y^{1-n}$ $(n\neq 0,1)$. $v'=(1-n)y^{-n}y'$ so $y'=y^n v'/(1-n)$ and the ODE becomes $\frac{y^n v'}{1-n}+py=qy^n$ or, dividing by y^n, $v'+(1-n)pv=(q)(1-n)$.

10. (b) $n=2$, so $v'+\frac{2}{x}v=-x^2$, $v(x)=e^{-\int\frac{2}{x}dx}\left(\int e^{\int\frac{2}{x}dx}(-x^2)\,dx+C\right)$
$=\frac{1}{x^2}\left(\int(-x^4)\,dx+C\right)=-\frac{x^3}{5}+\frac{C}{x^2}$, so $y=\frac{1}{v}=\frac{5x^2}{A-x^5}$ (where $A=5C$).

11. $y'=py^2+qy+r$, $y=Y+\frac{1}{u}$ gives $Y'-\frac{u'}{u^2}=p\left(Y+\frac{1}{u}\right)^2+q\left(Y+\frac{1}{u}\right)+r$. Using $Y'=pY^2+qY+r$ to cancel terms gives $-\frac{u'}{u^2}=2p\frac{Y}{u}+\frac{p}{u^2}+\frac{q}{u}$, or
$u'+(2pY+q)u=-p$.

12. (b) $y'=y^2-xy+1$ so $p=1, q=-x, r=1$ and (11.3) is $u'+xu=-1$,
$u=e^{-x^2/2}\left(\int e^{x^2/2}(-1)\,dx+C\right)$ or $u(x)=e^{-x^2/2}\left(C-\int_0^x e^{t^2/2}dt\right)$, say.
Thus, (11.2) gives $y(x)=x+e^{x^2/2}/\left(C-\int_0^x e^{-t^2}dt\right)$.
(e) Find $Y(x)=ax^b=x^2$. (f) Use $Y(x)=1$ or $Y(x)=2$ (h) $Y(x)=2$ or $Y(x)=0$

13. (c) (13.3) is $\frac{dx}{dp}-\frac{1+2p}{p-p-p^2}x=0$, $x'+\frac{1+2p}{p^2}x=0$, $x(p)=Ce^{-\int\left(\frac{1+2p}{p^2}\right)dp}$

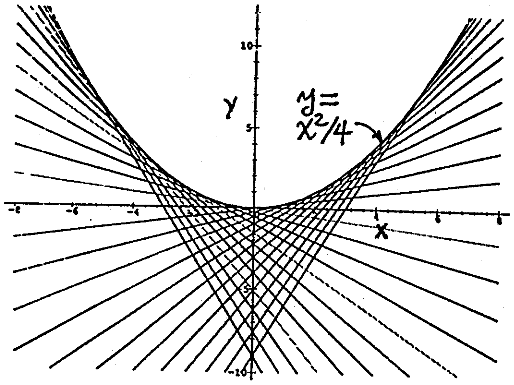
so the parametric solution is $x(p)=Ce^{1/p}/p^2$, $y(p)=x(p+p^2)$
$=Ce^{1/p}\left(1+\frac{1}{p}\right)$.

(d) Putting (13.4) into (13.1) gives $P_0x+g(P_0)=xf(P)+g(P)$, which is satisfied if $p=\text{constant}=p_0$, since $f(P_0)=P_0$.

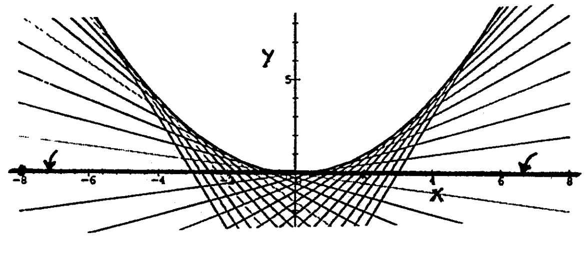
14. (b) $f(p)=p$ in (13.2) gives $0=[xf'(p)+g'(p)]\frac{dp}{dx}$, which is satisfied by $p=\text{constant}\equiv C$ [hence (14.1) gives (14.2)] <u>or</u> by $xf'(p)+g'(p)=0$. Since $f(p)=p$, the latter gives

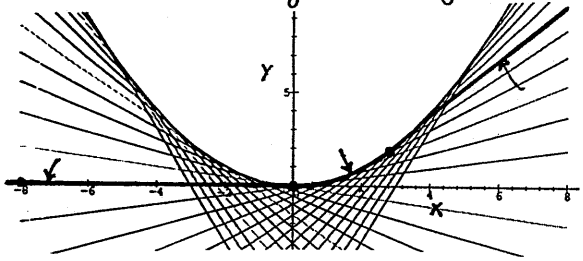
$$\left.\begin{aligned} &x=-g'(p) \\ \text{and (14.1) gives}\quad &y=xp+g(p)=-pg'(p)+g(p)\end{aligned}\right\}*$$

(c) In this case $g(p)=-p^2$ so $*$ gives $x=2p$, $y=2p^2-p^2=p^2$. In this case we are able to eliminate p between these two equations and obtain $y=x^2/4$.



(d) The point is that the Clairaut equation (14.1) admits both the family of straight-line solutions (14.2) and the additional solution (14.3). Geometrically, the integral curve given parametrically by (14.3) is an "envelope" of the family of straight lines; for the case in part (c), the envelope is the parabola $y=x^2/4$, as displayed at the right. Observe the breakdown in **uniqueness** which is in sharp contrast with the linear equation $y'+p(x)y=q(x)$, solutions of which are unique (subject to continuity conditions on $p(x)$ and $q(x)$; see Theorem 2.2.1, pg 26). For example, consider the solution(s) through the initial point $(-8,0)$. The solution curve through that point follows the x axis to $x=-\infty$. To the right, it follows the x axis to the origin, where it becomes tangent to the solution curve $y=x^2/4$. At that point it "has a choice": it can continue along the x axis to $x=+\infty$ or it can then move along the parabola $y=x^2/4$, getting off (or not) at any point along the straight line solution that is tangent to the parabola at that point, and proceeding along that line to $x=+\infty$. Two such solutions are shown below by the heavy lines.



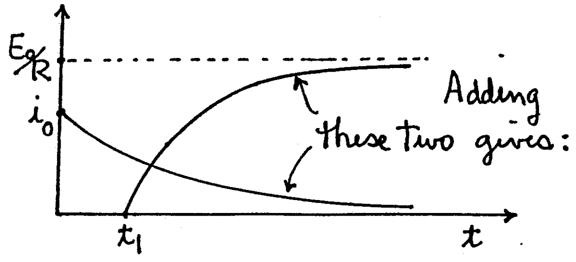


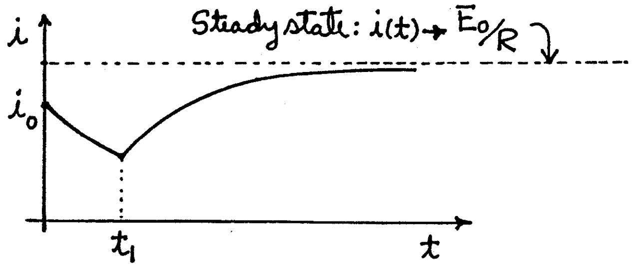
Section 2.3

2.(b) $i(t) = e^{-\int_0^t \frac{R}{L}d\tau}\left(\int_0^t e^{\int_0^\tau \frac{R}{L}d\mu}\,\frac{E(\tau)}{L}\,d\tau+i_0\right)$

If $t<t_1$ then $E(\tau)=0$ in the integral, so $i(t)=e^{-Rt/L}(0+i_0)=i_0e^{-Rt/L}$.
If $t>t_1$ then

$$i(t)=e^{-Rt/L}\left(\int_{t_1}^t e^{R\tau/L}\frac{E_0}{L}d\tau+i_0\right)=\frac{E_0}{R}\left(1-e^{-\frac{R}{L}(t-t_1)}\right)+i_0e^{-Rt/L}$$





4: $i(t)=\frac{E_0\omega L}{R^2+(\omega L)^2}\left[e^{-Rt/L}+\frac{1}{\omega L}\underbrace{(R\sin\omega t-\omega L\cos\omega t)}_{*}\right]$. To change * from two terms to one, write $A\sin(\omega t-\phi) = A(\sin\omega t\cos\omega\phi - \sin\phi\cos\omega t)$. Identify (by comparing with $*$)

$$\left.\begin{array}{l} A\sin\phi = \omega L \\ A\cos\phi = R \end{array}\right\}$$ Dividing gives $\tan\phi = \omega L/R$ or $\phi = \tan^{-1}(\omega L/R)$, and squaring and adding gives $A^2 = R^2+(\omega L)^2$ so $A = \sqrt{R^2+(\omega L)^2}$.

Thus, $i(t) = \frac{E_0\omega L}{R^2+(\omega L)^2}e^{-Rt/L} + \frac{E_0}{\sqrt{R^2+(\omega L)^2}}\sin(\omega t-\phi)$.

6. $m(t) = m_0e^{-kt}$ so $8 = 10e^{-60k}$ gives $-60k = \ln 0.8$, $k = 0.00372$ so $m(t) = 10e^{-0.00372t}$. $2 = 10e^{-0.00372t}$ gives $t = 432.6$ yrs, and $0.1 = 10e^{-0.00372t}$ gives $t = 1237.9$ yrs.

7. $m(t) = m_0e^{-kt}$. $0.8\cancel{m_0} = \cancel{m_0}e^{-70k}$ gives $k = 0.003188$. Then, $0.5\cancel{m_0} = \cancel{m_0}e^{-0.003188T}$ gives $T = 217.4$ days.

12. (a) $mv' = mg - cv$; $v(0) = 0$. Then $v' + \frac{c}{m}v = g$ (first-order linear eqn.) so $v(t) = e^{-\int c\,dt/m}\left(\int e^{\int c\,dt/m}g\,dt + A\right) = e^{-ct/m}\left(g\int e^{ct/m}dt + A\right)$ $= \frac{mg}{c} + Ae^{-ct/m}$. Then $v(0) = 0 = \frac{mg}{c} + A$ gives $A = -mg/c$ and $v(t) = \frac{mg}{c}(1-e^{-ct/m})$. As $t\to\infty$, $v(t)\to\frac{mg}{c}$ = "terminal velocity".

(b) $mv' = mg - cv^2$ is now a Riccati equation (see Exercise 11 in Sec. 2.2) with x, y changed to t, v, and $p(t) = -c/m$, $q(t) = 0$, $r(t) = g$. Observing the particular solution $\sqrt{mg/c}$, change dependent variable according to $v(t) = \sqrt{mg/c} + \frac{1}{u(t)}$. Then the ODE becomes $0 - \frac{u'}{u^2} = g - \frac{c}{m}\left(\sqrt{\frac{mg}{c}} + \frac{1}{u}\right)^2$ $= \cancel{g} - \frac{c}{m}\left(\cancel{\frac{mg}{c}} + 2\sqrt{\frac{mg}{c}}\frac{1}{u} + \frac{1}{u^2}\right)$ or $u' - 2\sqrt{\frac{gc}{m}}u = \frac{c}{m}$ with solution $u(t) = -\frac{1}{2}\sqrt{\frac{c}{mg}} + Ae^{2\sqrt{gc/m}\,t}$. Then, $v(0) = 0 = \sqrt{\frac{mg}{c}} + \frac{1}{u(0)}$ gives $u(0) = -\sqrt{\frac{c}{mg}}$ $= -\frac{1}{2}\sqrt{\frac{c}{mg}} + A$ gives $A = -\frac{1}{2}\sqrt{\frac{c}{mg}}$. Finally, $v(t) = \sqrt{\frac{mg}{c}} + \frac{1}{u(t)}$

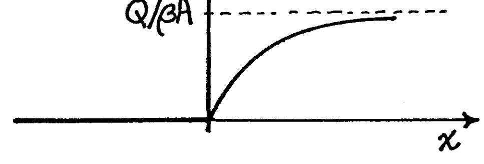
$$= \sqrt{\frac{mg}{c}} + \frac{1}{-\frac{1}{2}\sqrt{\frac{c}{mg}} - \frac{1}{2}\sqrt{\frac{c}{mg}}e^{2\sqrt{gc/m}\,t}} = \sqrt{\frac{mg}{c}}\left(1 - \frac{2}{1+e^{2\sqrt{gc/m}\,t}}\right)$$

and the terminal velocity is $\sqrt{mg/c}$.

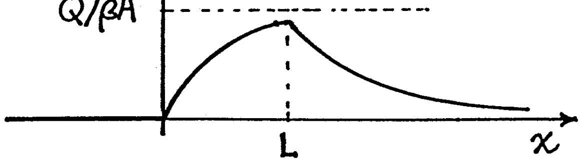
13. This problem is worked in the Answers to Selected Exercises. Here, we just wish to mention that to help the student feel more comfortable about the physical process of light extinction it might be useful to note the gradual extinction of light as we proceed deeper and deeper into the ocean.

14. NOTE: This problem is nice for use in class or lecture, especially in view of its environmental interest. Later on it will also make a nice example for the application of the Fourier transform, especially if the source is modeled as Q times a delta function at x = 0. The solution is given in the Answers to Selected Exercises, so here we will just give sketches of the results and (for possible class discussion) give a brief formal derivation of the governing ODE.

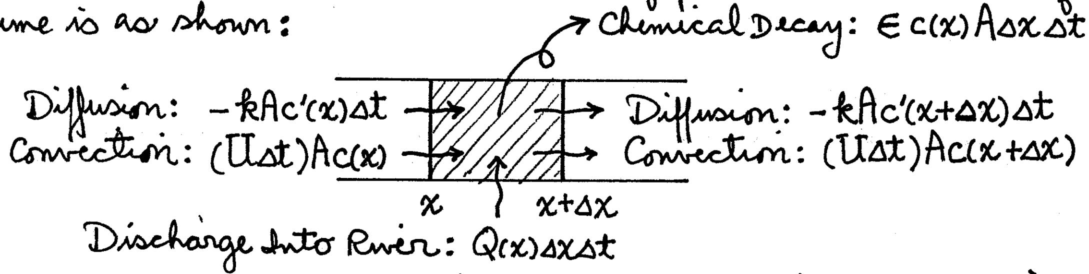
(a)



(b)



To derive the governing ODE carry out a mass balance for an arbitrary section of the river, between x and $x+\Delta x$. **Fick's law** of diffusion says that the flow of mass (of pollutant) across the "window" of area A at x is proportional to the area A and the concentration gradient $-c'(x)$ (minus because the flux will be from high concentration to low concentration, so $c'(x)>0$ will cause a flux, by diffusion, to the left and $c'(x)<0$ will cause a flux to the right) with a constant of proportionality k which is a diffusivity constant specific to the medium. Over a time Δt the movement of pollutant in and out of the control volume is as shown:



where the loss due to chemical decay is ϵ per unit mass per unit time and $Q(x)$ is the discharge into the river per unit x length per unit time. Now,

Decrease in mass of pollutant in control volume by decay, over time Δt = mass in − mass out,

so
$$\epsilon c(x) A\Delta x\Delta t = [-kAc'(x)\Delta t-(U\Delta t)Ac(x)] - [-kAc'(x+\Delta x)\Delta t-(U\Delta t)Ac(x+\Delta x)+Q(x)\Delta x\Delta t]$$

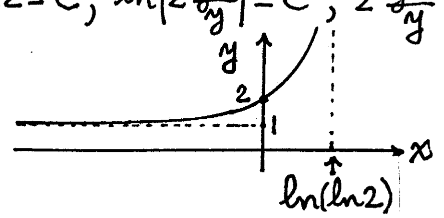
Dividing by $A\Delta x\Delta t$ and letting $\Delta x\to 0$ gives
$$kc''-Uc'-\boxed{\frac{\epsilon}{A}}c=-\frac{Q(x)}{A}$$
Let us call this β

15. (a) $\frac{du}{dt}+ku=kU$ gives $u(t)=e^{-\int k dt}\left(\int e^{\int k dt}kU\,dt+C\right)=U+Ce^{-kt}$.
$u(0)=u_0=U+C$ gives $C=u_0-U$, so $u(t)=u_0+U(1-e^{-kt})$.

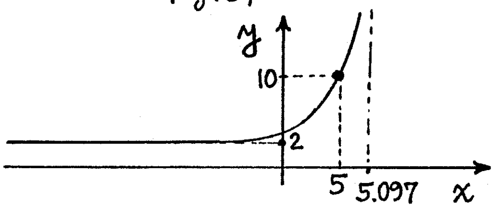
16. (a) $S(t)=S_0\left(1+\frac{k}{n}\right)^{nt}=S_0\left(1+\frac{1}{n/k}\right)^{(\frac{n}{k})kt}=S_0\left(1+\frac{1}{\text{“}m\text{”}}\right)^{mkt}\to S_0e^{kt}$ as $m\to\infty$.

Section 2.4

1. (b) $y'=6x^2+5$, $\int dy=\int(6x^2+5)dx$, $y=2x^3+5x+C$, $y(0)=0=C$, $y(x)=2x^3+5x$
(c) $y'+4y=0$, $\int\frac{dy}{y}+4\int dx=0$, $\ln y+4x=A$, $y=e^{A-4x}=Ce^{-4x}$,
$y(-1)=0=Ce^4$ gives $C=0$, so $y(x)=0$.
(e) $y'=(y^2-y)e^x$, $\int\frac{dy}{y(y-1)}=\int e^x dx$, partial fractions → $-\int\frac{dy}{y}+\int\frac{dy}{y-1}=e^x+C$,
$\ln\left|\frac{y-1}{y}\right|=e^x+C$, $y(0)=2\to -\ln 2=C$, $\ln\left|2\frac{y-1}{y}\right|=e^x$, $2\frac{y-1}{y}=e^{e^x}$,
$y(x)=\frac{2}{2-e^{e^x}}$ on $-\infty<x<\ln(\ln 2)$.

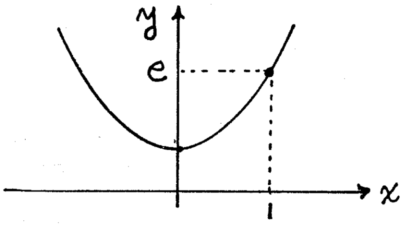


(f) $y'=y^2+y-6$, $\frac{dy}{(y-2)(y+3)}=dx$, $\frac{1}{5}\int\frac{dy}{y-2}-\frac{1}{5}\int\frac{dy}{y+3}=\int dx$, $\frac{1}{5}\ln\left|\frac{y-2}{y+3}\right|=x+C$,
$y(5)=10$ gives $C=\frac{1}{5}\ln\frac{8}{13}-5$, so
$\ln\left|\frac{13}{8}\frac{y-2}{y+3}\right|=5x-25$, $y(x)=\frac{26+24e^{5(x-5)}}{13-8e^{5(x-5)}}$



$\to +2$ as $x\to-\infty$ and $\to+\infty$ as $13-8e^{5(x-5)}\to 0$,
i.e., as $x\to\frac{1}{5}\ln\frac{13}{8}+5\approx 5.097$ from the left.

(h) $y'=6\frac{y\ln y}{x}$, $\int\frac{dy}{y\ln y}=6\int\frac{dx}{x}$. Let $\ln y=u$.
Then $\ln(\ln y)=6\ln x+\overset{\text{ln for convenience}}{\ln}C$, $\ln(\ln y)=\ln Cx^6$,
$\ln y=Cx^6$, $y=e^{Cx^6}$. $y(1)=e=e^C\to C=1$,
so $y(x)=e^{x^6}$.



2. (a) dsolve({diff(y(x),x) - 3*x^2 * exp(-y(x))=0, y(0)=0}, y(x)); gives
$y(x)=\ln(x^3+1)$.

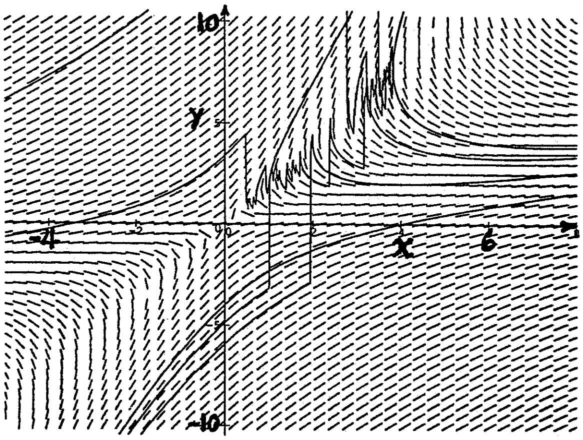
3. $\frac{du}{dt}=k(U-u)$, $\frac{du}{u-U}=-k\,dt$, $\ln(u-U)=-kt+A$, $u(t)=U+e^{-kt+A}=U+Ce^{-kt}$.
$u(0)=u_0=U+C$ gives $C=u_0-U$ so $u(t)=U+(u_0-U)e^{-kt}$.

5. $y'+py=qy^n$, where p & q are nonzero constants. $\frac{dy}{py-qy^n}=dx$.
Change variables by $v=y^{1-n}$ (consider $n\neq 0,1$ here). Then $\int\frac{dv}{(1-n)(pv-q)}$
$=\int dx$ gives $\frac{1}{p(1-n)}\ln(v-\frac{q}{p})=x+A$, $v-\frac{q}{p}=e^{p(1-n)(x+A)}$,
$y(x)=\left(\frac{q}{p}+Ce^{p(1-n)x}\right)^{1/1-n}$.

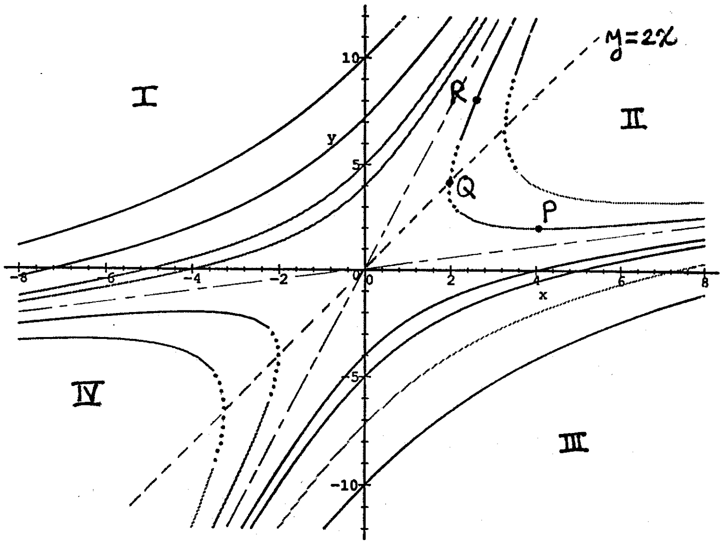
6. (b) $y'=(6x^2+1)/(y-1)$, $(y-1)dy=(6x^2+1)dx$, $y^2-y=2x^3+x+C$. $y(0)=4$ gives
$16-4=0+0+C$ so $C=12$, $y^2-y-(2x^3+x+12)=0$, $y=\frac{1\pm\sqrt{8x^3+4x+49}}{2}$.
Of these two solutions choose the + so $y(0)=4$. Thus, $y(x)=[1+\sqrt{8x^3+4x+49}]/2$.

9. (a) $y'=\frac{y}{x}$ is separable, $y'=\sin(\frac{y}{x})$ is not.
(b) $y=vx$, $y'=v'x+v=f(v)$ gives $v'=\frac{f(v)-v}{x}$.

10. (b) $y'=\frac{2y-x}{y-2x}=\frac{2v-1}{v-2}\equiv f(v)$, so $v'=\frac{\frac{2v-1}{v-2}-v}{x}=\frac{2v-1-v^2+2v}{x(v-2)}$,
$\frac{(v-2)dv}{v^2-4v+1}=-\frac{dx}{x}$, $\frac{1}{2}\ln|v^2-4v+1|$
$=-\ln x+C$
so $v=(2x\pm\sqrt{3x^2+C^2})/x$ and, since
v is y/x, $y(x)=2x\pm\sqrt{3x^2+A}$. $(A\equiv C^2)$



To understand the ± choice we've used phaseportrait to show the direction field and integral curves through a few points: [4,0], [4,2], [4,4], [4,6], but we get some zig zag "garbage"– evidently where the integral curves have vertical tangents, namely, as seen from the ODE $y'=(2y-x)/(y-2x)$, along the line $y=2x$. Thus, instead, let us use implicitplot to plot the solution $y(x)=2x\pm\sqrt{3x^2+A}$ through the representative initial points [4,-2], [4,0], [4,2], [4,4],



[0,5], [0,10], [0,-5], [0,-10], [-4,2], [-4,0], [-4,-2], [-4,-4]. For each initial point we need to choose A and the + or − sign. Since $(y-2x)^2 = 3x^2 + A$, $A = (y-2x)^2 - 3x^2$ and the points listed give the A values $A = 52, 16, -12, -32, 25, 100$. For each of these use + and then −, giving 12 curves, as shown at the right. Even using the numpoints = 2500 in the command

```
with(plots):
implicitplot({y=2*x+sqrt(x^2+52),
y=2*x-sqrt(x^2+52), and ten more
of these}, x=-8..8, y=-12..12, numpoints = 2500);
```

still there are gaps in the curves where the curve crosses the line $y=2x$. We have filled in those gaps by hand with dots. The two asymptotes $y \sim 2x \pm \sqrt{3}x = (2 \pm \sqrt{3})x$ (shown as — — —) are important. In the regions I and III, between these asymptotes, through each initial point there exists a unique solution defined on $-\infty < x < \infty$, such as the integral curves through [0,10] and [0,-10]. But consider initial pts. in II and IV: through P there exists a unique solution over $x_Q < x < \infty$, through Q there is no solution ($y' = \infty$ there), and through R there exists a unique solution over $x_Q < x < \infty$. Similarly in IV.

NOTE: The preceding problem, 2.4/10b, or one like it, is recommended for discussion in class, even including the problems encountered with phaseportrait.

11. (c) With $x = u + h$, $y = v + k$ the equation $y' = (1-y)/(x+4y-3)$ becomes $\frac{dv}{du} = \frac{1-v-k}{u+h+4v+4k-3}$ so set $1-k=0$ and $h+4k-3=0$; hence, $k=1$ and $h=-1$. Then $\frac{dv}{du} = -\frac{v}{u+4v}$. With $w = \frac{v}{u}$, $v = uw$, the latter becomes

$$\frac{dv}{du} = u\frac{dw}{du} + w = -\frac{w}{1+4w} \quad \text{so} \quad u\frac{dw}{du} = -\frac{2w+4w^2}{1+4w} \quad \text{so} \quad \int \frac{1+4w}{2w(1+2w)}\,dw = -\int \frac{du}{u}$$

$$\text{so} \quad \frac{1}{2}\ln[w(1+2w)] = -\ln u + \boxed{\frac{1}{2}\ln C} \leftarrow \text{for convenience}$$

$$\text{so} \quad \ln[w(1+2w)] = \ln(C/u^2) \quad \text{so} \quad w(1+2w) = \frac{C}{u^2}.$$ Putting back $w = v/u$, where $u = x+1$ and $v = y-1$, gives

$$2y^2 + (x-3)y - x = A \qquad *$$

where A is an arbitrary constant. We can solve $*$ for x as a single valued function of y or for y as a double valued function of x. The situation is similar to the one discussed in Exercise 10b and can be illuminated further using implicitplot.

(f) $y' = \frac{x+2y-1}{2x+4y-1}$. Let $x+2y = v$ so $\frac{dv}{dx} = 1+2\frac{dy}{dx} = 1+2\frac{v-1}{2v-1}$. Thus, $\frac{dv}{dx} = \frac{4v-3}{2v-1}$

$\int \frac{2v-1}{4v-3}\,dv = \int dx$ so $\frac{1}{2}v + \frac{1}{8}\ln(8v-6) = x + C$, or, $4(x+2y) + \ln(8x+16y-6) = 8x+A$ gives the solution in implicit form.

12. $dN/dt = kN^p$, $N^{-p}dN = k\,dt$, $\frac{N^{1-p}}{1-p} = kt + C \ (p \neq 1)$, $N(t) = [(1-p)kt + A]^{\frac{1}{1-p}}$.

For $p<1$, $N(t) \sim [(1-p)kt]^{\frac{1}{1-p}} = \alpha t^{\beta}$ where $\beta = \frac{1}{1-p} \to \begin{cases} 1 \text{ as } p \to 0 \\ \infty \text{ as } p \to 1 \end{cases}$

For $p>1$, $N(t) = \frac{1}{[A-(p-1)kt]^{\frac{1}{p-1}}} \to \infty$ as $t \to \frac{A}{(p-1)k}$, where A can be expressed in terms of N_0 since $N_0 = A^{\frac{1}{1-p}}$ gives $A = N_0^{1-p}$. Thus, $N(t) \to \infty$ as $t \to T$, where $T = 1/[(p-1)kN_0^{p-1}]$.

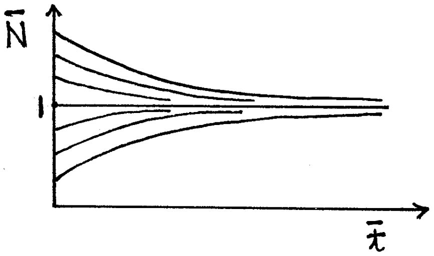
13. $dN/dt = (a-bN)N$, $N(0) = N_0$. With $\bar{t} = at$ and $\bar{N} = bN/a$,

$\frac{\frac{a}{b}d\bar{N}}{\frac{1}{a}d\bar{t}} = (a - b\frac{a}{b}\bar{N})\frac{a}{b}\bar{N}$ or $\frac{d\bar{N}}{d\bar{t}} = (1-\bar{N})\bar{N}$; $\frac{a}{b}\bar{N}(0) = N_0$ or $\bar{N}(0) = \frac{bN_0}{a} \equiv \beta$

$\frac{d\bar{N}}{\bar{N}(1-\bar{N})} = d\bar{t}$, $\ln\bar{N} - \ln(\bar{N}-1) = \bar{t} + A$, $\frac{\bar{N}}{\bar{N}-1} = Ce^{\bar{t}}$, $\bar{N}(0) = \beta$ gives $C = \frac{\beta}{\beta-1}$.

Thus, $\bar{N}(t) = -\frac{Ce^{\bar{t}}}{1-Ce^{\bar{t}}} = -\frac{\frac{\beta}{\beta-1}e^{\bar{t}}}{1-\frac{\beta}{\beta-1}e^{\bar{t}}}$

$$= \frac{\beta}{\beta + (1-\beta)e^{-\bar{t}}}$$



14. Let $\mathcal{F}, \mathcal{L}, \mathcal{T}$ stand for force, length and time. By Newton's 2nd law, mass is not independent: mass $= \frac{\text{force}}{\text{accel}} = \frac{\mathcal{F}\mathcal{T}^2}{\mathcal{L}}$

Now,

Variable	Dimension	Parameter	Dimension
t	$\mathcal{T}$	m	$\mathcal{F}\mathcal{T}^2/\mathcal{L}$
x	$\mathcal{L}$	c	$\mathcal{F}\mathcal{T}/\mathcal{L}$
		k	$\mathcal{F}/\mathcal{L}$
		F	$\mathcal{F}$
		ω	$1/\mathcal{T}$
		x_0	$\mathcal{L}$
		x_0'	$\mathcal{L}/\mathcal{T}$

To nondimensionalize t we need a combination of the parameters that has units of $\mathcal{T}$, such as $1/\omega$, x_0/x_0', m/c, or c/k; the choice is not unique. Let us use $1/\omega$, say. That is, $\bar{t} \equiv \frac{t}{1/\omega} = \omega t$.

To nondimensionalize x we need a combination of the parameters that has units of $\mathcal{L}$, such as x_0, x_0'/ω, F/k, and so on. Let us use x_0, say: $\bar{x} \equiv \frac{x}{x_0}$.

Noting that $dt = \frac{1}{\omega}d\bar{t}$, the ODE becomes

$$m\frac{d}{\frac{1}{\omega}d\bar{t}}\frac{d}{\frac{1}{\omega}d\bar{t}}x_0\bar{x}(\bar{t}) + c\frac{d}{\frac{1}{\omega}d\bar{t}}x_0\bar{x}(\bar{t}) + kx_0\bar{x}(\bar{t}) = F\sin\bar{t}\,; \quad x_0\bar{x}(0) = x_0,$$

$$\frac{d}{\frac{1}{\omega}d\bar{t}}x_0\bar{x}(0) = x_0'$$

or $m\omega^2 x_0 \frac{d^2\bar{x}}{d\bar{t}^2} + c\omega x_0 \frac{d\bar{x}}{d\bar{t}} + k x_0 \bar{x} = F \sin \bar{t}$; $\bar{x}(0)=1$, $\bar{x}'(0) = \frac{x_0'}{\omega x_0}$,

or $$\frac{d^2\bar{x}}{d\bar{t}^2} + \underset{\alpha}{\boxed{\frac{c}{m\omega}}}\frac{d\bar{x}}{d\bar{t}} + \underset{\beta}{\boxed{\frac{k}{m\omega^2}}}\bar{x} = \underset{\gamma}{\boxed{\frac{F}{m\omega^2 x_0}}}\sin\bar{t}; \quad \bar{x}(0)=1,\ \bar{x}'(0) = \underset{\delta}{\boxed{\frac{x_0'}{\omega x_0}}}$$

Thus, the nondimensionalized system contains only <u>four</u> (nondimensional) parameters $\alpha, \beta, \gamma, \delta$ rather than the original seven (dimensional) parameters. How can we see that $\alpha, \beta, \gamma, \delta$ are nondimensional? The simplest way is to use the fact that all terms in the final equation (or, indeed, in <u>any</u> equation) must have the same units. Since $d^2\bar{x}/d\bar{t}^2$ is dimensionless the other terms must be too. Since $d\bar{x}/d\bar{t}$ is dimensionless α must be. Similarly for the other terms and initial conditions. As noted above, the nondimensionalization is not unique. However, the final number of nondimensional parameter <u>is</u> unique – i.e., independent of the choices made in the nondimensionalization.

Section 2.5

1. (b) $M_y=0,\ N_x=0$ ✓ $\frac{\partial F}{\partial x} = x^2 \to F(x,y) = \int x^2 \partial x = \frac{x^3}{3} + A(y)$
$\frac{\partial F}{\partial y} = y^2 = 0 + A'(y)$ so $A(y) = \int y^2 dy = \frac{y^3}{3} + C$
so $F(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + C$ = constant gives $x^3 + y^3 = B$, say.
Then, $y(9) = -1$ gives $9^3 - 1 = B$ so $B = 728$, so $x^3 + y^3 = 728$.

(f) $M_z = 1,\ N_y = 1$ ✓ $\frac{\partial F}{\partial y} = e^y + z \to F(y,z) = \int (e^y + z)\partial y = e^y + yz + A(z)$
$\frac{\partial F}{\partial z} = y - \sin z = y + A'(z)$ so $A(z) = -\int \sin z\, dz = \cos z + C$
so $F(y,z) = e^y + yz + \cos z + C$ = const. gives $e^y + yz + \cos z = B$, say.
Then, $z(0) = 0$ gives $e^0 + 0 + \cos 0 = B$ gives $B = 1$, so $e^y + yz + \cos z = 1$.

(h) $M_y = \cos y + \cos x,\ N_x = \cos x + \cos y$ ✓
$\partial F/\partial x = \sin y + y\cos x \to F(x,y) = \int (\sin y + y\cos x)\partial x = x\sin y + y\sin x + A(y)$
$\partial F/\partial y = \cancel{\sin x} + x\cancel{\cos y} = x\cancel{\cos y} + \cancel{\sin x} + A'(y)$ so $A(y) = \int 0\, dy = C$
so $F(x,y) = x\sin y + y\sin x + C$ = const. gives $x\sin y + y\sin x = B$, say.
Then, $y(2) = 3$ gives $2\sin 3 + 3\sin 2 = B$, so $x\sin y + y\sin x = 2\sin 3 + 3\sin 2$.

4. $M_y = b,\ N_x = A$, so the equation will be exact if $A = b$.

5. (b) $M = y,\ N = x\ln x,\ M_y \neq N_x$. $\frac{M_y - N_x}{N} = \frac{1 - \ln x - 1}{x\ln x} = -\frac{1}{x}$ = fn of x alone,
so $\sigma(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$. Thus, scale the ODE as $\frac{y}{x}dx + \ln x\, dy = 0$.
$\partial F/\partial x = \frac{y}{x} \to F(x,y) = \int \frac{y}{x}\partial x = y\ln x + A(y)$
$\partial F/\partial y = \cancel{\ln x} = \cancel{\ln x} + A'(y)$ so $A(y) = C$. Thus, $F(x,y) = y\ln x + C$ = const. gives $y\ln x = B$ or $y(x) = B/\ln x$.

(e) $M = 1,\ N = x,\ M_y \neq N_x$. $\frac{M_y - N_x}{N} = \frac{0-1}{x}$ = fn of x alone, so $\sigma(x) = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

Thus, scale the ODE as $\frac{1}{x}dx + dy = 0$.

$\partial F/\partial x = \frac{1}{x} \rightarrow F(x,y) = \int \frac{1}{x} \partial x = \ln x + A(y)$

$\partial F/\partial y = 1 = 0 + A'(y)$ so $A(y) = y + C$

Thus, $F(x,y) = \ln x + y + C = \text{const.}$ gives $\ln x + y = B$, or, $y(x) = -\ln x + B$.

(h) Here, "y" is z. $M = 1-x-z, N = 1, M_z \neq N_y$. $\frac{M_z - N_x}{N} = \frac{-1-0}{1} = -1 =$ fn. of x alone so $\sigma(x) = e^{\int -dx} = e^{-x}$. Thus, scale the ODE as

$$e^{-x}(1-x-z)dx + e^{-x}dz = 0$$

$\partial F/\partial x = e^{-x}(1-x-z) \rightarrow F(x,z) = \int e^{-x}(1-x-z)\partial x = e^{-x}(x+z) + A(z)$

$\partial F/\partial z = \cancel{e^{-x}} = \cancel{e^{-x}} + A'(z)$ so $A(z) = C$. Thus, $F(x,z) = e^{-x}(x+z) + C = \text{const.}$ gives $e^{-x}(x+z) = B$ or, if we wish, $z(x) = Be^{x} - x$

6. $\underbrace{e^{\int p dx}(py-q)}_{M}dx + \underbrace{e^{\int p dx}}_{N}dy = 0$; $M_y = pe^{\int p dx}$, $N_x = pe^{\int p dx}$ ✓

$\partial F/\partial x = e^{\int p dx}(py-q) \rightarrow F(x,y) = \int e^{\int p dx}(py-q)\partial x + A(y)$

$\partial F/\partial y = e^{\int p dx} = \underbrace{\int p e^{\int p dx}dx} + A'(y)$

↑ This is $d(e^{\int p dx})$, so this integral gives $e^{\int p dx}$, which cancels with the like term on the left, giving $0 = A'(y)$, $A(y) = \text{const.}$

Thus, $F(x,y) = \int e^{\int p dx}(py-q)dx + \text{const.} = \text{const.}$ gives

$$y\underbrace{\int p e^{\int p dx}dx}_{\uparrow \text{this} = e^{\int p dx}, \text{ as noted above}} - \int e^{\int p dx} q\,dx = C$$

Thus, $ye^{\int p dx} = \int e^{\int p dx}q\,dx + C$ or $y(x) = e^{-\int p dx}\left(\int e^{\int p dx} q\,dx + C\right)$

NOTE: Observing that $\int p e^{\int p dx}dx = \int d(e^{\int p dx}) = e^{\int p dx}$ is tricky. If we reverse the order the solution is simpler:

$\partial F/\partial y = e^{\int p dx} \rightarrow F(x,y) = ye^{\int p dx} + B(x)$

$\partial F/\partial x = e^{\int p dx}(\cancel{py} - q) = \cancel{ype^{\int p dx}} + B'(x)$ gives $B(x) = -\int e^{\int p dx} q\,dx + \text{const.}$

so $F(x,y) = \text{const.}$ gives $ye^{\int p dx} - \int e^{\int p dx} q\,dx + \text{const.} = \text{const.}$, which gives the same result, but more easily.

7. (b) $(M_y - N_x)/N = (3x + 4y - 6x - 4y)/(3x^2 + 4xy) \neq$ fn. of x alone,

(")/M = (")$/(3xy + 2y^2) \neq$ " " y ", so $\sigma(x)$ and $\sigma(y)$ do not exist. Try $\sigma = x^a y^b$: $\underbrace{x^a y^b(3xy+2y^2)}_{\text{new } M}dx + \underbrace{x^a y^b(3x^2+4xy)}_{\text{new } N}dy = 0$

Set $M_y = N_x$, i.e., $3x^{a+1}(b+1)y^b + 2x^a(b+2)y^{b+1} = 3(a+2)x^{a+1}y^b + 4(a+1)x^a y^{b+1}$ which can be satisfied by setting $3(b+1) = 3(a+2)$ and $2(b+2) = 4(a+1)$, i.e., $a=1$ and $b=2$. Then our exact equation is

$$(3x^2y^3 + 2xy^4)dx + (3x^3y^2 + 4x^2y^3)dy = 0$$

$\partial F/\partial x = 3x^2y^3 + 2xy^4 \rightarrow F = \int(3x^2y^3 + 2xy^4)\partial x = x^3y^3 + x^2y^4 + A(y)$

$\partial F/\partial y = \cancel{3x^3y^2 + 4x^2y^3} = \cancel{3x^3y^2 + 4x^2y^3} + A'(y) \rightarrow A(y) = \text{const.}$

so $F(x,y) = \text{Const.}$ gives the solution $x^3y^3 + x^2y^4 = C$.

8. The idea is that $f(x)dx + g(y)dy = 0$ is exact, for any functions $f(x)$ and $g(y)$. Thus, $h(y)dx + i(x)dy = 0$ can be made exact, easily, by dividing by $i(x)$ and $h(y)$, to obtain $\frac{1}{i(x)}dx + \frac{1}{h(y)}dy = 0$. That is, $\sigma(x,y) = 1/[i(x)h(y)]$.

(b) Thus, $e^{-3x}dx - y^{-2}dy = 0$. We can say $\partial F/\partial x = e^{-3x}$ so $F = \int e^{-3x}\,\partial x =$ etc and $\partial F/\partial y = -y^{-2}$ so ... etc, but it is simpler (and equivalent) to merely integrate: $\int e^{-3x}dx - \int y^{-2}dy = 0$, $\frac{e^{-3x}}{-3} + \frac{1}{y} = C$, or, $y(x) = 1/(C + \frac{1}{3}e^{-3x})$.

(c) $\cot x\,dx - e^{-2y}dy = 0$, $\int \cos x\,dx/\sin x - \int e^{-2y}dy =$ const., $\ln(\sin x) + \frac{1}{2}e^{-2y} = C$ or, $y(x) = -\frac{1}{2}\ln[A - 2\ln(\sin x)]$ ($2C \to A$, for convenience)

9. (b) $\underbrace{(2r\sin\theta+1)}_{M(r,\theta)}dr + \underbrace{r^2\cos\theta}_{N(r,\theta)}\,d\theta = 0$, $M_\theta = 2r\cos\theta = N_r$ so exact.

$\partial F/\partial r = 2r\sin\theta + 1 \to F(r,\theta) = \int(2r\sin\theta+1)\,\partial r = r^2\sin\theta + r + A(\theta)$

$\partial F/\partial\theta = r^2\cancel{\cos\theta} = r^2\cancel{\cos\theta} + A'(\theta)$ gives $A(\theta) =$ const., so $F(r,\theta) =$ const. gives the solution $r^2\sin\theta + r = C$ (could solve for $r(\theta)$ or $\theta(r)$, if desired).

(c) $(2xy - e^y)dx + x(x - e^y)dy = 0$, $M_y = 2x - e^y = N_x$, so exact.

$\partial F/\partial x = 2xy - e^y \to F(x,y) = \int(2xy - e^y)\,\partial x = x^2y - xe^y + A(y)$

$\partial F/\partial y = x^2 - xe^y = x^2 - xe^y + A'(y)$ gives $A(y) =$ const, so $x^2y - xe^y = C$.

10. $\sigma = 1$ (or any nonzero constant)

11. (b) Not necessarily. For ex. if $M(x,y) = e^{xy}$ and $M(y,x) = e^{yx}$, then $M_y(x,y) = xe^{xy}$ whereas $M_x(y,x) = ye^{xy} \neq xe^{xy}$.

12. $F(a,b) = C$, so particular solution is $F(x,y) = F(a,b)$.

13. Does $(M+P)_y = (N+Q)_x$? Yes, because it gives $\cancel{M_y} + \cancel{P_y} = \cancel{N_x} + \cancel{Q_x}$ or $0 = 0$ ✓

CHAPTER 3

Section 3.2

1. (b) a set is LD if it is not LI, so it cant be both. NO.

2. (b) $\{x^2, x^2+x, x^2+x+1, x-1\}$. $(x^2+x)-(x^2) = x$

$$= \tfrac{1}{2}[(x^2+x+1)-(x^2)+(x-1)]$$

i.e., $1(x^2+x)-\frac{1}{2}(x^2+x+1)-\frac{1}{2}(x^2)-\frac{1}{2}(x-1) = 0$.

(g) $6(0)+0(x)+0(x^3)=0$, where 6,0,0 are not all zero.

(h) $6(x)-3(2x)+0(x^2)=0$, where 6,-3,0 are not all zero.

3. (b) Use Theorem 3.2.2:

$$W[e^{a_1x},\dots,e^{a_nx}] = \begin{vmatrix} e^{a_1x} & e^{a_2x} & \cdots & e^{a_nx} \\ a_1e^{a_1x} & a_2e^{a_2x} & \cdots & a_ne^{a_nx} \\ \vdots & \vdots & & \vdots \\ a_1^{n-1}e^{a_1x} & a_2^{n-1}e^{a_2x} & \cdots & a_n^{n-1}e^{a_nx} \end{vmatrix}.$$

By property D7 in Section 10.4, this

$$= e^{a_1x}e^{a_2x}\cdots e^{a_nx} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

The latter determinant is a Vandermonde determinant (see Exercise 17, Section 10.4) so if the a_j's are distinct then that determinant, and hence W (since the e^{a_jx} factors are nonzero for all x), is nonzero. It follows from Theorem 3.2.2 that if the a_j's are distinct then $\{e^{a_1x},\dots,e^{a_nx}\}$ is LI. Surely, if the a_j's are not distinct then the set is LD. For suppose $a_1=a_3$, for instance. Then $4e^{a_1x}+0e^{a_2x}-4e^{a_3x}+0e^{a_4x}+\dots+0e^{a_nx}=0$ with the coefficients 4,0,-4,0,…,0 not all 0.

(c)

$$W[1,1+x,1+x^2] = \begin{vmatrix} 1 & 1+x & 1+x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = \text{etc} = 2 \neq 0$$ so (Theorem 3.2.2) LI

(e) $$W[\sin x, \cos x, \sinh x] = \begin{vmatrix} \sin x & \cos x & \sinh x \\ \cos x & -\sin x & \cosh x \\ -\sin x & -\cos x & \sinh x \end{vmatrix} = \text{etc} = -2\sinh x,$$ which is not identically 0 on any interval. Hence (Theorem 3.2.2), LI.

(f)

$W[x,x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$, which is not identically 0 on any interval. Hence (Theorem 3.2.2), LI. Since there are only two functions in the set, it is simpler to use Theorem 3.2.4: neither is a scalar multiple of the other; hence, they are LI.

(g) LI by Theorem 3.2.4.

4. (b)

$W[\sin 2x, \cos 2x] = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$ so (Thm. 3.2.3) LI.

(c) (as in (b), we'll omit the straightforward verification that the functions are indeed solutions of the ODE.) $W = \begin{vmatrix} e^x & xe^x & e^{4x} \\ e^x & e^x+xe^x & 4e^{4x} \\ e^x & 2e^x+xe^x & 16e^{4x} \end{vmatrix} = (e^x)(e^x)(e^{4x}) \begin{vmatrix} 1 & x & 1 \\ 1 & 1+x & 4 \\ 1 & 2+x & 16 \end{vmatrix}$,

by property D7 (Section 10.4), $= e^{6x}(9) \neq 0$ so (Thm. 3.2.3) LI. Of course, we don't <u>need</u> property D7, we could simply use (B5c) in Appendix B.

5. (a)
$$\begin{aligned} W'(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' &= (y_1y_2' - y_1'y_2)' = y_1y_2'' + \cancel{y_1'y_2'} - \cancel{y_1'y_2'} - y_1''y_2 \\ &= y_1y_2'' - y_1''y_2 \\ &= y_1(-p_1y_2' - \cancel{p_2y_2}) - (-p_1y_1' - \cancel{p_2y_1})y_2 \quad \text{since } y_1''+p_1y_1'+p_2y_1=0 \text{ and } y_2''+p_1y_2'+p_2y_2=0 \\ &= p_1(y_1'y_2 - y_1y_2') \\ &= -p_1 \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -p_1W(x) \end{aligned}$$

Then, (9) in Section 2.2 gives $W(x) = W(\xi)\, e^{-\int_\xi^x p_1(t)dt}$. ✓

6. (a) If they are LD then $a_1u_1(x) + a_2u_2(x) = 0$ (on I) with a_1, a_2 not both 0. Thus a_1 and/or a_2 are nonzero. Let $a_2 \neq 0$, say. Then we can divide by a_2 and obtain $u_2(x) = -\frac{a_1}{a_2}u_1(x)$, so u_1 is expressible as a multiple of u_2. Conversely, suppose one (say u_1) can be expressed as a multiple of the other: $u_1 = \alpha u_2$. Then $1u_1(x) - \alpha u_2(x) = 0$ where not both coefficients are zero (since the first is 1); hence u_1, u_2 are LD.

(b) Let $u_2(x) = 0$, say. Then surely $0u_1(x) + 5u_2(x) + 0u_3(x) + \cdots + 0u_n(x) = 0$ with the coefficients $0, 5, 0, \ldots, 0$ not all 0. Hence, the set is LD.

(c) $a_1u_1(x) + \cdots + a_nu_n(x) = b_1u_1(x) + \cdots + b_nu_n(x)$ gives $(a_1-b_1)u_1(x) + \cdots + (a_n-b_n)u_n(x) = 0$. Since $u_1, \ldots, u_n$ are LI, it follows that $a_1-b_1 = 0, \ldots, a_n-b_n = 0$; i.e., $a_1 = b_1, \ldots, a_n = b_n$.

7. No, it does <u>not</u> follow. For ex., 1 and x are LI (Thm 3.2.4), 1 and $1+2x$ are LI, and x and $1+2x$ are LI, yet $\{1, x, 1+2x\}$ is LD since $1(1) - 2(x) + 1(1+2x) = 0$.

8. No, because the theorem does not apply since its conditions are not met. Specifically, $p_1(x) = -4/x$ and $p_2(x) = 6/x^2$ are not continuous on any interval containing the point $x=0$.

Section 3.3

1. (b) $e^x - e^{2x}$ and e^x are solutions (as is easily verified by substitution) and they are LI (one is not a multiple of the other), so $C_1(e^x - e^{2x}) + C_2e^x$ <u>is</u> a general solution.

(c) $e^{-x} + e^{2x}$ is a solution, but we need <u>two</u> LI solutions for a general solution.

(e) No, we need <u>three</u> LI solutions.

(f) Yes. (g) No (h) Yes (i) Yes

2. (b) e^{3x} and $\cosh 3x$ <u>are</u> solutions, they are LI, and there are two of them. Hence $\{e^{3x}, \cosh 3x\}$ is a basis for $y''-9y=0$.
   (c) No, because $\sinh 3x$ and $2\cosh 3x$ are not solutions of the ODE.
   (e) Yes, they are 3 LI solutions so they constitute a basis. (f) Yes

3. (c) On $0<x<\infty$? Yes. On $-\infty<x<0$? Yes.

4. (b) No; neither e^{x} nor e^{-x} is a solution of the ODE
   (d) $x+x\ln|x|$ and $x-x\ln|x|$ are LI solutions of the ODE on any interval not containing the origin — such as $-\infty<x<0$, $0<x<\infty$, and $6<x<10$.

5. (b) It is not, because it contains only 6 LI solutions; e.g., the $\sinh x$ is a linear combination of the e^{x} and the e^{-x} and the $\cosh 2x$ is a linear combination of the e^{2x} and the e^{-2x}.

6. Yes, $y(x)=3$ is a solution. No contradiction; when we say that Thm 3.3.2 does not hold for nonlinear or nonhomogeneous we are saying that if $y_1(x)$ and $y_2(x)$ are solutions of a nonlinear " " equation (the ODE in this exercise is nonlinear) then $C_1y_1(x)+C_2y_2(x)$ is not <u>necessarily</u> a solution too — it <u>could</u> be, by coincidence, as in this case.

8. (b) The answer is $y(x)=-1-2x^2-\frac{2}{3}x^4-\frac{4}{45}x^6+\cdots$, as can be checked using these Maple commands:

```
Order := 8;
dsolve({diff(y(x),x,x)-4*y(x)=0, y(0)=-1, D(y)(0)=0},
        y(x), type=series);
```

   (c) $y(x)=2-5x+\frac{13}{2}x^2-\frac{35}{6}x^3+\frac{97}{24}x^4-\frac{55}{24}x^5+\cdots$
   (e) $y(x)=2-3x-\frac{1}{6}x^4+\frac{3}{20}x^5+\cdots$

9. (b) The ODE is of the type (5a) and the conditions are initial conditions like (5b). $p_1(x)=2$ and $p_2(x)=3$ are continuous for all x so, by Thm 3.3.1, the problem admits a unique solution on $-\infty<x<\infty$.
   (f) $p_1(x)=x/\sin x$ is continuous on $-\pi<x<\pi$ (containing the initial point $x=2$) as are $p_2(x)=p_3(x)=p_4(x)=0$, so, by Thm 3.3.1, the problem admits a unique solution on that interval.

11. (c) $y(x)=C_1\cos x+C_2\sin x$, $y(1)=1=C_1\cos 1+C_2\sin 1$
$y(2)=2=C_1\cos 2+C_2\sin 2$
has a unique solution for C_1, C_2 because $\begin{vmatrix}\cos 1 & \sin 1\\ \cos 2 & \sin 2\end{vmatrix}=\cos 1\sin 2-\sin 1\cos 2=\sin(2-1)$ $=\sin 1\neq 0$. Namely, $C_1=-0.920$, $C_2=1.779$. Thus, the boundary-value problem has the unique solution $y(x)=-0.920\cos x+1.779\sin x$.

13. Surely (10) implies (13.1a) (by choosing $\alpha=\beta=1$) and (13.1b) (by choosing $\beta=0$), but we also need to show that (13.1a,b) imply (10), which we do next:
$L[\alpha u+\beta v]=L[\alpha u]+L[\beta v]$ (by 13.1a) $=\alpha L[u]+\beta L[v]$ (by 13.1b).

14. If (11) holds for k, then $L[\alpha_1u_1+\cdots+\alpha_{k+1}u_{k+1}]=L[1(\alpha_1u_1+\cdots+\alpha_ku_{k+1})+\alpha_{k+1}u_{k+1}]$ $=1L[\alpha_1u_1+\cdots+\alpha_ku_k]+\alpha_{k+1}L[u_{k+1}]$ by (11) with $k=2$. Further, from (11) this $=\alpha_1L[u_1]+\cdots+\alpha_kL[u_k]+\alpha_{k+1}L[u_{k+1}]$, so (11) holds for $k+1$. Hence, $P(k)$ holds for <u>all</u> $k\geq 1$.

Section 3.4

4. (b) $y(x) = A + Be^{x}$

(c) $y(x) = A + Be^{-x}$, $y(0)=3$ and $y'(0)=0$ give $A+B=3$, $-B=0$ so $B=0, A=3$, $y(x)=3$.

(n) $y = e^{\lambda x} \rightarrow \lambda^4 - 1 = 0$, $\lambda^4 = 1$, $\lambda^2 = \pm 1$, $\lambda = \pm 1, \pm i$ so $y(x) = Ae^{x} + Be^{-x} + Ce^{ix} + De^{-ix}$ or $y(x) = E\cosh x + F\sinh x + G\cos x + H\sin x$, for example.

(o) $y = e^{\lambda x} \rightarrow \lambda^4 - 2\lambda^2 - 3 = 0$, $\lambda^2 = (2 \pm \sqrt{4+12})/2 = 1 \pm 2 = 3, -1$; $\lambda = \pm\sqrt{3}$ and $\pm i$ so $y(x) = Ae^{\sqrt{3}x} + Be^{-\sqrt{3}x} + C\cos x + D\sin x$.

5. (e) dsolve({diff(y(x), x, x) − 4*diff(y(x), x) − 5*y(x) = 0, y(1)=1, D(y)(1)=0}, y(x)); gives $y(x) = \frac{5}{6}\frac{e^{-x}}{e^{-1}} + \frac{1}{6}\frac{e^{5x}}{e^{5}}$

(n) dsolve(diff(y(x), x, x, x, x) − y(x) = 0, y(x)); gives
$y(x) = C_1 e^{x} + C_2 \cos x + C_3 \sin x + C_4 e^{-x}$

6. (b) $y(x) = (A+Bx)e^{-3x}$, $\left.\begin{array}{l} y(1) = e = (A+B)e^{-3} \\ y'(1) = -2 = -(3A+2B)e^{-3} \end{array}\right\} \Rightarrow \begin{array}{l} A = 2e^{3}(1-e), \\ B = e^{3}(3e-2) \end{array}$

so $y(x) = [2(1-e) + (3e-2)x]e^{-3(x-1)}$

(c) $y(x) = A + Bx + Cx^2$, $y(0) = 3 = A$, $y'(0) = -5 = B$, $y''(0) = 1 = 2C$, so $y(x) = 3 - 5x + \frac{1}{2}x^2$

8. (b) $(\lambda - 2i)(\lambda + 2i) = \lambda^2 + 4$, so the ODE is $y'' + 4y = 0$. $y(x) = Ae^{i2x} + Be^{-i2x}$ or $C\cos 2x + D\sin 2x$.

(c) $(\lambda - (4-2i))(\lambda - (4+2i)) = \lambda^2 - 8\lambda + 20$, so the ODE is $y'' - 8y' + 20y = 0$ with general solution $y(x) = Ae^{(4-2i)x} + Be^{(4+2i)x} = e^{4x}(C\cos 2x + D\sin 2x)$

(f) $(\lambda-1)^2(\lambda+2) = \lambda^3 - 3\lambda + 2$, so the ODE is $y''' - 3y' + 2y = 0$ with general solution $y(x) = (A+Bx)e^{x} + Ce^{-2x}$.

9. (b) $\lambda^2 - 3i\lambda - 2 = 0$ gives $\lambda = (3i \pm \sqrt{-9+8})/2 = i, 2i$ so $y(x) = Ae^{ix} + Be^{i2x}$

(c) $\lambda^2 + i\lambda - 1 = 0$ gives $\lambda = (-i \pm \sqrt{-1+4})/2 = (-i \pm \sqrt{3})/2$ so $y(x) = e^{-ix/2}(Ae^{\sqrt{3}x/2} + Be^{-\sqrt{3}x})$

10. Remember that, in maple, $i = \sqrt{-1}$ is written as I.

11. (a) $(D-\lambda_1)\underbrace{(D-\lambda_2)y}_{u} = 0$. $u' - \lambda_1 u = 0$ gives $u_1 = Ae^{\lambda_1 x}$. Then $(D-\lambda_2)y = u$ becomes $y' - \lambda_2 y = Ae^{\lambda_1 x}$, which, being first-order linear, gives

$$y(x) = e^{-\int -\lambda_2 dx}\left(\int e^{-\int \lambda_2 dx} Ae^{\lambda_1 x} dx + B\right) = e^{\lambda_2 x}\left(\int Ae^{(\lambda_1 - \lambda_2)x} dx + B\right)$$
$$= e^{\lambda_2 x}\left(\frac{A}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2)x} + B\right) = Ce^{\lambda_1 x} + Be^{\lambda_2 x} \quad \text{(B, C arbitrary constants)}$$

12. (b) $\lambda \approx -2.52, -0.239 \pm 0.858i$. Each $\operatorname{Re}\lambda < 0$, so stable. The maple command used was fsolve (x^3 + 3*x^2 + 2*x + 2 = 0, x, complex);

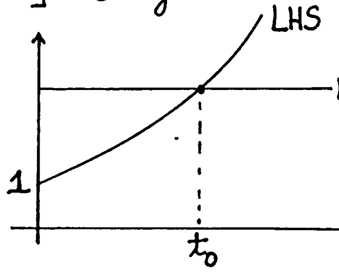
(e) $\lambda \approx -0.793 \pm 0.458i, -0.297 \pm 2.236i, +0.590 \pm 0.597i$ hence unstable, because of the +0.590. This result is in accord with Theorem 3.4.4 because the polynomial in λ has mixed signs.

Section 3.5

1. (b) $3\cos 6t - 4\sin 6t = E\sin(\omega t + \phi)$, $E = \sqrt{3^2+4^2} = 5$, $\phi = \tan^{-1}(\frac{3}{-4}) = -0.6435$ rad

5. It is striking that the frequency $\omega = \sqrt{k/m}$ is fixed; i.e., it is independent of x_0 (and x_0'). While true for the linear oscillator $m\ddot{x} + kx = 0$, it is not true for nonlinear oscillators, as we will see in Chapter 7.

6. (a) The form $x(t) = e^{-\alpha t}(Ae^{\sqrt{\Gamma}t} + Be^{-\sqrt{\Gamma}t})$ will be convenient, where α is $c/2m$ and $\sqrt{\Gamma}$ is $\sqrt{\alpha^2 - \omega^2}$; A, B are, of course, dictated by the initial conditions. Then $x'(t) = e^{-\alpha t}[-\alpha Ae^{\sqrt{\Gamma}t} - \alpha Be^{-\sqrt{\Gamma}t} + A\sqrt{\Gamma}e^{\sqrt{\Gamma}t} - B\sqrt{\Gamma}e^{-\sqrt{\Gamma}t}] = 0$ gives $e^{2\sqrt{\Gamma}t} = \frac{B}{A}\frac{\sqrt{\Gamma}+\alpha}{\sqrt{\Gamma}-\alpha}$. The graphs of the LHS and RHS are sketched at the right. Since the LHS is a monotone function of t and the RHS is a constant, we have exactly one flatspot (at t_0) if the initial conditions are such that RHS > 1 and none if RHS < 1. The foregoing is for the overdamped case. For the critically damped case $x(t) = (A+Bt)e^{-\alpha t}$ and $x'(t) = (-\alpha A + B - \alpha Bt)e^{-\alpha t} = 0$ gives "t_0" $= (B - \alpha A)/(\alpha B)$. If the latter is negative then there are no flat spots on $0 \leq t < \infty$, and if it is positive then there is one flat spot on $0 \leq t < \infty$.



(b) Let $m = k = 1$ and $c = c_{cr} = \sqrt{4mk} = 2$. Then $\alpha = c/2m = 2/2 = 1$ so $x(t) = (A+Bt)e^{-t}$. $t_0 = (B - \alpha A)/(\alpha B) = (B - A)/B$. If $B = 1$ and $A = 2$ then $t_0 < 0$ so there are no flatspots; in this case $x(0) = x_0 = 2$ and $x'(0) = x_0' = -1$. (C) If instead we let $B = 1$ and $A = -1$, then $t_0 = 2 > 0$ so there is one flatspot; in this case $x(0) = x_0 = -1$ and $x'(0) = x_0' = 2$. Of course these choices are by no means uniq

NOTE that this is a "**design**" question – how to design the physical system (i.e., how to choose m, c, k, x_0, x_0') so as to achieve a certain behavior.

7. (a) $x(t) = e^{-\alpha t}(A\cos\sqrt{\Gamma}t + B\sin\sqrt{\Gamma}t)$, where α is $c/2m$ and $\sqrt{\Gamma}$ is $\sqrt{\omega^2 - (c/2m)^2}$. $x'(t) = 0$ gives $\tan\sqrt{\Gamma}t = (\sqrt{\Gamma}B - \alpha A)/(\alpha B + \sqrt{\Gamma}A) \equiv *$, say. The latter has roots $\sqrt{\Gamma}t = \sqrt{\Gamma}t_0 + n\pi$ (where $t_0 = \tan^{-1}\frac{*}{\sqrt{\Gamma}}$ in $-\frac{\pi}{2} < t_0 < \frac{\pi}{2}$). But successive flat spots are max, min, max, ..., so to consider successive maxima change the $n\pi$ to $2n\pi$ and write $\sqrt{\Gamma}t = \sqrt{\Gamma}t_0 + 2n\pi$. Then, if x_n and x_{n+1} are successive maxima of $x(t)$,

$$r = \frac{x_n}{x_{n+1}} = \frac{\exp[-\alpha(t_0 + 2n\pi/\sqrt{\Gamma})][A\cos(\sqrt{\Gamma}t_0 + 2n\pi) + B\sin(\sqrt{\Gamma}t_0 + 2n\pi)]}{\exp[-\alpha(t_0 + 2(n+1)\pi/\sqrt{\Gamma})][A\cos(\sqrt{\Gamma}t_0 + 2(n+1)\pi) + B\sin(\sqrt{\Gamma}t_0 + 2(n+1)\pi)]}$$

$= \exp(+2\pi\alpha/\sqrt{\Gamma})$ is a constant (i.e., doesn't change with r)

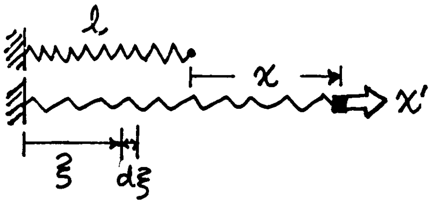
(b) logarithmic decrement $\delta = \ln r = \ln \exp(\frac{2\pi\alpha}{\sqrt{\Gamma}}) = \frac{2\pi\alpha}{\sqrt{\Gamma}} = \frac{2\pi c/(2m)}{\sqrt{\omega^2 - (c/2m)^2}}$

8. If $\epsilon \ll 1$, then $\theta'' + \epsilon\theta' + \frac{g}{L}\theta = 0$ is underdamped and its solution is given by (12) with $m \to 1$, $c \to \epsilon$, $k \to g/L$:

$$\theta(t) = e^{-\epsilon t/2}\left[A\cos\sqrt{\tfrac{g}{L} - (\tfrac{\epsilon}{2})^2}\,t + B\sin\sqrt{\tfrac{g}{L} - (\tfrac{\epsilon}{2})^2}\,t\right]$$

The oscillation frequency, $\sqrt{(g/L) - (\epsilon/2)^2}$, is a constant, even as the magnitude damps out due to the $\exp(-\epsilon t/2)$ factor.

9. (b)

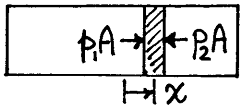


$$\text{KE in spring} = \int_{\xi=0}^{\xi=l+x} \frac{1}{2}\left(\frac{d\xi}{l+x} m_s\right)\left[\frac{\xi}{l+x} x'\right]^2$$

$$= \frac{1}{2}\frac{m_s x'^2}{(l+x)^3}\frac{(l+x)^3}{3} = \frac{1}{6} m_s x'^2$$

Including this spring KE gives (9.2), and d/dt of (9.2) gives (9.3).

10. (a)



Newton's 2nd law → $mx'' = (p_1 - p_2)A$.

Boyle's law → $p_2(L-x)A = p_1(L+x)A = p_0LA$

gives $p_1 = \frac{p_0L}{L+x}$, $p_2 = \frac{p_0L}{L-x}$

so $mx'' + p_0LA\left(\frac{1}{L-x} - \frac{1}{L+x}\right) = 0$,

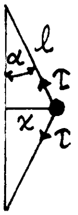
$mx'' + \frac{2p_0ALx}{L^2 - x^2} = 0$

(b) Nonlinear due to the $x/(L^2-x^2)$ term.

(c) Taylor series: $x/(L^2-x^2) = \frac{x}{L^2}\frac{1}{1-(\frac{x}{L})^2} = \frac{x}{L^2}\left(1 + \frac{x^2}{L^2} + \frac{x^4}{L^4} + \cdots\right) \sim \frac{x}{L^2}$ gives the linearized version for small x (i.e., for $|x/L| \ll 1$): $mx'' + 2\frac{p_0A}{L}x = 0$.

(d) freq $= \sqrt{\frac{2p_0A}{mL}} \frac{\text{rad}}{\text{sec}} \frac{1\,\text{cycle}}{2\pi\,\text{rad}} = \frac{1}{2\pi}\sqrt{\frac{2p_0A}{mL}} \frac{\text{cycles}}{\text{sec}}$.

(e) Yes



11. (a) $mx'' = -2\tau\sin\alpha$ (see sketch at right) $= -2\tau(l)x/l$

so $mx'' + 2\frac{\tau(\sqrt{l_0^2+x^2})}{\sqrt{l_0^2+x^2}}x = 0$

(b) Nonlinear because $\tau(\sqrt{l_0^2+x^2})x/\sqrt{l_0^2+x^2}$ is not a linear function of x.

(c) $\tau[l(x)] = \tau[l(0)] + \left.\frac{d\tau}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2\tau}{dx^2}\right|_{x=0} x^2 + \text{etc.}$

$\hookrightarrow \frac{d\tau}{dx} = \frac{d\tau}{dl}\frac{dl}{dx} = \frac{d\tau}{dl}\frac{1}{2}\frac{2x}{l}$, $\frac{d^2\tau}{dx^2} = \frac{d\tau}{dl}\frac{1}{l} + \frac{x}{l}\frac{d^2\tau}{dl^2}\frac{1}{2}\frac{2x}{l}$

so $\left.\frac{d\tau}{dx}\right|_x = 0$ and $\left.\frac{d^2\tau}{dx^2}\right|_{x=0} = \tau'(l_0)/l_0 + 0$

Thus, $\tau[l(x)] = \tau(l_0) + 0x + \frac{1}{2!}\frac{\tau'(l_0)}{l_0}x^2 + \cdots$.

It *might* be clearer to proceed, instead, like this:

$\tau[l(x)] = \tau(\sqrt{l_0^2+x^2}) = \tau\{l_0[1+(\frac{x}{l_0})^2]^{1/2}\} = \tau\{l_0(1 + \frac{1}{2}\frac{x^2}{l_0^2} - \frac{1}{8}\frac{x^4}{l_0^4} + \cdots)\}$

$= \tau[l_0 + \underbrace{(\frac{1}{2}\frac{x^2}{l_0} + \cdots)}_{\text{Call this } Z}] = \tau(l_0 + Z) = \tau(l_0) + \tau'(l_0)Z + \frac{1}{2!}\tau''(l_0)Z^2 + \cdots$

$= \tau(l_0) + \tau'(l_0)(\frac{1}{2}\frac{x^2}{l_0} + \cdots) + \frac{1}{2!}\tau''(l_0)(\frac{1}{2}\frac{x^2}{l_0} + \cdots)^2 + \cdots$

Rearranging (formally) in ascending powers of x gives

$\tau[l(x)] = \tau(l_0) + \tau'(l_0)\frac{x^2}{2l_0}$ + terms of order $x^4, x^6, \ldots$.

Since we want the Taylor series of $\tau[l(x)]/l(x)$ we also need to expand the $1/l(x)$ factor and then multiply its series into the series for $\tau[l(x)]$.

$$\frac{1}{\ell(x)} = (\ell_0^2+x^2)^{-1/2} = \frac{1}{\ell_0}\left[1+\left(\frac{x}{\ell_0}\right)^2\right]^{-1/2} = \frac{1}{\ell_0}\left[1-\frac{1}{2}\frac{x^2}{\ell_0^2}+\cdots\right], \text{ so}$$

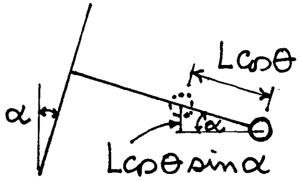
$$\frac{\tau[\ell(x)]}{\ell(x)} = \left[\tau(\ell_0)+\tau'(\ell_0)\frac{x^2}{2\ell_0}+\cdots\right]\frac{1}{\ell_0}\left(1-\frac{1}{2}\frac{x^2}{\ell_0^2}+\cdots\right) = \frac{\tau(\ell_0)}{\ell_0} + \left[\tau'(\ell_0)\frac{1}{2\ell_0^2}-\frac{\tau(\ell_0)}{2\ell_0^3}\right]x^2+\cdots$$

(d) Linearizing (i.e., keeping terms through x to the first power), $\frac{\tau[\ell(x)]}{\ell(x)}x \sim \frac{\tau(\ell_0)}{\ell_0}x$ so the linearized ODE is $mx'' + \boxed{2\frac{\tau(\ell_0)}{\ell_0}}x = 0$ ← "$k_{equiv.}$"

$$\text{Frequency} = \sqrt{k_{eq.}/m}\ \frac{\text{rad}}{\text{sec}} = \sqrt{\frac{2\tau_0}{\ell_0 m}}\ \frac{\text{rad}}{\text{sec}}\ \frac{1\text{ cycle}}{2\pi\text{ rad}} = \frac{1}{2\pi}\sqrt{\frac{2\tau_0}{m\ell_0}}\ \text{cycles/sec}$$

12. Σ Vertical forces $=0$ gives $N_1+N_2=mg$
Σ Moments about left-hand cylinder gives $N_2L = mg(x+\frac{L}{2})$ } so $N_2 = mg(\frac{1}{2}+\frac{x}{L})$, $N_1 = mg(\frac{1}{2}-\frac{x}{L})$
Then, $mx'' = \Sigma$ Horizontal forces
$= \mu N_1 - \mu N_2 = \mu mg(\frac{1}{2}-\frac{x}{L}) - \mu mg(\frac{1}{2}+\frac{x}{L})$
or, $mx'' + \frac{2mg\mu}{L}x = 0$.
Frequency $= \sqrt{2mg\mu/mL} = \sqrt{2g\mu/L}$ rad/sec

13. (a) Let potential energy (due to gravity) $=0$ when m is in the position shown.



so $PE = mg(L\cos\theta\sin\alpha)$
$KE = \frac{1}{2}m(L\theta')^2$
so $PE+KE = mgL\cos\theta\sin\alpha + \frac{1}{2}m(L\theta')^2 = \text{const}$.
d/dt gives $-mgL\sin\theta\,\theta'\sin\alpha + \frac{1}{2}mL^2 2\theta''\theta' = 0$
$\theta'' + \frac{g}{L}\sin\alpha\sin\theta = 0$

(b) Linearized, $\theta'' + \frac{g\sin\alpha}{L}\theta = 0$ so freq. $= \sqrt{\frac{g\sin\alpha}{L}}\ \frac{\text{rad}}{\text{sec}} = \frac{1}{2\pi}\sqrt{\frac{g\sin\alpha}{L}}\ \frac{\text{cycles}}{\text{sec}}$

Section 3.6

1.(b) $y=x^\lambda$ gives $\lambda-1=0$ so $\lambda=1$, $y=Ax$, $y(2)=5=2A$ so $A=5/2$ and $y(x)=5x/2$ $(-\infty<x<\infty$

(c) $\lambda^2-\lambda+\lambda=0$, $\lambda=0,0$, $y(x)=(A+B\ln|x|)x^0 = A+B\ln|x| = \begin{cases} A+B\ln x & \text{for } 0<x<\infty \\ A+B\ln(-x) & \text{for } -\infty<x<0 \end{cases}$

(e) $\lambda^2-\lambda+\lambda-9=0$, $\lambda=\pm3$, $y=Ax^3+Bx^{-3}$. $y(2)=1=8A+B/8$ and $y'(2)=2=12A-3B/16$
so $y(x)=\frac{7}{48}x^3-\frac{4}{3}x^{-3}$ on $0<x<\infty$

(f) $\lambda^2-\lambda+\lambda+1=0$, $\lambda=\pm i$, $y=A\cos(\ln x)+B\sin(\ln x)$.
$y(1)=1=A$, $y'(1)=0=B$, so $y(x)=\cos(\ln x)$ on $0<x<\infty$.

(h) $\lambda=2,-1$, $y=Ax^2+B/x$. $y(-5)=3=25A-B/5$, $y'(-5)=0=-10A-B/25$; $A=1/25$, $B=-10$,
so $y(x)=x^2/25-10/x$ on $-\infty<x<0$

(m) $\lambda(\lambda-1)(\lambda-2)-2\lambda=0$, $\lambda=0,0,3$; $y(x)=A+B\ln|x|+Cx^3$. $y(1)=2=A+C$, $y'(1)=0=B+3C$.
$y''(1)=0=-B+6C$ gives $A=2$, $B=C=0$, $y(x)=2$ on $-\infty<x<\infty$.

(o) $\lambda^2-\lambda+\lambda-K^2=0$, $\lambda=\pm K$, $y(x)=A|x|^K+B|x|^{-K} = \begin{cases} Ax^K+Bx^{-K} & \text{on } 0<x<\infty \\ Ax^K+B(-x)^{-K} & \text{on } -\infty<x<0 \end{cases}$

(q) $\lambda(\lambda-1)(\lambda-2)+2\lambda-2=0$, $\lambda=1, 1\pm i$; $y(x)=Ax+x[B\cos(\ln|x|)+C\sin(\ln|x|)]$ on $0<x<\infty$ or on $-\infty<x<0$.

2. (m) dsolve({x^2*diff(y(x),x,x,x)-2*diff(y(x),x)=0, y(1)=2, D(y)(1)=0, D(D(y))(1)=0}, y(x)); gives the solution $y(x)=2$. Note the D(y)(1) and D(D(y))(1) designations for $y'(1)$ and $y''(1)$.

6. Recall that two functions are LI if and only if one is not a constant multiple of the other. Thus, the two solutions in (33) are LI if and only if $\star = \int Y(x)^{-2}e^{-\int a_1(x)dx}dx \neq$ constant. Well, $\frac{d}{dx}\int Y(x)^{-2}e^{-\int a_1 dx}dx = e^{-\int a_1(x)dx}/Y^2(x)=0$ is impossible (since the exponential function is nowhere 0 and $Y^2(x)\neq\infty$) so $\star \neq$ constant and the solutions are LI.

7. (a) $x^2y''-xy'-3y=0$.
$x=e^t$, $dx/dt=e^t$, $dt/dx=e^{-t}$
$dy/dx=dY/dt\; dt/dx=e^{-t}\,dY/dt$
$d^2y/dx^2=\frac{d}{dt}\left(e^{-t}\frac{dY}{dt}\right)\frac{dt}{dx}=\left(-e^{-t}\frac{dY}{dt}+e^{-t}\frac{d^2Y}{dt^2}\right)e^{-t}$
so $e^{2t}\left(-e^{-t}\frac{dY}{dt}+e^{-t}\frac{d^2Y}{dt^2}\right)e^{-t}-e^t\left(e^{-t}\frac{dY}{dt}\right)-3Y=0$,
$d^2Y/dt^2-2dY/dt-3Y=0$, $Y(t)=Ae^{-t}+Be^{3t}$. But $x=e^t\rightarrow t=\ln x$,
so $y(x)=Ae^{-\ln x}+Be^{3\ln x}=A/x+Bx^3$.

8. (a) We saw in 7(a) that $xDy=DY$. Then $xD(xDy)=D^2Y$ gives

$$x^2D^2y+\underbrace{xDy}_{DY}=D^2Y \quad \text{or} \quad x^2D^2y=D^2Y-DY=D(D-1)Y.$$

Next,

$$xD(x^2D^2y)=DD(D-1)Y$$
$$x^3D^3y+\underbrace{2x^2D^2y}_{D(D-1)Y}=D^2(D-1)Y$$

so

$$x^3D^3y=D^2(D-1)Y-2D(D-1)Y$$
$$=D(D-1)(D-2)Y,$$

and so on.

9. (b) $\Phi=A+B\ln r$, $\Phi'(r_1)=0=B/r_1\Rightarrow B=0$ so $\Phi(r)=A$. Then $\Phi(r_2)=\Phi_2=A$, so $\Phi(r)=\Phi_2$

10. (b) $u=A+B/r$, $u'(r_1)=3=-B/r_1^2\Rightarrow B=-3r_1^2$ so $u(r)=A-3r_1^2/r$. Then $u(r_2)=0=A-3r_1^2/r_2$ gives $A=3r_1^2/r_2$ so $u(r)=3r_1^2\left(\frac{1}{r_2}-\frac{1}{r}\right)$

11. (b) Seek $y(x)=A(x)x$. $y'=A+A'x$, $y''=A'+A'+A''x$ so
$x(2A'+A''x)+x(\cancel{A}+A'x)-\cancel{A}x=0$, $x^2A''+(2x+x^2)A'=0$ or, with $A'=p$,
$\frac{dp}{p}+\left(\frac{2}{x}+1\right)dx=0$, $\ln p+2\ln x+x=B$, $p=A'=e^{B-x-2\ln x}=Ce^{-x}/x^2$

so $A(x)=C\int e^{-x}x^{-2}dx$. Thus, $y(x)=Ax+Cx\int e^{-x}dx/x^2$.

12. (b) dsolve(x*diff(y(x),x,x)+x*diff(y(x),x)-y(x)=0, y(x)); gives
$y(x)=C_1x+C_2(-e^{-x}+\mathrm{Ei}(1,x)x)$.
Is this equivalent to our solution in (11b)? ?Ei gives us the Maple definition of the exponential integral function as

$$\mathrm{Ei}(n,x)=\int_1^\infty\frac{e^{-xt}}{t^n}dt$$

Integrating by parts, $y(x) = Ax + Cx\int e^{-x}\frac{dx}{x^2} = C_1x + C_2x\int_x^\infty \frac{e^{-\xi}}{\xi^2}d\xi \quad \left(\begin{smallmatrix}u = e^{-\xi}\\ dv = d\xi/\xi^2\end{smallmatrix}\right)$

$= C_1x + C_2x\left[-e^{-\xi}\frac{1}{\xi}\Big|_x^\infty - \int_x^\infty(-\frac{1}{\xi})(-e^{-\xi}d\xi)\right] \overset{\text{Let } \xi = xt}{=} C_1x + C_2x\left[\frac{e^{-x}}{x} - \int_1^\infty \frac{e^{-xt}}{xt}x\,dt\right]$

$= C_1x + C_2(e^{-x} - x\int_1^\infty e^{-xt}dt/t) = C_1x + C_2(e^{-x} - x\,\mathrm{Ei}(1,x))$, which agrees with the Maple solution. ✓

13. (48) says $y'' + a_1y' + a_2y = y'' - (a+b)y' + (ab-b')y$. Since this identity is to hold for all (twice-differentiable) functions, we can let $y = 1$ and x, in turn. These give $0a_1 + a_2 = ab - b'$ and $a_1 + \cancel{xa_2} = -(a+b) + (ab\cancel{-b'})x$, so $a_2 = ab - b'$, $a_1 = -(a+b)$.

15. $(D+x)\underbrace{(D-x)y}_{u} = 0 \rightarrow \frac{du}{dx} + xu = 0$, $\frac{du}{u} = -x\,dx$, $u = Ae^{-x^2/2}$, $y' \underbrace{-x}_{p(x)} y = \underbrace{Ae^{-x^2/2}}_{q(x)}$

$y(x) = e^{-\int -x\,dx}\left[\int e^{\int -x\,dx}Ae^{-x^2/2}dx + B\right] = Be^{+x^2/2} + Ae^{x^2/2}\int e^{-x^2}dx$, which (with A and B interchanged) is the same as (57).

16. If a, b are constants then (50a,b) become

$$a' = a^2 + a_1a + a_2, \quad b' = -b^2 - a_1b - a_2,$$

both of which are satisfied if a and b are constants, namely, solutions of $\lambda^2 + a_1\lambda + a_2 = 0$, say λ_1, λ_2. Then $(D-\lambda_1)\underbrace{(D-\lambda_2)y}_{u} = 0$. $(D-\lambda_1)u = 0$ gives

$u(x) = Ae^{\lambda_1x}$. Then $(D-\lambda_2)y = Ae^{\lambda_1x}$, or $y' - \lambda_2y = Ae^{\lambda_1x}$, so

$y(x) = e^{-\int -\lambda_2dx}\left[\int e^{\int -\lambda_2dx}Ae^{\lambda_1x}dx + B\right] = e^{\lambda_2x}(A\int e^{(\lambda_1-\lambda_2)x}dx + B)$

$= e^{\lambda_2x}\frac{A}{\lambda_1-\lambda_2}e^{(\lambda_1-\lambda_2)x} + Be^{\lambda_2x} = Ce^{\lambda_1x} + Be^{\lambda_2x}$ if $\lambda_1 \neq \lambda_2$. If $\lambda_1 = \lambda_2 = \lambda$,

then the foregoing gives $y(x) = e^{\lambda x}(A\int e^{0x}dx + B) = (Ax+B)e^{\lambda x}$. These results are the same as obtained by the elementary methods given in Section 3.4.

17. (b) $x^2y'' + xy' + 9y = 0$, so $a_1(x) = 1/x$, $a_2(x) = 9/x^2$. Then (50a,b) give

$a' = a^2 + \frac{1}{x}a + (\frac{9}{x^2} + \frac{1}{x^2})$ and $b' = -b^2 - \frac{1}{x}b - \frac{9}{x^2}$. Try $a = \alpha/x$ and $b = \beta/x$.

Then $-\frac{\alpha}{x^2} = \frac{\alpha^2}{x^2} + \frac{\alpha}{x^2} + \frac{10}{x^2}$ and $\cancel{-\frac{\beta}{x^2}} = -\frac{\beta^2}{x^2} - \cancel{\frac{\beta}{x^2}} - \frac{9}{x^2}$, so $\alpha = -1 \pm 3i$, $\beta = \pm 3i$.

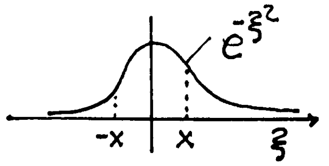
Choose $a(x) = \frac{-1+3i}{x}$ and $b(x) = -\frac{3i}{x}$, say. Then the factored ODE is $(D - \frac{-1+3i}{x})\underbrace{(D + \frac{3i}{x})y}_{u} = 0$. $u' - \frac{-1+3i}{x}u = 0$ gives $u = Ax^{3i-1}$. Then

$(D + \frac{3i}{x})y = u$ becomes $y' + \frac{3i}{x}y = Ax^{3i-1}$, so $y(x) = e^{-\int\frac{3i}{x}dx}\left(\int e^{\int\frac{3i}{x}dx}Ax^{3i-1}dx + B\right)$

$= x^{-3i}(\int x^{3i}Ax^{3i-1}dx + B) = x^{-3i}(\frac{A}{6i}x^{6i} + B) = Cx^{3i} + Bx^{-3i}$, which is the same result as is obtained by seeking $y(x) = x^\lambda$.

18. $\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x}e^{-\xi^2}d\xi = -\frac{2}{\sqrt{\pi}}\int_{-x}^0 e^{-\xi^2}d\xi$

$= -\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi = -\mathrm{erf}(x)$ because the graph of the integrand is symmetric about $x = 0$.



19. (a) $\ln x^a = \int_1^{x^a} \frac{dt}{t} \overset{t=u^a}{=} \int_1^x \frac{a u^{a-1}}{u^a}\,du = a\int_1^x \frac{du}{u} = a\ln x$

(b) $\ln xy = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \boxed{\int_x^{xy} \frac{dt}{t}} \leftarrow$ Let $t = x\tau$

$= \ln x + \int_1^y \frac{x\,d\tau}{x\tau} = \ln x + \ln y$

Section 3.7

1. (b) Yes, $\cos x \sinh 2x \to \{\cos x \sinh 2x, \sin x \sinh 2x, \cos x \cosh 2x, \sin x \cosh 2x\}$

(c) No, $\ln x \to \ln x, 1/x, 1/x^2, 1/x^3, \ldots$ without end.

2.(b) $y' + y = x^4 + 2x$; $y_h = C_1 e^{-x}$; putting $y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$ into the ODE gives $4Ax^3 + 3Bx^2 + 2Cx + D + Ax^4 + Bx^3 + Cx^2 + Dx + E = x^4 + 2x$.

$$\left.\begin{array}{rl} x^4: & A = 1 \\ x^3: & 4A + B = 0 \\ x^2: & 3B + C = 0 \\ x: & 2C + D = 2 \\ 1: & D + E = 0 \end{array}\right] \begin{array}{l} A = 1, B = -4, C = 12, D = -22, E = 22 \\ \text{so } y(x) = C_1 e^{-x} + x^4 - 4x^3 + 12x^2 - 22x + 22 \end{array}$$

(c) $y' + 2y = 3e^{2x} + 4\sin x$; $y_h = C_1 e^{-2x}$; putting $y_p = \overbrace{Ae^{2x}}^{\text{for } 3e^{2x}} + \overbrace{B\sin x + C\cos x}^{\text{for } 4\sin x}$ into the ODE gives $2Ae^{2x} + B\cos x - C\sin x + 2Ae^{2x} + 2B\sin x + 2C\cos x = 3e^{2x} + 4\sin x$.

$$\left.\begin{array}{rl} e^{2x}: & 2A + 2A = 3 \\ \sin x: & -C + 2B = 4 \\ \cos x: & B + 2C = 0 \end{array}\right\} \begin{array}{l} A = 3/4, B = 8/5, C = -4/5 \\ \text{so } y(x) = C_1 e^{-2x} + \frac{3}{4}e^{2x} + \frac{8}{5}\sin x - \frac{4}{5}\cos x \end{array}$$

(h) $y'' + y' = 4xe^x + 3\sin x$. $y_h = C_1 + C_2 e^{-x}$.
$4xe^x \to \{xe^x, e^x\}$, $3\sin x \to \{\sin x, \cos x\}$. No duplication, so seek $y_p = Axe^x + Be^x + C\sin x + D\cos x$. Putting this in the ODE gives

$Axe^x + Ae^x + Be^x + C\cos x - D\sin x$
$+ Axe^x + Ae^x + Ae^x + Be^x - C\sin x - D\cos x = 4xe^x + 3\sin x$.

$$\left.\begin{array}{rl} xe^x: & 2A = 4 \\ e^x: & 3A + 2B = 0 \\ \cos x: & C - D = 0 \\ \sin x: & -D - C = 3 \end{array}\right] \begin{array}{l} A = 2, B = -3, C = -3/2, D = -3/2 \\ \text{so } y(x) = C_1 + C_2 e^{-x} + 2xe^x - 3e^x - \frac{3}{2}\sin x - \frac{3}{2}\cos x \end{array}$$

(l) $y'' + 2y' = x^2 + 4e^{2x}$. $y_h = C_1 + C_2 e^{-2x}$
$x^2 \to \{x^2, x, 1\}$, $4e^{2x} \to \{e^{2x}\}$ so try $y_p = (Ax^2 + Bx + C) + (De^{2x})$. But the C term duplicates the C_1 term in y_h, so try, instead, $y_p = x(Ax^2 + Bx + C) + (De^{2x})$. There is no more duplication so we accept $y_p = Ax^3 + Bx^2 + Cx + De^{2x}$ and proceed. Putting the latter into the ODE gives

$2(3Ax^2 + 2Bx + C + 2De^{2x}) + (6Ax + 2B + 4De^{2x}) = x^2 + 4e^{2x}$

x^2: $6A = 1$, x: $4B + 6A = 0$, 1: $2C + 2B = 0$, e^{2x}: $4D + 4D = 4$ gives
$A = 1/6, B = -1/4, C = 1/4, D = 1/2$, so
$y(x) = C_1 + C_2 e^{-2x} + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2}e^{2x}$

(m) $y''-2y'+y=x^2e^x$. $y_h=(C_1+C_2x)e^x$. $x^2e^x \to \{x^2e^x, xe^x, e^x\}$ so try $y_p=Ax^2e^x+Bxe^x+Ce^x$. But the Bxe^x term duplicates the C_2xe^x term and the Ce^x term duplicates the C_1e^x term, so try $y_p=Ax^3e^x+Bx^2e^x+Cxe^x$. Still, the Cxe^x term duplicates the C_2xe^x term, so try $y_p=Ax^4e^x+Bx^3e^x+Cx^2e^x=(Ax^4+Bx^3+Cx^2)e^x$. Putting this in the ODE gives

$$(Ax^4+Bx^3+Cx^2)e^x-2(4Ax^3+3Bx^2+2Cx+Ax^4+Bx^3+Cx^2)e^x$$
$$+(12Ax^2+6Bx+2C+4Ax^3+3Bx^2+2Cx+4Ax^3+3Bx^2+2Cx+Ax^4+Bx^3+Cx^2)e^x$$
$$=x^2e^x$$

$$\left.\begin{aligned}
x^4e^x&: \; A-2A+A=0\\
x^3e^x&: \; B-8A-2B+4A+4A+B=0\\
x^2e^x&: \; C-6B-2C+12A+3B+3B+C=1\\
xe^x&: \; -4C+6B+2C+2C=0\\
e^x&: \; 2C=0
\end{aligned}\right\} \begin{aligned}&A=1/12,\ B=0,\ C=0,\\ &\text{so } y(x)=(C_1+C_2x)e^x+\tfrac{1}{12}x^4e^x\end{aligned}$$

(p) $y'''-y'=25\cos 2x$. $y_h=C_1+C_2e^x+C_3e^{-x}$. Try $y_p=A\cos 2x+B\sin 2x$.

$$(8A\sin 2x-8B\cos 2x)-(-2A\sin 2x+2B\cos 2x)=25\cos 2x$$

$$\left.\begin{aligned}
\sin 2x&: \; 8A+2A=0\\
\cos 2x&: \; -8B-2B=25
\end{aligned}\right\} \begin{aligned}&A=0,\ B=-5/2,\\ &\text{so } y(x)=C_1+C_2e^x+C_3e^{-x}-\tfrac{5}{2}\sin 2x\end{aligned}$$

3. (a) `dsolve(diff(y(x),x)-3*y(x)=x*exp(2*x)+6,y(x));` gives

$$y(x)=C_1e^{3x}-\tfrac{1}{2}xe^{x}-\tfrac{1}{4}e^{x}-2$$

(q) `dsolve(diff(y(x),x,x,x)-diff(y(x),x,x)=6*x+2*cosh(x),y(x));` gives quite a messy expression, and the simplify command doesnt help, so let us proceed instead as follows: Integration of

$y'''-y''=6x+2\cosh x$ gives

$y''-y'=3x^2+2\sinh x+A$. Integrating again gives

$y'-y=x^3+2\cosh x+Ax+B$.

Now use general solution of linear first-order equation:

$$y(x)=e^x\left[\int e^{-x}(x^3+e^x+e^{-x}+Ax+B)dx+C\right]$$
$$=e^x\left[\int(x^3e^{-x}+1+e^{-2x}+Axe^{-x}+Be^{-x})dx+C\right]$$
$$=e^{-x}\left[(-x^3-3x^2-6x-6)e^{-x}+x-\tfrac{1}{2}e^{-2x}+A(-x-1)e^{-x}-Be^{-x}+C\right]$$

4. (b) $y'-y=xe^x+1$. $y_h=Ae^x$ so seek $y_p=A(x)e^x$. Putting that into the ODE gives $A'e^x+Ae^x-Ae^x=xe^x+1$ or $A'=x+e^{-x}$, $A(x)=\frac{x^2}{2}-e^{-x}+C$ so $y(x)=(\frac{x^2}{2}-e^{-x}+C)e^x=Ce^x+\frac{x^2}{2}e^x-1$

(c) $xy'-y=x^3$. $y_h=Ax$ so seek $y_p=A(x)x$. Putting that into the ODE gives $x(A'x+\cancel{A})-\cancel{A}x=x^3$, $A'=x$, $A(x)=x^2/2+C$, so $y(x)=(\frac{x^2}{2}+C)x=Cx+x^3/2$.

(h) $y''-2y'+y=6x^2$. $y_h=(A+Bx)e^x$ so seek $y_p=[A(x)+B(x)x]e^x$. That gives

$$\left.\begin{aligned}
&A'+xB'=0\\
&A'+(1+x)B'=6x^2e^{-x}
\end{aligned}\right\}\to \begin{aligned}&A'=-6x^3e^{-x},\ A(x)=-6e^{-x}(-x^3-3x^2-6x-6)+C\\ &B'=6x^2e^{-x},\ B(x)=6e^{-x}(-x^2-2x-2)+D,\text{ so}\end{aligned}$$

$$y(x)=(A+Bx)e^x=6(x^3+3x^2+6x+6)+Ce^x-6x(x^2+2x+2)+Dxe^x$$
$$=(C+Dx)e^x+6x^2+24x+36$$

(n) $x^2y'' - xy' - 3y = 4x$. $y_h = Ax^3 + Bx^{-1}$ so seek $y_p = A(x)x^3 + B(x)x^{-1}$. Obtain

$$\left.\begin{aligned} x^3A' + x^{-1}B' &= 0 \\ 3x^4A' - B' &= 4x \end{aligned}\right\} \quad \begin{aligned} A' &= x^{-3}, \quad A(x) = -x^{-2}/2 + C \\ B' &= -x, \quad B(x) = -x^2/2 + D, \text{ so} \end{aligned}$$

$$y(x) = (-x^{-2}/2 + C)x^3 + (-x^2/2 + D)x^{-1} = Cx^3 + Dx^{-1} - x$$

6. At most, $\int^x \frac{W_1(\xi)}{W(\xi)}\,d\xi$ and $\int_{\alpha_1}^x \frac{W_1(\xi)}{W(\xi)}\,d\xi$ differ by a constant, and that constant times $y_1(x)$ does not hurt because $y_1(x)$ is a homogeneous solution. Similarly for the other term.

7. Sure it would work. Rather than consider the general case, let us illustrate the effect of this change by re-working problem 4(n), shown at the top of this page.

$x^2y'' - xy' - 3y = 4x$. $y_h = Ax^3 + Bx^{-1}$ so seek $y_p = A(x)x^3 + B(x)x^{-1}$. Then

$$y_p' = \underbrace{A'x^3 + B'x^{-1}}_{\leftarrow \text{ Instead of setting this} = 0, \text{ set it} = 6, \text{ say.}} + 3x^2A - x^{-2}B \quad = 6 + 3x^2A - x^{-2}B$$

$y_p'' = 3x^2A' + 6xA - x^{-2}B' + 2x^{-3}B$ and putting these in the ODE gives

$$(3x^4A' + \cancel{6x^3A} - B' + \cancel{2x^{-1}B}) - (6x + \cancel{3x^3A} - \cancel{x^{-1}B}) - (\cancel{3x^3A} + \cancel{3x^{-1}B}) = 4x$$

so
$$\left.\begin{aligned} x^3A' + x^{-1}B' &= 6 \\ 3x^4A' - B' &= 10x \end{aligned}\right\} \quad \begin{aligned} A' &= 4x^{-3}, \quad A(x) = -2x^{-2} + C \\ B' &= 2x, \quad B(x) = x^2 + D, \text{ so} \end{aligned}$$

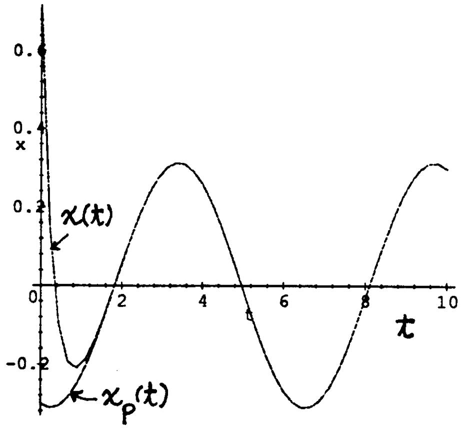
$$y(x) = \left(-\frac{2}{x^2} + C\right)x^3 + (x^2 + D)\frac{1}{x} = -2x + Cx^3 + x + Dx^{-1} = Cx^3 + Dx^{-1} - x, \text{ as before.}$$

Section 3.8

6. (a) $mx'' + cx' + kx = F_0\cos\Omega t$. Arbitrarily, let $m = 1$, $k = 32$, $c = c_{cr} = \sqrt{2mk} = 8$, $\Omega = 1$, $F_0 = 10$. Then $\omega = \sqrt{k/m} = \sqrt{32}$ and (16) and (19) give the solution as

$$x(t) = e^{-4t}(A + Bt) + \frac{10}{\sqrt{(32-1)^2 + 8^2}}\cos\Big(t + \underbrace{\tan^{-1}\frac{8}{1-32}}_{\substack{\text{the } \tan^{-1} \text{ in} \\ \text{the interval} \\ (0,\pi)}}\Big)$$

$$= e^{-4t}(A + Bt) + 0.3123\cos(t + 2.889).$$



Rather than set $x(0)$ and $x'(0)$ and solve for A,B, it is more convenient to do the reverse: set $A = 1$, $B = 0.5$, say. Then $x(0) = 0.6976$ and $x'(0) = -3.578$. To plot, use these maple commands and obtain the plot shown above:

```
> with(plots):
> implicitplot({x=(1+0.5*t)*exp(-4*t)+0.3123*cos(t+2.889),x=0.3123*c
  os(t+2.889)},t=0..10,x=-2..2,numpoints=2000);
```

For the underdamped case shown in Fig. 7 the approach to steady state (shown there as dotted) is oscillatory, but for the critically damped case it is not.

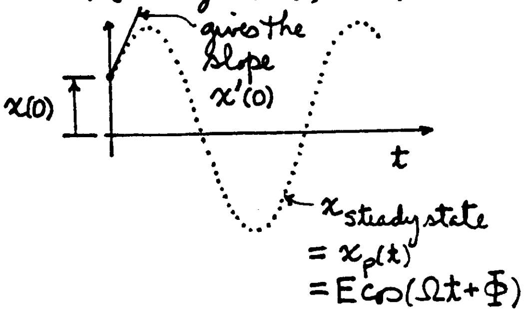
7. $$\lim_{\Omega\to\omega} x(t) = -\frac{F_0}{m}\lim_{\Omega\to\omega}\frac{\cos\omega t - \cos\Omega t}{\omega^2 - \Omega^2} \underset{\substack{\uparrow \\ \text{l'Hôpital}}}{=} -\frac{F_0}{m}\lim_{\Omega\to\omega}\frac{t\sin\Omega t}{-2\Omega} = \frac{F_0 t}{2m\omega}\sin\omega t$$

9.(a) $x(t) = x_h(t) + E\cos(\Omega t+\Phi)$. We can have $x_h(t) \equiv 0$ by imposing on $x_h(t)$ the initial conditions $x_h(0)=0$, $x_h'(0)=0$; these will give $A=B=0$ in (16). These initial conditions on $x_h(t)$ imply conditions on $x(t)$ through (20), as follows:

$$x(0) = \cancel{x_h(0)} + E\cos\Phi = E\cos\Phi$$

$$x'(0) = \cancel{x_h'(0)} - \Omega E\sin\Phi = -\Omega E\sin\Phi.$$

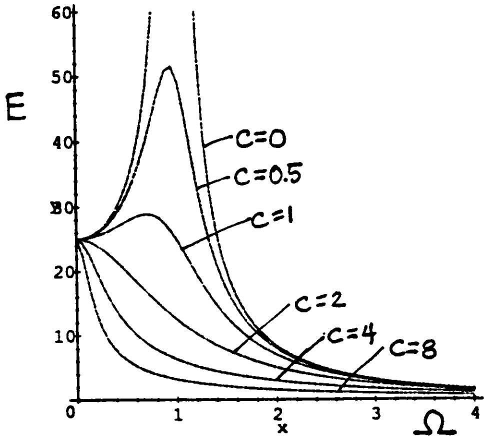
(b) That is, if the steady-state solution is shown as dotted (at the right), then the initial conditions $x(0), x'(0)$ simply start us out along that curve.



10. $mx'' = -k(x-\delta)$ so $mx''+kx = k\delta_0\cos\Omega t$

11. (a) $m=k=1$ so $\omega=\sqrt{k/m}=1$, $F_0=25$, $c_{cr}=2\sqrt{mk}=2$ so $c=0, 0.5, 1, 2, 4, 8$.

```
> with(plots):
> implicitplot({y=25/sqrt((1-x^2)^2+0*x^2),y=25/sqrt((1-x^2)^2+0.25*
  x^2),y=25/sqrt((1-x^2)^2+1*x^2),y=25/sqrt((1-x^2)^2+4*x^2),y=25/sq
  rt((1-x^2)^2+16*x^2),y=25/sqrt((1-x^2)^2+64*x^2)},x=0..4,y=0..60,n
  umpoints=4000);
```



12.(a) We want to solve $L[x] = F_0\cos\Omega t$. Consider instead $L[w] = F_0e^{i\Omega t}$.

Then $L[\text{Re}\,w + i\,\text{Im}\,w] = F_0\cos\Omega t + iF_0\sin\Omega t$

$L[\text{Re}\,w] + i\,L[\text{Im}\,w] =$ " " (by the linearity of L)

Equating real and imaginary parts,

$L[\text{Re}\,w] = F_0\cos\Omega t$, $L[\text{Im}\,w] = F_0\sin\Omega t$

so $x(t) = \text{Re}\,w(t)$.

(d) $w'+3w=5e^{i2t}$. $w_p = Ae^{i2t}$, $i2Ae^{i2t}+3Ae^{i2t}=5e^{i2t}$ gives $A=5/(3+2i)$, so $x(t)=\text{Re}\left(\frac{5}{3+2i}e^{i2t}\right) = 5\,\text{Re}\,\frac{3-2i}{(3+2i)(3-2i)}(\cos2t+i\sin2t)$

$= \frac{5}{13}(3\cos2t+2\sin2t)$.

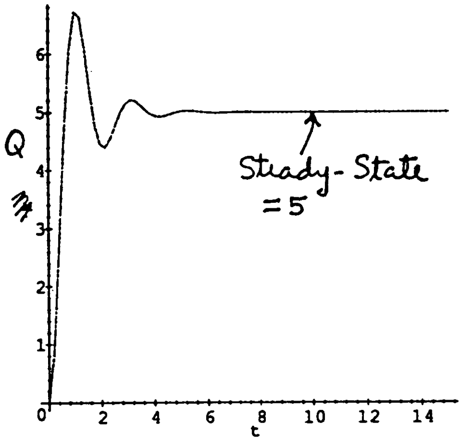
(g) $w''+5w'+w=3e^{i4t}$. $w_p=Ae^{i4t}$, $(-16+20i+1)Ae^{i4t}=3e^{i4t}$ gives $A=3/(-15+20i)$, so $x(t)=\text{Im}\left(\frac{3}{-15+20i}\,\frac{-15-20i}{-15-20i}(\cos4t+i\sin4t)\right)$

$= (-60\cos4t-45\sin4t)/125 = -(12\cos4t+9\sin4t)/25$

13.(a) $2Q''+4Q'+20Q=100,\ Q(0)=Q'(0)=0$

Obtain $Q(t)=5-e^{-t}(5\cos 3t+\frac{5}{3}\sin 3t)$ by maple dsolve command or by hand.
Steady-state solution is $Q(t)\to 5$

```
> with(plots):
> implicitplot(x=5-exp(-t)*(5*cos(3*t)
 +(5/3)*sin(3*t)),t=0..15,x=0..
  10,numpoints=6000);
```



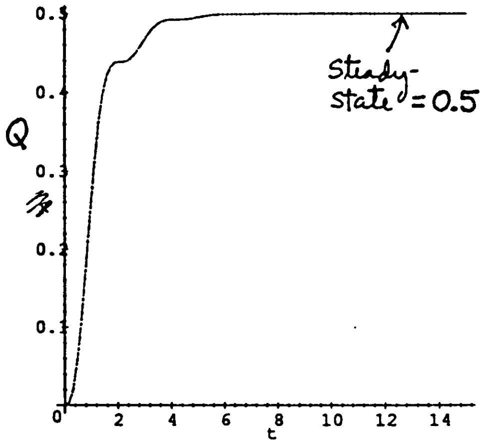
(e) $2Q''+4Q'+20Q=10(1-e^{-t}),\ Q(0)=Q'(0)=0$

The maple dsolve solution is very messy, so let's solve by hand.
$Q_h=e^{-t}(C_1\cos 3t+C_2\sin 3t)$ and seek $Q_p=A+Be^{-t}$. This gives $A=1/2, B=-5/9$ so $Q(t)=\frac{1}{2}-\frac{5}{9}e^{-t}+e^{-t}(C_1\cos 3t+C_2\sin 3t)$
$Q(0)=0$ and $Q'(0)=0$ give $C_1=1/18$ and $C_2=-1/6$, so

$$Q(t)=\frac{1}{2}-\frac{5}{9}e^{-t}+e^{-t}\left(\frac{\cos 3t}{18}-\frac{\sin 3t}{6}\right)$$

Steady-state solution is
$Q(t)\to 1/2$

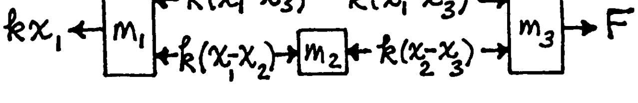
```
> implicitplot(x=(1/2)-(5/9)*exp(-t)
  +exp(-t)*((1/18)*cos(3*t)-(1/6)*
     sin(3*t)),t=0..15,
   x=0..10,numpoints=9000);
```



Section 3.9

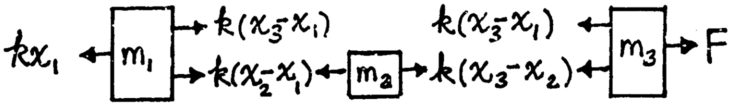
We call your attention especially to Example 8, on the free vibration of a two-mass system. We return to that problem in Section 11.3 and study it there in terms of the matrix eigenvalue problem. It is an important problem, and you may wish to give it added emphasis by discussing it in class, both for Section 3.9 and Section 11.3, and even comparing the two lines of approach to the solution.

3. Assuming $x_1>x_2>x_3>0$:
$m_1x_1''=-kx_1-k(x_1-x_3)-k(x_1-x_2)$
$m_2x_2''=k(x_1-x_2)-k(x_2-x_3)$
$m_3x_3''=k(x_1-x_3)+k(x_2-x_3)$



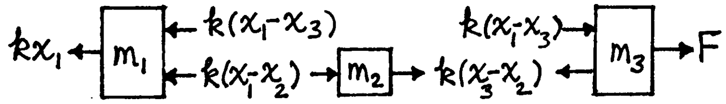
Assuming $x_3 > x_2 > x_1 > 0$:

$$m_1x_1'' = -kx_1 + k(x_2 - x_1) + k(x_3 - x_1)$$
$$m_2x_2'' = -k(x_2 - x_1) + k(x_3 - x_2)$$
$$m_3x_3'' = -k(x_3 - x_1) - k(x_3 - x_2)$$

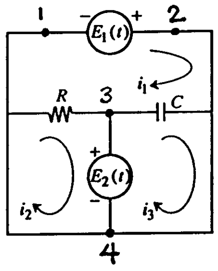


Assuming $x_1 > x_3 > x_2 > 0$:

$$m_1x_1'' = -kx_1 - k(x_1 - x_3) - k(x_1 - x_2)$$
$$m_2x_2'' = k(x_1 - x_2) + k(x_3 - x_2)$$
$$m_3x_3'' = -k(x_3 - x_2) + k(x_1 - x_3) + F$$



The three sets of equations are seen to be identical. Similarly for <u>any</u> such assumption, such as $x_3 < x_1 < 0$ and $x_2 > 0$, and so on.



4.(a) Merely for definiteness, suppose $i_1 > i_2 > i_3 > 0$. Then

Kirchoff voltage law (loop 1): $E_2 - E_1 + E_3 - E_2 + E_1 - E_3 = 0$ or (see p.35)

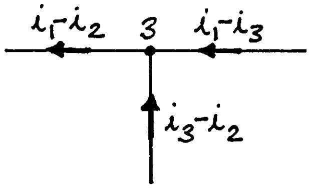
$$E_1(t) - \frac{1}{C}\int(i_1 - i_3)dt - R(i_1 - i_2) = 0 \quad ①$$

Kirch. volt. law (loop 2): $E_4 - E_3 + E_3 - E_4 = 0$ or

$$-E_2(t) - R(i_2 - i_1) = 0 \quad ②$$

Kirch. volt. law (loop 3): $E_3 - E_4 + E_4 - E_3 = 0$ or

$$E_2(t) - \frac{1}{C}\int(i_3 - i_1)dt = 0 \quad ③$$



Kirch. current law (junction at 3): By choosing to work with loop currents we do not need to invoke Kirchoff's current law. For ex., suppose we invoke that law at point 3. Then, from the diagram at the right, we have

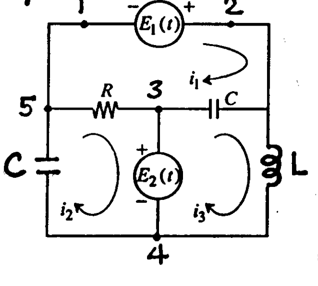
$$(i_1 - i_3) + (i_3 - i_2) = (i_1 - i_2),$$

but the latter is merely an identity and is therefore <u>automatically</u> satisfied.

Taking d/dt of the loop 1,3 equation, to eliminate the integral signs gives this system:

$$\frac{1}{C}(i_1 - i_3) + R(i_1' - i_2') = E_1'(t) \quad ④$$
$$R(i_1 - i_2) = E_2(t) \quad ⑤$$
$$\frac{1}{C}(i_3 - i_1) = E_2'(t) \quad ⑥$$

NOTE: This problem does illustrate the application of Kirchoff's laws but it contains two drawbacks. First, two of the three resulting equations ④-⑥ are algebraic rather than differential. Second, rather than $E_1(t)$ and $E_2(t)$ being arbitrary functions, as I intended, summing equations ①-③ reveals that $E_1(t)$ is necessarily 0, as can also be seen by applying Kirchoff's voltage law to the outer loop. These drawbacks disappear if we include one or more additional elements in the outer loop, for instance as shown at the right.→



In that case, application of Kirchoff's voltage law gives:

loop 1: $E_1 - \frac{1}{C}\int(i_1 - i_3)dt - R(i_1 - i_2) = 0$

loop 2: $-E_2 - \frac{1}{C}\int i_2 dt - R(i_2 - i_1) = 0$

loop 3: $E_2 - \frac{1}{C}\int(i_3 - i_1)dt - Li_3' = 0$

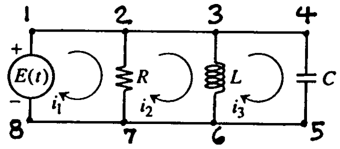
or, taking d/dt of these,

$$\frac{1}{C}(i_1-i_3)+R(i_1'-i_2') = E_1'(t)$$
$$-\frac{1}{C}i_2-R(i_2'-i_1') = E_2'(t)$$
$$\frac{1}{C}(i_3-i_1)+Li_3'' = E_2'(t).$$

(b) $\left.\begin{array}{l} -i_1R-\frac{1}{C}\int i_1\,dt-E=0 \\ E-\frac{1}{C}\int i_2\,dt-L\frac{d}{dt}i_2=0 \end{array}\right\}$ or, $\begin{array}{l} Ri_1'+\frac{1}{C}i_1=-E'(t) \\ Li_2''+\frac{1}{C}i_2=E'(t) \end{array}$

Observe that these equations are not coupled.

(c)



Let us apply Kirchoff's voltage law to 3 loops. We could use loops 12781, 13681, 14581 or loops 12781, 23672, 34563. Let us use the former:

12781: $-R(i_1-i_2)+E=0$ $\quad R(i_1-i_2)=E(t)$ ①

13681: $-L\frac{d}{dt}(i_2-i_3)+E=0$ or, $L(i_2'-i_3')=E(t)$ ②

14581: $-\frac{1}{C}\int i_3\,dt+E=0$ $\quad \frac{1}{C}i_3=E'(t)$ ③

Equations ①-③ are coupled, but the coupling is trivial; we can solve ③ for i_3, then put that i_3 into ② and solve ② for i_2, then put that i_2 into ① and solve ① for i_1.

5. (b) $(D-1)x+2Dy=0$ ①

$(D+1)x+4Dy=0$ ②

The simplest way to solve by elimination is to subtract twice the first equation from the second, giving $(-D+3)x=0$. Thus, $x=Ae^{3t}$. Putting this in first equation then gives $y'=-\frac{1}{2}(x'-x)=-\frac{1}{2}(3Ae^{3t}-Ae^{3t})=-Ae^{3t}$, $y=-\frac{A}{3}e^{3t}+B$. Or, by Cramer's rule,

$$x=\frac{\begin{vmatrix} 0 & 2D \\ 0 & 4D \end{vmatrix}}{\begin{vmatrix} D-1 & 2D \\ D+1 & 4D \end{vmatrix}}=\frac{0}{2D^2-6D},$$ to be understood as $(2D^2-6D)x=0$, so $x=A+Be^{3t}$ ③

$$y=\frac{\begin{vmatrix} D-1 & 0 \\ D+1 & 0 \end{vmatrix}}{\begin{vmatrix} D-1 & 2D \\ D+1 & 4D \end{vmatrix}}=\frac{0}{2D^2-6D},$$ to be understood as $(2D^2-6D)y=0$, so $y=C+Ee^{3t}$. ④

A, B, C, E are not independent constants. To determine how they are related, put ③ and ④ into ① (the same result is obtained if we put them into ②):

$$(D-1)(A+Be^{3t})+2D(C+Ee^{3t})=0$$

or $\quad 3Be^{3t}-A-Be^{3t}+6Ee^{3t}=0$ or $-A+(2B+6E)e^{3t}=0$.

Since 1 and e^{3t} are linearly independent, we must have $-A=0$ and $2B+6E=0$ or $A=0$ and $E=-\frac{1}{3}B$, with C remaining arbitrary, so ③ and ④ become

$$x(t)=Be^{3t}, \quad y(t)=C-\frac{B}{3}e^{3t}$$

(B, C arbitrary constants), which is the same result as obtained above.

(c) $\left.\begin{array}{l} Dx+(D-1)y=5 \\ 2(D+1)x+(D+1)y=0 \end{array}\right\}$ — elimination gives → $[2(D+1)(D-1)-D(D+1)]y=2(D+1)(5)-(D)(0)$ and $[(D+1)D-2(D-1)(D+1)]x=(D+1)(5)-(D-1)(0)$

or, $\quad (D^2-D-2)x=-5$

$\quad (D^2-D-2)y=10$

Solving these (uncoupled) equations gives $x(t)=\frac{5}{2}+Ae^{-t}+Be^{2t}$

$y(t)=-5+Ce^{-t}+Ee^{2t}$

To determine any relations among A, B, C, E put these solutions into either of the original ODE's, say the first: $Dx + (D-1)y = 5$ becomes

$$(-Ae^{-t}+2Be^{2t})+(-Ce^{-t}+2Ee^{2t})-(-5+Ce^{-t}+Ee^{2t})=5$$

or, $(-A-2C)e^{-t}+(2B+2E-E)e^{2t}=0$ so $A=-2C$ and $E=-2B$.

Thus, $x(t)=\frac{5}{2}-2Ce^{-t}+Be^{2t}$

$y(t)=-5+Ce^{-t}-2Be^{2t}$

(e) $\left.\begin{aligned} Dx+y&=\sin t\\ 9x+Dy&=4\end{aligned}\right\}$ Elimination gives $\begin{aligned}(D^2-9)x&=D(\sin t)-4 = \cos t-4\\ (D^2-9)y&=-9\sin t+D(0) = -9\sin t\end{aligned}$

with solutions $x(t)=Ae^{3t}+Be^{-3t}-\frac{1}{10}\cos t+\frac{4}{9}$

$y(t)=Ce^{3t}+Ee^{-3t}+\frac{9}{10}\sin t$

To determine any relations among A, B, C, E, put these solutions into either of the original ODE's, say the first: $Dx+y=\sin t$ becomes

$$(3Ae^{3t}-3Be^{-3t}+\tfrac{1}{10}\cancel{\sin t})+(Ce^{3t}+Ee^{-3t}+\tfrac{9}{10}\cancel{\sin t})=\cancel{\sin t}$$

so $3A+C=0$ and $-3B+E=0$ or, $C=-3A$ and $E=3B$. Thus,

$x(t)=Ae^{3t}+Be^{-3t}-\frac{1}{10}\cos t+\frac{4}{9}$

$y(t)=-3Ae^{3t}+3Be^{-3t}+\frac{9}{10}\sin t$

(f) $x(t)=-\frac{8}{3}t^2-\frac{16}{27}-4Ae^{3t}+2Be^{-3t}$

$y(t)=\frac{2}{3}t-\frac{2}{27}-\frac{1}{3}t^2+Ae^{3t}+Be^{-3t}$

(h) $x(t)=Ae^{9t}+4Be^{-t}$, $y(t)=-Ae^{9t}+Be^{-t}$

(i) $x(t)=\frac{52}{49}-\frac{4}{7}t-\frac{4}{3}Ae^{7t}+4Be^{-t}$, $y(t)=\frac{10}{49}-\frac{1}{14}t+Ae^{7t}+Be^{-t}$

(l) $x(t)=A\sin\sqrt{3}t+B\cos\sqrt{3}t+2C+2Et$, $y(t)=2A\sin\sqrt{3}t+2B\cos\sqrt{3}t+C+Et$

(m) $x(t)=\frac{1}{18}t^4-\frac{5}{9}t^2-\frac{8}{27}+A\sin\sqrt{3}t+B\cos\sqrt{3}t+2C+2Et$,

$y(t)=-\frac{11}{18}t^2+\frac{1}{36}t^4+\frac{11}{27}+2A\sin\sqrt{3}t+2B\cos\sqrt{3}t+C+Et$

6.(q) $(2D^2+3)x+(2D+1)y=4e^{3t}-7$

$Dx+(D-2)y=2$

```
deq1:= 2*diff(x(t),t,t)+3*x(t)+2*diff(y(t),t)+y(t)=4*exp(3*t)-7:
deq2:= diff(x(t),t)+diff(y(t),t)-2*y(t)=2:
dsolve({deq1, deq2}, {x(t), y(t)});
```

gives $x(t)=-2+\frac{1}{5}te^{3t}-\frac{3}{25}e^{3t}+Ae^{3t}+(-B+2C)\sin t+(-2B-C)\cos t$

$y(t)=-1-\frac{3}{5}te^{3t}+\frac{9}{25}e^{3t}+(\frac{2}{5}-3A)e^{3t}+B\sin t+C\cos t.$

7. $x_1(t)=G\sin(t+\phi)+H\sin(\sqrt{3}t+\psi)$

$x_2(t)=G\sin(t+\phi)-H\sin(\sqrt{3}t+\psi)$

(a) $x_1(0)=1=G\sin\phi+H\sin\psi$ ①

$x_2(0)=1=G\sin\phi-H\sin\psi$ ②

$x_1'(0)=0=G\cos\phi+\sqrt{3}H\cos\psi$ ③

$x_2'(0)=0=G\cos\phi-\sqrt{3}H\cos\psi$ ④

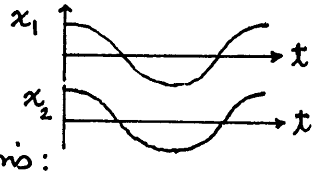
Observe that ①-④ are NOT linear algebraic equations in G, H, ϕ, ψ; they are nonlinear. Nevertheless they are solved easily, as follows:

$$\left.\begin{array}{l} \text{eqn ①} + \text{eqn ②} \rightarrow G\sin\phi = 1 \\ \text{eqn ①} - \text{eqn ②} \rightarrow H\sin\psi = 0 \\ \text{eqn ②} + \text{eqn ③} \rightarrow G\cos\phi = 0 \\ \text{eqn ②} - \text{eqn ③} \rightarrow \sqrt{3}\,H\cos\psi = 0 \end{array}\right\} \begin{array}{l} \rightarrow \phi = \pi/2, H = 1 \\ \rightarrow H = 0, \psi \text{ is therefore irrelevant} \end{array}$$

Thus, $x_1(t) = \sin(t + \pi/2) = \cos t$ (Recall that $\sin(A+B) = \sin A\cos B + \sin B\cos A$)

$x_2(t) = \sin(t + \pi/2) = \cos t$,

as given by (38). Here, the two masses swing in unison at the low frequency 1, as sketched:



"Low mode" motion

Next, consider the second set of initial conditions:

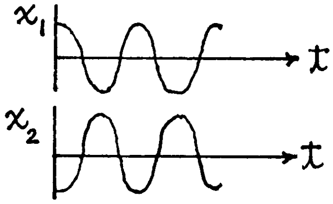
$$\left.\begin{array}{l} x_1(0) = 1 = G\sin\phi + H\sin\psi \\ x_2(0) = -1 = G\sin\phi - H\sin\psi \\ x_1'(0) = 0 = G\cos\phi + \sqrt{3}H\cos\psi \\ x_2'(0) = 0 = G\cos\phi - \sqrt{3}H\cos\psi \end{array}\right]$$

Solving these as above gives $G=0$, ϕ irrelevant, $H=1$, $\psi = \pi/2$, so

$x_1(t) = \sin(\sqrt{3}t + \pi/2) = \cos\sqrt{3}t$

$x_2(t) = -\sin(\sqrt{3}t + \pi/2) = -\cos\sqrt{3}t$,

as given by (39). Here, the two masses swing in opposition at the high frequency $\sqrt{3}$, as sketched:



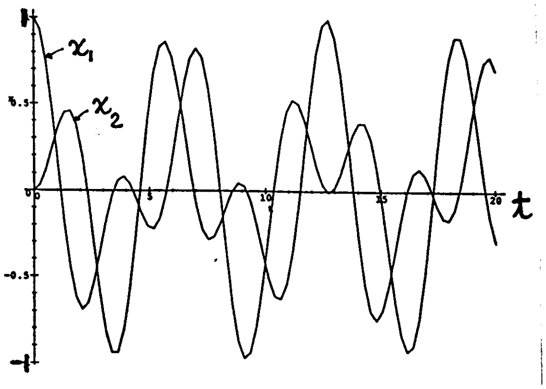
"High mode" motion

(b) This time the initial conditions will excite both modes.

$$\left.\begin{array}{l} x_1(0) = 1 = G\sin\phi + H\sin\psi \\ x_2(0) = 0 = G\sin\phi - H\sin\psi \\ x_1'(0) = 0 = G\cos\phi + \sqrt{3}H\cos\psi \\ x_2'(0) = 0 = G\cos\phi - \sqrt{3}H\cos\psi \end{array}\right] \text{manipulating these as above gives} \longrightarrow \begin{array}{l} G\sin\phi = 1/2 \\ H\sin\psi = 1/2 \\ G\cos\phi = 0 \\ \sqrt{3}H\cos\psi = 0 \end{array} \quad \begin{array}{l} \text{so } \phi = \psi = \pi/2, \\ G = H = 1/2 \end{array}$$

so $x_1(t) = \frac{1}{2}\sin(t + \pi/2) + \frac{1}{2}\sin(\sqrt{3}t + \pi/2) = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{3}t$

$x_2(t) = \frac{1}{2}\sin(t + \pi/2) - \frac{1}{2}\sin(\sqrt{3}t + \pi/2) = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{3}t$



Plot obtained using the Maple command
with(plots):
implicitplot({x=0.5*cos(t)
+0.5*cos(sqrt(3)*t), x=
0.5*cos(t)-0.5*cos(sqrt(3)*t)}
t=0..20, x=-2..2,
numpoints=6000);

8. (a) $x'+\alpha x=0$ gives $x(t)=Ae^{-\alpha t}$. Then $x+y+z=\gamma$ gives $y=\gamma-x-z=\gamma-Ae^{-\alpha t}-z$ and putting this into $z'=\beta y$ gives the ODE $z'+\beta z=\beta\gamma-\beta Ae^{-\alpha t}$ on z. $z_h=Be^{-\beta t}$, and seeking $z_p=P+Qe^{-\alpha t}$ gives $P=\gamma$ and $Q=-\beta A/(\beta-\alpha)$ so $z(t)=Be^{-\beta t}+\gamma-\frac{\beta A}{\beta-\alpha}e^{-\alpha t}$. Then,

$$\left.\begin{aligned}&z(0)=0=B+\gamma-\beta A/(\beta-\alpha)\\&z'(0)=0=-\beta B+\alpha\beta A/(\beta-\alpha)\end{aligned}\right\}\text{ gives } A=\gamma,\ B=\alpha\gamma/(\beta-\alpha)$$

so $$z(t)=\gamma\left[\frac{\alpha e^{-\beta t}-\beta e^{-\alpha t}}{\beta-\alpha}+1\right]$$

$$y(t)=\gamma\left[\frac{\beta e^{-\alpha t}-\alpha e^{-\beta t}}{\beta-\alpha}-e^{-\alpha t}\right]$$

$$x(t)=\gamma e^{-\alpha t}$$

(b) If $\beta=\alpha$ the above expressions for $z(t)$ and $y(t)$ are indeterminate, namely, 0/0. L'Hôpital's rule (as $\beta\to\alpha$) gives $z(t)=\gamma[-\alpha te^{-\alpha t}-e^{-\alpha t}+1]$

$y(t)=\gamma[e^{-\alpha t}+\alpha te^{-\alpha t}-e^{-\alpha t}]$.

Or, of course, we could set $\beta=\alpha$ and re-solve:

$x'+\alpha x=0$ gives $x(t)=Ae^{-\alpha t}$

$z'=\alpha y$

$x+y+z=\gamma$ gives $y=\gamma-Ae^{-\alpha t}-z$

so $z'=\alpha y$ becomes $z'=\alpha(\gamma-Ae^{-\alpha t}-z)$

$$z'+\alpha z=\alpha\gamma-\alpha Ae^{-\alpha t}$$

$z_h=Be^{-\alpha t}$ and this time seek $z_p=P+Qte^{-\alpha t}$. Putting this into ODE gives

$$Qe^{-\alpha t}-\cancel{\alpha Qte^{-\alpha t}}+\alpha P+\cancel{\alpha Qte^{-\alpha t}}=\gamma\alpha-\alpha Ae^{-\alpha t}$$

$e^{-\alpha t}$ terms: $Q=-\alpha A$

Constant terms: $\alpha P=\gamma\alpha$ gives $P=\gamma$

Thus, $z(t)=Be^{-\alpha t}+\gamma-\alpha Ate^{-\alpha t}$

$$\left.\begin{aligned}&z(0)=0=B+\gamma\\&z'(0)=0=-\alpha B-\alpha A\end{aligned}\right\}\rightarrow B=-\gamma,\ A=\gamma,\text{ so } z(t)=-\gamma e^{-\alpha t}+\gamma-\alpha\gamma te^{-\alpha t}$$

$$x(t)=\gamma e^{-\alpha t}$$

$$y(t)=\gamma-x(t)-z(t)=\text{etc.},$$

which agrees with the solution obtained using l'Hôpital's rule.

9. (a) Let $qB/m\equiv\alpha$. Then

$$x''-\alpha y'=0$$
$$\alpha x'+y''=0$$
$$z''=0$$

We might as well integrate these equations once, once, twice, respectively, before proceeding:

$Dx-\alpha y=E$ ①

$\alpha x+Dy=G$ ②

$z=H+It$ ③

Using elimination on the first two of these gives these uncoupled equations

$x''+\alpha^2x=\alpha G$ so $x(t)=J\sin\alpha t+K\cos\alpha t+G/\alpha$ ④

$y''+\alpha^2y=-\alpha E$ so $y(t)=M\sin\alpha t+N\cos\alpha t-E/\alpha$ ⑤

To determine any relations among the integration constants put ④ and ⑤ into

① or ②, say ①: that step gives $\alpha J\cos\alpha t - \alpha K\sin\alpha t - \alpha M\sin\alpha t - \alpha N\cos\alpha t + \cancel{E} = \cancel{E}$ so $N = J$ and $M = -K$. Thus, the general solution is

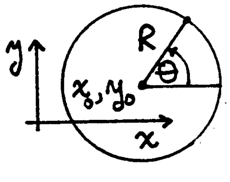
$x(t) = J\sin\alpha t + K\cos\alpha t + G/\alpha$ ⑥

$y(t) = -K\sin\alpha t + J\cos\alpha t - E/\alpha$ ⑦

$z(t) = H + It,$ ⑧

with the arbitrary integration constants J, K, G, E, H, I. (6 arbitrary independent constants)

(b) For the circle shown at the right, $x = x_0 + R\cos\theta$
$y = y_0 + R\sin\theta$



Comparing these equations with ⑥ and ⑦ we see that we need to have initial conditions such that $J = 0$, $G/\alpha = x_0$, $E/\alpha = -y_0$, and $K = R$. That will cause ⑥ and ⑦ to be $x(t) = x_0 + R\cos\alpha t = x_0 + R\cos(-\alpha t)$ ⑨

$y(t) = y_0 - R\sin\alpha t = y_0 + R\sin(-\alpha t)$ ⑩

That is, $\theta = -\alpha t$ so the motion is <u>clockwise</u> with angular velocity α. Initial conditions that will result in that motion can be found directly from ⑨ and ⑩: $x(0) = x_0 + R$, $x'(0) = 0$, $y(0) = y_0$, $y'(0) = -\alpha R$.

(c) If $z'(0) \neq 0$ then $z(t) = H + It$ where $I \neq 0$ and in that case the circular x, y motion plus the linear z motion will produce a <u>helix</u>.

NOTE: We showed in (b) that ⑥ and ⑦ <u>can</u> give a circular motion. In fact, the x, y motion is <u>necessarily</u> circular motion at constant angular velocity, for, recalling eqns. (7)-(10) in Section 3.5, ⑥ and ⑦ give

$x = x_0 + \sqrt{J^2+K^2}\sin(\alpha t + \phi)$, where $\phi = \tan^{-1}(K/J)$

$y = y_0 + \sqrt{J^2+K^2}\sin(\alpha t + \psi)$, where $\psi = \tan^{-1}(-J/K)$

Thus $\tan\phi = K/J$ and $\tan\psi = -J/K$. Hence ψ and ϕ are 90° apart (since the slope $-J/K$ is the negative reciprocal of the slope K/J). If $\psi = \phi + 90°$ then

$x = x_0 + R\sin(\alpha t + \phi)$

$y = y_0 + R\sin(\alpha t + \phi + \pi/2) = y_0 + R\cos(\alpha t + \phi)$

and if $\psi = \phi - 90°$ then

$x = x_0 + R\sin(\alpha t + \phi)$

$y = y_0 + R\sin(\alpha t + \phi - \pi/2) = y_0 - R\cos(\alpha t + \phi)$.

Either way, the trajectory is a circle, traversed clockwise or counterclockwise) at constant angular velocity $\alpha = qB/m$.

CHAPTER 4

Section 4.2

1\. (b) $|a_{n+1}/a_n| = |(-1)^{n+1}(n+1)^{1000}/(-1)^n n^{1000}| = (1+\frac{1}{n})^{1000} \to 1$ as $n\to\infty$ so, by (7a), $R=1$; i.e., the series converges in $|x|<1$.

(c) $|a_{n+1}/a_n| = e^{n+1}/e = e$ so $R=1/e$; convergence in $|x|<1/e$

(e) $|a_{n+1}/a_n| = (1/2)^{n+1}/(1/2)^n = 1/2$ so $R=2$; convergence in $|x+3|<2$ $(1<x<5)$
Or, $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = 1/2$ so $R = 1/(1/2) = 2$ again

(f) $|a_{n+1}/a_n| = n^3/(n-1)^3 \to 1$ as $n\to\infty$ so $R=1$; convergence in $|x-5|<1$ $(4<x<6)$

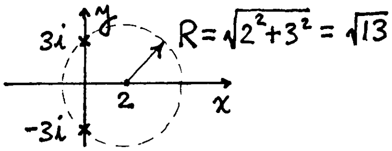
(h) This one is trickier. $|a_{n+1}/a_n| = [\ln(n+1)]^{n+2}/[\ln n]^{n+1} = \underbrace{\left(\frac{\ln(n+1)}{\ln n}\right)^{n+1}}_{*}\ \underbrace{\ln(n+1)}_{\star}$
As $n\to\infty$, $*$ tends to at least 1, and $\star\to\infty$, so $|a_{n+1}/a_n|\to\infty$ and $R=1/\infty=0$.

(i) $|a_{n+1}/a_n| = (1/4^{n+1})/(1/4^n) = 1/4$ for all n, so $R=1/(1/4)=4$. Hence, $\sum_3^\infty[(-1)^n/4^n][(x+2)^3]^n$ converges in $|(x+2)^{\underset{\text{NOTE}}{3}}|<4$ or $|x+2|<\sqrt[3]{4}$

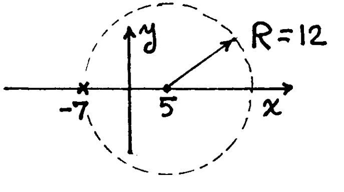
(j) $|a_{n+1}/a_n| = [(n+1)/2^{n+1}]/[n/2^n] = \frac{n+1}{n}\cdot 2 \to 2$ as $n\to\infty$ so $\sum_0^\infty \frac{n}{2^n}[(x-5)^2]^n$ converges in $|(x-5)^2|<1/2$ or $|x-5|<1/\sqrt{2}$

(k) $|a_{n+1}/a_n| = [(n+1)^6/(3^{n+1}+n+1)]/[n^6/(3^n+n)] = \left(\frac{n+1}{n}\right)^6 \frac{3^n+n}{3^{n+1}+n+1} \to \frac{1}{3}$ as $n\to\infty$ so $(x+4)\sum_0^\infty \frac{n^6}{3^n+n}[(x+4)^8]^n$ converges in $|(x+4)^8|<3$ or $|x+4|<3^{1/8}$. NOTE: We used the fact that $3^n+n\sim 3^n$ as $n\to\infty$. How to justify that? One way is as follows: $3^n+n = 3^n(1+\frac{n}{3^n})$. Now, $\frac{n}{3^n} = \frac{n}{e^{n\ln 3}} = \frac{n}{1+(n\ln 3)+\frac{1}{2!}(n\ln 3)^2+\cdots} < \frac{n}{\frac{1}{2}n^2(\ln 3)^2} \to 0$, so $\frac{n}{3^n}\to 0$ as $n\to\infty$.

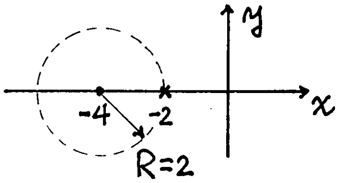
2\. (b) $z^2+9=0$ at $\pm 3i$



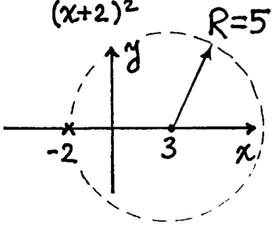
(c) First, note that x^3-2x+1 does not contain an $x+7$ factor that could be canceled.



(e) $\frac{(x+1)^5}{x^2+3x+2} = \frac{(x+1)^4}{x+2}$



(f) $\frac{x^2-3x+1}{(x+2)^2}$



(h) $\frac{x^2-3x+2}{x-1} = \frac{(x-1)(x-2)}{x-1} = x-2$ is analytic everywhere, so $R=\infty$.

3\. (b) $e^{-x} = e^2 - \frac{e^2}{1!}(x+2) + \frac{e^2}{2!}(x+2)^2 - \cdots + (-1)^n\frac{e^2}{n!}(x+2)^n + \cdots$ so $|a_{n+1}/a_n| = \frac{1}{n+1}\to 0$ as $n\to\infty$, so $R=\infty$.

(c) A shortcut is as follows: $\sin x = \sin[(x-\pi)+\pi] = \sin(x-\pi)\cancelto{-1}{\cos\pi} + \cancelto{0}{\sin\pi}\cos(x-\pi)$ $= -\sin(x-\pi) = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \cdots + (-1)^n \frac{(x-\pi)^{2n-1}}{(2n-1)!} + \cdots$ so $|a_{n+1}/a_n| =$

$\{1/[2(n+1)-1]!\}/\{1/[2n-1]!\} = (2n-1)!/(2n+1)! = 1/[(2n+1)(2n)] \to 0$ as $n \to \infty$, so $R = \infty$.

(h) $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n+1}\frac{1}{n}(x-1)^n + \cdots$ so $|a_{n+1}/a_n| = \frac{n}{n+1} \to 1$ as $n \to \infty$, so $R = 1$. That is, the series converges in $0 < x < 2$, the limitation being due to $\ln x$'s singular behavior at $x = 0$ (where $\ln x$ and every one of its derivatives fails to exist).

(i) $x^2 = 9 + 6(x-3) + (x-3)^2$. Clearly, the series converges for all x because it terminates; i.e., $a_n = 0$ for $n \geq 3$.

(k) $\cos(x-2) = 1 - \frac{1}{2!}(x-2)^2 + \frac{1}{4!}(x-2)^4 - \cdots$, $R = \infty$

(l) In (l), (m), and (n) the idea is to use a change of variables and to then do a Taylor series in the new variable. In (l), let $x^{10} \equiv t$. An expansion in powers of t (i.e. about $t = 0$) will <u>give</u> powers of x, as desired. That is,

$1/(1-x^{10}) = 1/(1-t) = 1 + t + t^2 + t^3 + \cdots$ (which Taylor series is the famous geometric series, which converges for $|t| < 1$)

$= 1 + x^{10} + x^{20} + x^{30} + \cdots$, which converges for $|x| < 1$ (because $|t| < 1$ implies $|x| < 1$). If, instead, we expand $1/(1-x^{10})$ in x directly, we would obtain the same result but would, wastefully, generate a great many terms that are 0; i.e., $1/(1-x^{10}) = \underline{\underline{1}} + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 + 0x^6 + 0x^7 + 0x^8 + 0x^9 + \underline{\underline{x^{10}}} + \cdots$.

5. (a) $1 = (1-x)(1 + x + x^2 + \cdots + x^{n-1}) + x^n = 1 + x + \cdots + x^{n-1} - x - \cdots - x^{n-1} - x^n + x^n$ ✓

6. (b) To expand $1/(x-1)$ about $x = 3$ it is simplest to write $\frac{1}{x-1} = \frac{1}{3+(x-4)}$ $= \frac{1}{3}\frac{1}{1+(x-4)/3} = \frac{1}{3}\left[1 - \frac{x-4}{3} + \frac{(x-4)^2}{3^2} - \frac{(x-4)^3}{3^3} + \cdots\right]$ in $|t| = |\frac{x-4}{3}| < 1$, i.e., in $1 < x < 7$.

7. (b) $y'' + 2y' = 0$, $x_0 = 0$. $p(x) = 2$ and $q(x) = 0$ analytic for all x so we will be able to find two LI solutions in power series form. $y(x) = \sum_0^\infty a_n x^n$ gives $\sum_0^\infty n(n-1)a_n x^{n-2} + 2\sum_0^\infty n a_n x^{n-1} = 0$ or $\sum_{-1}^\infty (n+1)n a_{n+1} x^{n-1} + 2\sum_0^\infty n a_n x^{n-1} = 0$ or $\sum_{-1}^\infty [n(n+1)a_{n+1} + 2na_n]x^{n-1} = 0$ where $a_{-1} \equiv 0$. Thus, $n(n+1)a_{n+1} + 2na_n = 0$. (Do not cancel the n's yet because of the case $n = 0$.) $n = -1 \to 0a_0 = 0$ so $a_0 =$ arb. $n = 0$ gives $0 = 0$ with $a_1 =$ arb. <u>Now</u> cancel the n's, so $a_{n+1} = -\frac{2}{n+1}a_n$ for $n \geq 1$: $a_2 = -2a_1/2$, $a_3 = -\frac{2}{3}(-\frac{2}{2}a_1) = \frac{2^2}{3!}a_1$, $a_4 = -\frac{2}{4}(\frac{2^2 a_1}{3!}) = -\frac{2^3}{4!}a_1, \ldots, a_n = (-1)^{n+1}\frac{2^{n-1}}{n!}a_1$ so

$$y(x) = a_0 + a_1 x - \frac{2a_1}{2}x^2 + \frac{2^2 a_1}{3!}x^3 - \frac{2^3 a_1}{4!}x^4 + \cdots + (-1)^{n+1}\frac{2^{n-1}a_1}{n!}x^n + \cdots$$
$$= a_0 + \frac{a_1}{2}\left[(2x) - \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} - \cdots\right].$$

In this case we can get the result into closed form by observing that the series is related to e^{2x}: $y(x) = a_0 + \frac{a_1}{2}\left[1 - 1 + (2x) - \frac{(2x)^2}{2!} + \cdots\right] = a_0 + \frac{a_1}{2}(1 - e^{2x})$ $= (a_0 + \frac{a_1}{2}) - \frac{a_1}{2}e^{2x} \equiv C_1 + C_2 e^{2x}$.

(e) $xy''-2y'+xy=0$, $x_0=1$. $p(x)=-2/x$ and $q(x)=1$ are analytic for all $x \neq 0$ so $R=1$, hence convergence in $0<x<2$. Seeking $y(x)=\sum_0^\infty a_n(x-1)^n$, it is convenient to let $z=x-1$ and $y(x)=y(z+1)\equiv Y(z)$. Then the ODE becomes

$$(z+1)\sum_0^\infty n(n-1)a_n z^{n-2}-2\sum_0^\infty na_n z^{n-1}+(z+1)\sum_0^\infty a_n z^n=0, \quad \text{or,}$$

$$\sum_0^\infty n(n-1)a_n z^{n-1}+\sum_0^\infty n(n-1)a_n z^{n-2}-2\sum_0^\infty na_n z^{n-1}+\sum_0^\infty a_n z^{n+1}+\sum_0^\infty a_n z^n=0 \quad \text{or, shifting indices}$$

$$\sum_{-1}^\infty (n+1)na_{n+1}z^n+\sum_{-2}^\infty (n+2)(n+1)a_{n+2}z^n-2\sum_{-1}^\infty (n+1)a_{n+1}z^n+\sum_1^\infty a_{n-1}z^n+\sum_0^\infty a_n z^n=0 \quad \text{or,}$$

$$\sum_0^\infty \left[(n+2)(n+1)a_{n+2}+(n+1)(n-2)a_{n+1}+a_n+a_{n-1}\right]z^n=0 \quad (\text{with } a_{-1}\equiv 0)$$

Then "$[\cdots]$"$=0$ gives the recursion formula. The latter gives

$$Y(z)=C_1\left(1-\tfrac{1}{2}z^2-\tfrac{1}{3}z^3+\tfrac{1}{24}z^4-\cdots\right)+C_2\left(z+z^2+\tfrac{1}{6}z^3-\tfrac{1}{6}z^4-\cdots\right)$$

or,

$$y(x)=C_1\left[1-\tfrac{1}{2}(x-1)^2-\tfrac{1}{3}(x-1)^3+\tfrac{1}{24}(x-1)^4+\cdots\right]+C_2\left[(x-1)+(x-1)^2+\tfrac{1}{6}(x-1)^3-\tfrac{1}{6}(x-1)^4-\cdots\right]$$

$$\equiv C_1y_1(x)+y_2(x)$$

(f) [This nonconstant coefficient ODE happens to be a Cauchy-Euler eqn. with closed form solution $y(x)=Ax^{(1+\sqrt{5})/2}+Bx^{(1-\sqrt{5})/2}$, so if we seek a power series solution we will obtain the Taylor series expansions of $x^{(1+\sqrt{5})/2}$ and $x^{(1-\sqrt{5})/2}$ about $x_0=2$.]

$x^2y''-y=0$, $x_0=2$. $p(x)=0$ and $q(x)=-1/x^2$ are analytic for all $x\neq 0$ so $R=2$, hence convergence in $0<x<4$. Since $x_0=2$, it is convenient to set $z=x-2$. Doing so, and seeking $y(x)=y(z+2)\equiv Y(z)=\sum_0^\infty a_n z^n$, the ODE becomes

$$(z+2)^2\sum_0^\infty n(n-1)a_n z^{n-2}-\sum_0^\infty a_n z^n=0$$

or,

$$\sum_0^\infty n(n-1)a_n z^n+\sum_0^\infty 4n(n-1)a_n z^{n-1}+\sum_0^\infty 4n(n-1)a_n z^{n-2}-\sum_0^\infty a_n z^n=0 \qquad ♀$$

$$\sum_0^\infty n(n-1)a_n z^n+\sum_{-1}^\infty 4(n+1)na_{n+1}z^n+\sum_{-2}^\infty 4(n+2)(n+1)a_{n+2}z^n-\sum_0^\infty a_n z^n=0 \qquad ☆$$

or,

$$\sum_0^\infty\left[n(n-1)a_n+4(n+1)na_{n+1}+4(n+2)(n+1)a_{n+2}-a_n\right]z^n=0$$

or,

$$\sum_0^\infty\left[4(n+2)(n+1)a_{n+2}+4(n+1)na_{n+1}+(n^2-n-1)a_n\right]z^n=0$$

Then "[ ]"$=0$ gives the recursion formula. The latter gives

$$Y(z)=C_1\left[1+\tfrac{1}{8}z^2-\tfrac{1}{24}z^3+\tfrac{7}{384}z^4-\cdots\right]+C_2\left[z+\tfrac{1}{24}z^3-\tfrac{1}{48}z^4+\tfrac{19}{1920}z^5-\cdots\right]$$

$$y(x)=C_1\left[1+\tfrac{1}{8}(x-2)^2-\tfrac{1}{24}(x-2)^3+\tfrac{7}{384}(x-2)^4-\cdots\right]+C_2\left[(x-2)+\tfrac{1}{24}(x-1)^3-\tfrac{1}{48}(x-1)^4+\tfrac{19}{1920}(x-1)^5-\cdots\right]$$

$$\equiv C_1y_1(x)+C_2y_2(x)$$

(h) $y''+y'+(1+x+x^2)y=0$, $x_0=0$. $p(x)=1$ and $q(x)=1+x+x^2$ are analytic for all x, so $R=\infty$, hence convergence in $-\infty<x<\infty$. Seeking $y(x)=\sum_0^\infty a_n x^n$ the ODE becomes

$$\sum_0^\infty n(n-1)a_n x^{n-2}+\sum_0^\infty a_n n x^{n-1}+\sum_0^\infty a_n x^n+\sum_0^\infty a_n x^{n+1}+\sum_0^\infty a_n x^{n+2}=0$$

or,

$$\sum_{-2}^\infty (n+2)(n+1)a_{n+2}x^n+\sum_{-1}^\infty a_{n+1}(n+1)x^n+\sum_0^\infty a_n x^n+\sum_1^\infty a_{n-1}x^n+\sum_2^\infty a_{n-2}x^n=0$$

or,

$$\sum_0^\infty\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+a_n+a_{n-1}+a_{n-2}\right]x^n=0 \qquad (a_{-1}=a_{-2}\equiv 0)$$

Recursion formula: ["]$=0$ for $n=0,1,2,\dots$. The latter gives

$$y(x)=C_1\left(1-\tfrac{1}{2}x^2-\tfrac{1}{24}x^4+\tfrac{1}{30}x^5-\cdots\right)+C_2\left(x-\tfrac{1}{2}x^2-\tfrac{1}{24}x^4-\tfrac{1}{60}x^5+\cdots\right)$$

8.(b) `dsolve(diff(y(x),x,x)+2*diff(y(x),x)=0, y(x), type=series);`

gives $y(x)=y(0)+D(y)(0)x-D(y)(0)x^2+\tfrac{2}{3}D(y)(0)x^3-\tfrac{1}{3}D(y)(0)x^4+\tfrac{2}{15}D(y)(0)x^5+O(x^6)$

or, if we call $y(0)=A$ and $D(y)(0)=B$,

$$y(x) = A + B\left(x - x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{3}x^4 + \tfrac{2}{15}x^5\right) + O(x^6),$$

which agrees with our result in Exercise 7(b) (where A is a_0 and B is $a_1/2$).

(f) For $x^2y'' - y = 0$, the command dsolve(x^2*diff(y(x),x,x)-y(x)=0, y(x), type=series); will give an expansion about $x=0$. To obtain an expansion about $x_0 = 2$, first set $z = x-2$ so the ODE becomes $(z+2)^2Y''(z) - Y(z) = 0$. Now the command

dsolve((z+2)^2*diff(Y(z),z,z)-Y(z)=0, Y(z), type=series);

gives $Y(z) = Y(0) + D(Y)(0)z + \frac{1}{8}Y(0)z^2 + \left(-\frac{1}{24}Y(0) + \frac{1}{24}D(Y)(0)\right)z^3$
$+ \left(\frac{7}{384}Y(0) - \frac{1}{48}D(Y)(0)\right)z^4 + \left(-\frac{1}{120}Y(0) + \frac{19}{1920}D(Y)(0)\right)z^5 + O(z^6)$

or, equivalently, $y(x) = C_1\left[1 + \frac{1}{8}(x-2)^2 - \frac{1}{24}(x-2)^3 + \frac{7}{384}(x-2)^4 - \frac{1}{120}(x-2)^5\right]$
$+ C_2\left[(x-2) + \frac{1}{24}(x-2)^3 - \frac{1}{48}(x-2)^4 + \frac{19}{1920}(x-2)^5\right] + O(x-2)^6$

10\. (a) Order := 9;

dsolve({diff(y(x),x,x)+4*diff(y(x),x)+y(x)=0, y(0)=1, D(y)(0)=0}, y(x), type=series);

gives $y(x) = 1 - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{5}{8}x^4 + \frac{7}{15}x^5 - \frac{209}{720}x^6 + \frac{13}{84}x^7 - \frac{2911}{40320}x^8 + O(x^9)$

11\. (b) $(n+1)a_{n+2} + 5na_{n+1} + a_n - a_{n-1} = 0$ gives $\dfrac{a_{n+2}}{a_{n+1}} = -\dfrac{5n}{n+1} - \dfrac{a_n - a_{n-1}}{(n+1)a_{n+1}} \sim -5$ as $n \to \infty$ so $\lim|a_{n+1}/a_n| = 5$ and $R = 1/5$. Does this term really drop out? We need to show that its dropping out is consistent with the (tentative) result $a_{n+2}/a_{n+1} \sim -5$. The latter implies that $a_{n+1}/a_n \sim -5$ and $a_n/a_{n-1} \sim -5$ so

$$-\frac{5n}{n+1} - \frac{a_n - a_{n-1}}{(n+1)a_{n+1}} \sim -5 - \frac{a_n - (-\frac{1}{5})a_n}{(n+1)a_{n+1}} = -5 - \frac{6}{5}\frac{a_n}{(n+1)a_{n+1}}$$
$$\sim -5 - \frac{6}{5}\frac{1}{(n+1)(-5)} \sim -5 \checkmark$$

(c) $(n+1)^2a_{n+2} + (2n^2+1)a_{n+1} - 4a_n = 0$ gives $\dfrac{a_{n+2}}{a_{n+1}} = -\dfrac{2n^2+1}{(n+1)^2} + \dfrac{4}{(n+1)^2}\dfrac{a_n}{a_{n+1}} \sim -2$ as $n \to \infty$ so $\lim\limits_{n\to\infty}|a_{n+2}/a_{n+1}| = \lim\limits_{n\to\infty}|a_{n+1}/a_n| = 2$ so $R = 1/2$. Need to check this term: If, tentatively, $a_{n+2}/a_{n+1} \sim -2$, then

$$\frac{a_{n+2}}{a_{n+1}} = -\frac{2n^2+1}{(n+1)^2} + \frac{4}{(n+1)^2}\frac{a_n}{a_{n+1}} \sim -2 + \frac{4}{n^2}\frac{1}{(-2)} \sim -2 \checkmark$$

(e) $na_{n+2} + 4na_{n+1} + 3a_n = 0$ gives $\dfrac{a_{n+2}}{a_{n+1}} = -4 - \dfrac{3}{n}\dfrac{a_n}{a_{n+1}} \sim -4$ so $R = 1/4$.

Checking our dropping of the last term, $\dfrac{a_{n+2}}{a_{n+1}} = -4 - \dfrac{3}{n}\dfrac{a_n}{a_{n+1}} \sim -4 - \dfrac{3}{n(-4)} \sim -4 \checkmark$

(f) $n^2a_{n+2} - 3(n+2)^2a_{n+1} + 3a_{n-1} = 0$ gives $\dfrac{a_{n+2}}{a_{n+1}} = \dfrac{3(n+2)^2}{n^2} - \dfrac{3}{n^2}\dfrac{a_{n-1}}{a_{n+1}} \sim 3$ gives $R = 1/3$. Checking [as we did in parts (b), (c), (e)], $\dfrac{a_{n+2}}{a_{n+1}} = \dfrac{3(n+2)^2}{n^2} - \dfrac{3}{n^2}\dfrac{a_{n-1}}{a_{n+1}} \sim 3 - \dfrac{3}{n^2}\dfrac{1}{9} \sim 3 \checkmark$

13. $y''+y=0$. $y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$ gives
$(2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots)+(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots)=0$
x^0: $2a_2+a_0=0 \rightarrow a_2=-a_0/2$
x^1: $6a_3+a_1=0 \rightarrow a_3=-a_1/6$
x^2: $12a_4+a_2=0 \rightarrow a_4=-a_2/12=a_0/24$
x^3: $20a_5+a_3=0 \rightarrow a_5=-a_3/20=a_1/120$,
etc. Thus far, $y(x)=a_0+a_1x-\frac{a_0}{2}x^2-\frac{a_1}{6}x^3+\frac{a_0}{24}x^4+\frac{a_1}{120}x^5-\cdots$
$=a_0(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots)+a_1(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots)$

15. From Exercise 5, $\frac{1}{1-x}=\underbrace{1+x+x^2+\cdots+x^{n-1}}_{S_n}+\frac{x^n}{1-x}$, so $S_n=\frac{1-x^n}{1-x}$. Thus,

$$\frac{S_1+\cdots+S_N}{N}=\frac{(1-x)+(1-x^2)+\cdots+(1-x^N)}{N(1-x)}=\frac{N-(x+x^2+\cdots+x^N)}{N(1-x)}=\frac{1}{1-x}-\frac{x}{N}\frac{1+x+\cdots+x^{N-1}}{1-x}$$

$=\frac{1}{1-x}-\frac{x}{N}\frac{1-x^N}{(1-x)^2}$ for all $x\neq 1$. If $|x|<1$ this $\rightarrow \frac{1}{1-x}-0\,\frac{1-0}{1-x}=\frac{1}{1-x}$.

If $|x|>1$ then the $x^N\rightarrow\infty$ as $N\rightarrow\infty$ and the series *diverges* in the Cesàro sense.

Section 4.3

1. (b) $p(x)=-\cos x/x$ and $q(x)=5/x$ are analytic for all $x\neq 0$. The singular point $x=0$ is regular because $xp(x)=-\cos x$ and $x^2q(x)=5x$ are analytic there. min $R=\infty$.

(c) $p(x)=0$, $q(x)=-1/(x^2-3)$. Singular points at $x=\pm\sqrt{3}$, regular because $(x-\sqrt{3})p(x)=0$ and $(x-\sqrt{3})^2q(x)=-(x-\sqrt{3})^2/(x^2-3)=-(x-\sqrt{3})/(x+\sqrt{3})$ is analytic at $x=+\sqrt{3}$; similarly for $x=-\sqrt{3}$.

(e) $p(x)=-4/(x+1)^2$, $q(x)=1/(x+1)^2$. Singular point at $x=-1$, irregular because $(x+1)p(x)=-4/(x+1)$ is *not* analytic at $x=-1$.

(f) $p(x)=\ln x$, $q(x)=2$. Singular point at $x=0$, irregular because $xp(x)=x\ln x$ is not analytic at $x=0$; i.e., although $x\ln x\rightarrow 0$ exists, as $x\rightarrow 0$, all derivatives of $x\ln x$ fail to exist at $x=0$.

2. (a) $y''+\sqrt{x}\,y=0$. $\sqrt{x}=t$, $x=t^2$, $dx=2t\,dt$,
$\frac{d}{dx}\frac{d}{dx}y+\sqrt{x}\,y=0=\frac{d}{2t\,dt}\frac{d}{2t\,dt}Y(t)+tY(t)=\frac{1}{2t}\left(\frac{1}{2t}Y'\right)'+tY$

$=\frac{1}{4t^2}Y''-\frac{1}{4t^3}Y'+tY=0$ or, $Y''-\frac{1}{t}Y'+4t^3Y=0$.

(b) Regular singular pt. at $t=0$ because $tP(t)=-1$ and $t^2Q(t)=4t^5$ are analytic at $t=0$.

(c) Put $Y=\sum_0^\infty a_nt^{n+r}$ into $Y''-\frac{1}{t}Y'+4t^3Y=0$:
$$\sum_0^\infty(n+r)(n+r-1)a_nt^{n+r-2}-\sum_0^\infty(n+r)a_nt^{n+r-2}+\sum_0^\infty 4a_nt^{n+r+3}=0$$
or $$\sum_0^\infty(n+r)(n+r-2)a_nt^{n+r-2}+\sum_0^\infty 4a_nt^{n+r+3}=0$$
Shift indices: $\sum_{-2}^\infty(n+r+2)(n+r)a_{n+2}t^{n+r}+\sum_3^\infty 4a_{n-3}t^{n+r}=0$ ‡

NOTE CAREFULLY: Recall the ✡ equation in the solution to Exercise 4.2/7f, for example. Having, by shifting the summation index in the 2nd and 3rd sums in ♀, gotten the same exponents (i.e., z^n) in each sum, we then got the summation limits to be the same by changing the -1 and -2 lower limits to 0. We could do that because the $n=-1$ term in $\sum_{-1}^{\infty} 4(n+1)na_{n+1}z^n$ is zero anyhow because of the $n+1$ factor; similarly, the $n=-2$ and $n=-1$ terms in $\sum_{-2}^{\infty} 4(n+2)(n+1)a_{n+2}z^n$ are zero anyhow because of the $n+2$ and $n+1$ factors. In the present example, however, although we can change the 3 to 0 in $\sum_3^{\infty}$ (merely by defining $a_{-3}=a_{-2}=a_{-1}\equiv 0$), we cannot change the -2 to 0 in $\sum_{-2}^{\infty}$ because the terms in the sum are not zero for $n=-2$ and $n=-1$. Instead, to pull the two sums together in ‡, keep the $\sum_{-2}^{\infty}$ and change the $\sum_3^{\infty}$ to $\sum_{-2}^{\infty}$ by defining $a_{-5}=a_{-4}=a_{-3}=a_{-2}=a_{-1}\equiv 0$. Thus, ‡ becomes

$$\sum_{-2}^{\infty}\left[(n+r+2)(n+r)a_{n+2}+4a_{n-3}\right]t^{n+r}=0,$$

so $\qquad (n+r+2)(n+r)a_{n+2}+4a_{n-3}=0 \quad$ each $n=-2,-1,0,1,\dots$

$n=-2$: $\qquad r(r-2)a_0=0 \Rightarrow r(r-2)=0, \quad r=0,2.$

Theorem 4.3.1 says the smaller root ($r=0$) will give both solutions or neither, while the larger ($r=2$) will give one solution, the other being given by (41c). It appears that $r=0$ will give neither solution, but let us give it a try and see for ourselves: With $r=0$ the recursion formula is

$$(n+2)na_{n+2}+4a_{n-3}=0 \quad (n=\overset{\text{We've already done } n=-2.}{\downarrow}\!\!-1,0,1,2,\dots)$$

$n=-2$: gave $a_0=$ arb.
$n=-1$: $-a_1=0$ so $a_1=0$
$n=0$: $0=0$ so $a_2=$ arb.
$n=1$: $3a_3=0$ so $a_3=0$
$n=2$: $8a_4=0$ so $a_4=0$
$n=3$: $15a_5+4a_0=0$ so $a_5=-\frac{4}{15}a_0$
$n=4$: $24a_6+0=0$ so $a_6=0$
$n=5$: $35a_7+4a_2=0$ so $a_7=-\frac{4}{35}a_2$
$n=6$: $48a_8+4a_3=0$ so $a_8=0$
$n=7$: $63a_9+4a_4=0$ so $a_9=-\frac{4}{63}a_4=0$
$n=8$: $80a_{10}+4a_5=0$ so $a_{10}=-\frac{1}{20}a_5=\frac{1}{75}a_0$
$n=9$: $99a_{11}+4a_6=0$ so $a_{11}=0$
$n=10$: $120a_{12}+4a_7=0$ so $a_{12}=-\frac{1}{30}a_7=\frac{2}{525}a_2$
etc.

so
$$Y(t)=a_0+a_2t^2-\frac{4}{15}a_0t^5-\frac{4}{35}a_2t^7+\frac{1}{75}a_0t^{10}+\frac{2}{525}a_2t^{12}+\cdots$$
$$=a_0\left(1-\frac{4}{15}t^5+\frac{1}{75}t^{10}-\cdots\right)+a_2\left(t^2-\frac{4}{35}t^7+\frac{2}{525}t^{12}-\cdots\right) \qquad ✡$$

or,

$$y(x)=C_1\left(1-\frac{4}{15}x^{5/2}+\frac{1}{75}x^{10/2}-\cdots\right)+C_2x\left(1-\frac{4}{35}x^{5/2}+\frac{2}{525}x^{10/2}-\cdots\right)$$

Whereas the $Y(t)$ solutions were of Frobenius type, the $y(x)$ solutions are not, because they are not of the form $x^r\sum_0^{\infty}a_nx^n$.

NOTE: So I was mistaken; I thought the smaller root, $r=0$, would not lead to any solutions because if it lead to both then we would have two <u>analytic</u> solutions for $Y(t)$ whereas the Y ODE has a regular singular point. The point is this: Theorem 4.2.4 says that if x_0 is a <u>regular</u> point of $y''+p(x)y+q(x)y=0$ then there are two LI solutions that are analytic at x_0. It does <u>NOT</u> say that if x_0 is a regular singular point then there cannot be two LI solutions that are analytic at x_0.

(d) The maple command dsolve(diff(Y(t),t,t)-(1/t)*diff(Y(t),t)+4*t^3*Y(t)=0, Y(t),

```
type=series);
```
gives

$$Y(t) = C_1 t^2\left(1-\frac{4}{35}t^5+O(t^6)\right) + C_2\left(\ln(t)(O(t^6)) + (-2+\frac{8}{15}t^5+O(t^6))\right).$$

It's not clear whether a ln(t) term will show up at higher orders, but if we increase the order we do begin to see the ✡ solution emerge. For instance,

```
Order := 20;
dsolve(diff(Y(t),t,t)-(1/t)*diff(Y(t),t)+4*t^3*Y(t)=0, Y(t), type=series);
```

gives

$$Y(t) = C_1 t^2\left(1-\frac{4}{35}t^5+\frac{2}{525}t^{10}-\frac{8}{133875}t^{15}+O(t^{20})\right)$$
$$+ C_2\left(\ln(t)(O(t^{20})) + \left(-2+\frac{8}{15}t^5-\frac{2}{75}t^{10}+\frac{8}{14625}t^{15}+O(t^{20})\right)\right).$$

3. (b) $(x^2-1)y''+y=0$, $x-1=t$ or $x=t+1$, gives $t(t+2)Y''(t)+Y(t)=0$

5. The relevant equations are (5), (8), (35).

(b) $r=3,3$. Then indicial equation is $(r-3)(r-3)=r^2-6r+9=0$ so, from (35), $p_0=-5$ and $q_0=9$. Thus, the simplest ODE having indicial roots 3,3 is, from (8), $x^2y''-5xy'+9y=0$. Another example would be $x^2y''-5\sin x\, y'+9e^x y=0$, because $\sin x = x - x^3/3!+\cdots \sim x$ and $e^x=1+x+\cdots\sim 1$.

(e) $r=2+3i, 2-3i$. Then indicial equation is $[r-(2+3i)][r-(2-3i)]=r^2-4r+13=0$ so, from (35), $p_0=-3$ and $q_0=13$. Thus, the simplest ODE having indicial roots $2\pm3i$ is, from (8), the Cauchy-Euler equation $x^2y''-3xy'+13y=0$. Other examples: $x^2y''-3x(1-5x^2)y'+13\cos x\, y=0$ and $x^2y''-3(1+2x)\sin x\, y'+13(1-x)y=0$.

6. (b) $xy''+y'-xy=0$. $y(x)=\sum_0^\infty a_n x^{n+r}$ gives

$$\sum_0^\infty (n+r)(n+r-1)a_n x^{n+r-1} + \sum_0^\infty (n+r)a_n x^{n+r-1} - \sum_0^\infty a_n x^{n+r+1} = 0$$
$$\sum_{-1}^\infty (n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{-1}^\infty (n+r+1)a_{n+1}x^{n+r} - \sum_1^\infty a_{n-1}x^{n+r} = 0$$

or, $\sum_{-1}^\infty\left[(n+r+1)^2a_{n+1}-a_{n-1}\right]x^{n+r}=0 \qquad (a_{-2}=a_{-1}\equiv 0)$

so $\qquad (n+r+1)^2a_{n+1}-a_{n-1}=0$ for $n=-1,0,1,2,\dots \qquad *$

$n=-1$: $r^2a_0=0$ so $r^2=0$, $r=0,0$ and $a_0=$ arb.

Thus, set $r=0$ in $*$: $\quad (n+1)^2a_{n+1}-a_{n-1}=0$

or $\quad a_{n+1}=\frac{1}{(n+1)^2}a_{n-1}$

$n=0$: $a_1=0$

$n=1$: $a_2=\frac{1}{2^2}a_0 \qquad n=4$: $a_5=\frac{1}{5^2}a_3=0$

$n=2$: $a_3=\frac{1}{3^2}a_1=0 \qquad n=5$: $a_6=\frac{1}{6^2}a_4=\frac{1}{2^24^26^2}a_0$

$n=3$: $a_4=\frac{1}{4^2}a_2=\frac{1}{2^24^2}a_0 \qquad$ etc

$$\text{so } y_1(x) = \overset{\nearrow 1\text{, say}}{\cancel{a_0}}\left(1+\frac{1}{2^2}x^2+\frac{1}{2^24^2}x^4+\frac{1}{2^24^26^2}x^6+\cdots\right) = \sum_0^\infty \frac{x^{2n}}{2^{2n}(n!)^2}$$

For y_2 we use (41b):

$$y_2(x) = y_1(x)\ln x + \sum_1^\infty c_n x^{n+\cancel{r}\nearrow 0}$$

where, recall, r is 0. Putting that form into the ODE gives

$$\left[-\cancel{\frac{1}{x}y_1}+2y_1'+y_1''x\ln x+\sum_1^\infty n(n-1)c_nx^{n-1}\right]+\left[\cancel{\frac{1}{x}y_1}+y_1'\ln x+\sum_1^\infty nc_nx^{n-1}\right]$$
$$-\left[xy_1\ln x+\sum_1^\infty c_nx^{n+1}\right]=0$$

or, $[\cancelto{0}{xy_1''+y_1'-xy_1}]\ln x + \sum_1^\infty n^2 c_n x^{n-1} - \sum_1^\infty c_n x^{n+1} = -2y_1'$

or, $(c_1+4c_2x+9c_3x^2+16c_4x^3+\cdots)-(c_1x^2+c_2x^3+\cdots) = -2(\frac{1}{2}x+\frac{1}{16}x^3+\cdots)$

x^0: $c_1=0$

x: $4c_2=-1$ so $c_2=-1/4$

x^2: $9c_3-c_1=0$ so $c_3=0$

x^3: $16c_4-c_2=-1/8$ so $c_4=-3/128$, and so on.

Thus,

$$y_2(x) = y_1(x)\ln x - \frac{1}{4}x^2 - \frac{3}{128}x^4 - \cdots.$$

NOTE: Actually, $xy''+y'-xy=0$ is the modified Bessel equation of order 0, with general solution $y(x) = C_1I_0(x)+C_2K_0(x)$, where I_0, K_0 are "modified Bessel functions" of order 0 and of the first and second kinds, respectively, as will be studied in Section 4.6. Our $y_1(x)$ is $I_0(x)$ and our $y_2(x)$ is a linear combination of I_0 and K_0.

NOTE: Before continuing, note that the different cases defined in Theorem 4.3.1 depend on whether the roots of the indicial equation are equal, differ by a number not an integer, or differ by an integer. Since we will not be solving all parts of this lengthy problem, let us at least give the indicial roots for all parts of the problem, to assist you in choosing which parts to assign.

(a) r = 0,1/2 (b) r = 0,0 (c) r = 0,0 (d) r = 0,0 (e) r = -1,1 (f) r = -1,2
(g) r = -1,1 (h) r = -1,1 (i) r = 0,1 (j) r = 0,2/3 (k) r = 0,1 (l) r = $(1\pm\sqrt{3})/2$
(m) r = -1,2 (n) r = 0,4/5 (o) r = 0,1 (p) r = 0,1/2 (q) r = -1/4,3/4 (r) r = -1/4,3/4
(s) r = 0,0

(e) $x^2y''+xy'-y=0$. $y(x)=\sum_0^\infty a_nx^{n+r}$ gives

$$\sum_0^\infty (n+r)(n+r-1)a_nx^{n+r} + \sum_0^\infty (n+r)a_nx^{n+r} - \sum_0^\infty a_nx^{n+r} = 0$$

or $\sum_0^\infty [(n+r)^2-1]a_nx^{n+r} = 0$ so $[(n+r)^2-1]a_n = 0$ for $n=0,1,2,\ldots$.

$n=0$: $(r^2-1)a_0=0$ and $r=\pm 1$ with a_0 arb.

$r=-1$: $[(n-1)^2-1]a_n=0$ gives $a_n=0$ for each $n=1,2,3,\ldots$ except $n=2$; $a_2=$ arb.

$r=1$: $[(n+1)^2-1]a_n=0$ gives $a_n=0$ " " " " "

Thus, the smaller root $r=-1$ gives the general solution $y(x)=a_0x^{0-1}+a_2x^{2-1}$

or, $y(x) = C_1x^{-1}+C_2x$.

(The larger root $r=2$ gives the single solution $y(x)=a_0x^{0+1}$.)

Of course, in this case the ODE is a Cauchy-Euler equation and could have been solved more easily by seeking $y(x)=x^\lambda$.

(f) $x^2y''-x^2y'-2y=0$. $y(x)=\sum_0^\infty a_nx^{n+r}$ gives

$$\sum_0^\infty (n+r)(n+r-1)a_nx^{n+r} - \sum_0^\infty (n+r)a_nx^{n+r+1} - \sum_0^\infty 2a_nx^{n+r} = 0,$$

or $\sum_0^\infty (n+r)(n+r-1)a_nx^{n+r} - \sum_1^\infty (n+r-1)a_{n-1}x^{n+r} - \sum_0^\infty 2a_nx^{n+r}$

or $\sum_0^\infty \{[(n+r)(n+r-1)-2]a_n-(n+r-1)a_{n-1}\}x^{n+r} = 0$ $(a_{-1}\equiv 0)$

so $[(n+r)(n+r-1)-2]a_n-(n+r-1)a_{n-1} = 0$ for $n=0,1,2,\ldots$ ✱

$n=0$: $[r(r-1)-2]a_0=0$ so $r=-1,2$ and $a_0=$ arb.

Set $r=-1$ (which will lead to both solutions or neither). Then ✱ becomes,

$(n^2-3r)a_n = (n-2)a_{n-1}\,. \quad (n=1,2,\dots)$

$n=1$: $-3a_1=-a_0$ so $a_1=a_0/3$

$n=2$: $-2a_2=0$ so $a_2=0$

$n=3$: $0a_3=a_1=a_0/3$ gives $a_0=0$, which is a contradiction. Thus, $r=-1$ does not lead to either solution.

Set $r=2$ (which will lead to one solution, according to Theorem 4.3.1). Then $*$ becomes

$(n^2+3n)a_n=(n+1)a_{n-1}\,. \quad (n=1,2,\dots)$

$n=1$: $4a_1=2a_0$ so $a_1=a_0/2$ $\qquad$ $n=4$: $28a_4=5a_3$ so $a_4=a_0/168$

$n=2$: $10a_2=3a_1$ so $a_2=3a_0/20$ $\qquad$ $n=5$: $40a_5=6a_4$ so $a_5=a_0/1120$

$n=3$: $18a_3=4a_2$ so $a_3=a_0/30$ $\qquad$ etc.

Thus, with $a_0=1$ say, $y_1(x)=\sum_0^\infty a_n x^{n+2} = x^2+\frac{1}{2}x^3+\frac{3}{20}x^4+\frac{1}{30}x^5+\frac{1}{168}x^6+\frac{1}{1120}x^7+\cdots$

To find the missing solution $y_2(x)$ use (41c):

$$y_2(x)=Ky_1(x)\ln x + x^{-1}\sum_0^\infty d_n x^n.$$

Putting the latter into the ODE gives

$$x^2\left[\cancel{Ky_1''\ln x}+2Ky_1'/x-Ky_1/x^2+\sum_0^\infty (n-1)(n-2)d_n x^{n-3}\right]$$
$$-x^2\left[\cancel{Ky_1'\ln x}+Ky_1/x+\sum_0^\infty (n-1)d_n x^{n-2}\right]-2\left[\cancel{Ky_1\ln x}+\sum_0^\infty d_n x^{n-1}\right]=0$$

or, $\sum_0^\infty (n-1)(n-2)d_n x^{n-1}-\sum_0^\infty (n-1)d_n x^n-2\sum_0^\infty d_n x^{n-1}=K(y_1+xy_1-2xy_1')$

or,
$$\left.\begin{array}{l}(\cancel{2d_0}x^{-1}+2d_3x^2+6d_4x^3+12d_5x^4+\cdots)\\ -(-d_0+d_2x^2+2d_3x^3+3d_4x^4+\cdots)\\ -2(\cancel{d_0}x^{-1}+d_1+d_2x+d_3x^2+d_4x^3+d_5x^4+\cdots)\end{array}\right\}=\left\{\begin{array}{l}K(x^2+\frac{1}{2}x^3+\frac{3}{20}x^4+\cdots\\ \quad +x^3+\frac{1}{2}x^4+\cdots\\ \quad -4x^2-3x^3-\frac{6}{5}x^4-\cdots)\end{array}\right.$$

x^0: $d_0-2d_1=0$ so $d_1=d_0/2$

x^1: $-2d_2=0$ so $d_2=0$

x^2: $\cancel{2d_3}-d_2-\cancel{2d_3}=K(1-4)$, but $d_2=0$, so we need $K=0$; $d_3=$ arb.

x^3: $6d_4-2d_3-2d_4=0$ so $d_4=d_3/2$

x^4: $12d_5-3d_4-2d_5=0$ so $d_5=3d_4/10=3d_3/20$

etc

so $y_2(x)=0\,y_1(x)\ln x+x^{-1}\left(d_0+\frac{d_0}{2}x+0x^2+d_3x^3+\frac{d_3}{2}x^4+\frac{3d_3}{20}x^5+\cdots\right)$

$\qquad = d_0\left(\frac{1}{x}+\frac{1}{2}\right)+d_3\left(x^2+\frac{1}{2}x^3+\frac{3}{20}x^4+\cdots\right)$

Having already obtained $y_1(x)$, we really weren't expecting $y_2(x)$ to give two more LI solutions. Indeed, it doesn't because the $d_3(x^2+\frac{1}{2}x^3+\frac{3}{20}x^4+\cdots)$ solution merely repeats y_1. Thus, with $d_0=1$, say,

$$y_2(x)=\frac{1}{x}+\frac{1}{2}\,.$$

(j) $3xy''+y'+y=0$. $\quad y(x)=\sum_0^\infty a_n x^{n+r}$ gives

$$\sum_0^\infty 3(n+r)(n+r-1)a_n x^{n+r-1}+\sum_0^\infty (n+r)a_n x^{n+r-1}+\sum_0^\infty a_n x^{n+r}=0$$

or $\qquad$ " $\qquad$ + $\qquad$ " $\qquad + \sum_1^\infty a_{n-1}x^{n+r-1}=0$

or $\quad \sum_0^\infty\left[3(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-1}\right]x^{n+r-1}=0 \quad (a_{-1}\equiv 0)$

so $\quad \left[3(n+r)^2-2(n+r)\right]a_n+a_{n-1}=0$ for $n=0,1,2,\dots$

$n=0$: $\quad (3r^2-2r)a_0=0$ gives $r=0,\,2/3$ and $a_0=$ arb.

Each r value will contribute one solution.

First let $r=0$: Then $*$ becomes $(3n^2-2n)a_n=-a_{n-1}$ for $n=1,2,\dots$

$n=1$: $a_1=-a_0$

$n=2$: $8a_2=-a_1=a_0$ so $a_2=a_0/8$

$n=3$: $21a_3=-a_2=-a_0/8$ so $a_3=-a_0/168$

$n=4$: $40a_4=-a_3=a_0/168$ so $a_4=a_0/6720$

etc

so, with $a_0=1$ say,

$$y_1(x)=1-x+\tfrac{1}{8}x^2-\tfrac{1}{168}x^3+\tfrac{1}{6720}x^4-\cdots$$

Next, let $r=2/3$: Then $*$ becomes $(3n^2+2n)a_n=-a_{n-1}$ for $n=1,2,\dots$

$n=1$: $5a_1=-a_0$ so $a_1=-a_0/5$

$n=2$: $16a_2=-a_1$ so $a_2=a_0/80$

$n=3$: $33a_3=-a_2$ so $a_3=-a_0/2640$

$n=4$: $56a_4=-a_3$ so $a_4=a_0/147840$

etc

so, with $a_0=1$ say,

$$y_2(x)=\sum_0^\infty a_n x^{n+2/3}=x^{2/3}\left(1-\tfrac{1}{5}x+\tfrac{1}{80}x^2-\tfrac{1}{2640}x^3+\tfrac{1}{147840}x^4-\cdots\right)$$

(m) $x^2y''-(2+3x)y=0$. $y(x)=\sum_0^\infty a_n x^{n+r}$ gives

$$\sum_0^\infty (n+r)(n+r-1)a_n x^{n+r}-\sum_0^\infty 2a_n x^{n+r}-3\sum_0^\infty a_n x^{n+r+1}=0$$

or $\sum_0^\infty (n+r)(n+r-1)a_n x^{n+r}-\sum_0^\infty 2a_n x^{n+r}-\sum_1^\infty 3a_{n-1}x^{n+r}=0$

or $\sum_0^\infty\{[(n+r)(n+r-1)-2]a_n-3a_{n-1}\}x^{n+r}=0 \quad (a_{-1}\equiv 0)$

so $[(n+r)(n+r-1)-2]a_n-3a_{n-1}=0$ for $n=0,1,2,\dots$ $\quad *$

$n=0$: $(r^2-r-2)a_0=0$ gives $r=-1,2$ and $a_0=$ arb.

First, let $r=-1$ (which will give both solutions or neither). Then $*$ becomes

$$(n^2-3n)a_n=3a_{n-1} \text{ for } n=1,2,\dots$$

$n=1$: $-2a_1=3a_0$ so $a_1=-3a_0/2$

$n=2$: $-2a_2=3a_1$ so $a_2=-3a_1/2=9a_0/4$

$n=3$: $0a_3=3a_2=27a_0/4$ gives $a_0=0$ which is a contradiction. Thus, $r=-1$ gives no solutions. We will get one solution from $r=2$ and the other from (41c).

$r=2$: Then $*$ becomes

$$(n^2+3n)a_n=3a_{n-1} \text{ for } n=1,2,\dots$$

$n=1$: $4a_1=3a_0$ so $a_1=3a_0/4$

$n=2$: $10a_2=3a_1$ so $a_2=3a_1/10=9a_0/40$

$n=3$: $18a_3=3a_2$ so $a_3=a_2/6=3a_0/80$

$n=4$: $28a_4=3a_3$ so $a_4=3a_3/28=9a_0/2240$

etc.

so, with $a_0=1$, say, $y_1(x)=\sum_0^\infty a_n x^{n+2}=x^2+\tfrac{3}{4}x^3+\tfrac{9}{40}x^4+\tfrac{3}{80}x^5+\tfrac{9}{2240}x^6+\cdots$

To find the missing solution $y_2(x)$ use (41c):

$$y_2(x)=Ky_1(x)\ln x+\sum_0^\infty d_n x^{n-1}.$$

Putting the latter into the ODE gives

$$x^2[K\cancel{y_1''\ln}x + 2Ky_1'/x - Ky_1/x^2 + \Sigma_0^\infty (n-1)(n-2)d_n x^{n-3}]$$
$$-2[K\cancel{y_1\ln}x + \Sigma_0^\infty d_n x^{n-1}] - 3[K\cancel{xy_1\ln}x + \Sigma_0^\infty d_n x^n] = 0$$

or, $\Sigma_0^\infty (n-1)(n-2)d_n x^{n-1} - 2\Sigma_0^\infty d_n x^{n-1} - 3\Sigma_0^\infty d_n x^n = -2Kxy_1' + Ky_1$

or, writing it out,

$$\left.\begin{array}{l} (\cancel{2d_0x^{-1}} + 2d_3x^2 + 6d_4x^3 + 12d_5x^4 + \cdots) \\ -2(\cancel{d_0x^{-1}} + d_1 + d_2x + d_3x^2 + d_4x^3 + d_5x^4 + \cdots) \\ -3(d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + \cdots) \end{array}\right\} = \left\{\begin{array}{l} K(-4x^2 - \frac{9}{2}x^3 - \frac{9}{5}x^4 - \cdots \\ \quad + x^2 + \frac{3}{4}x^3 + \frac{9}{40}x^4 + \cdots) \\ \end{array}\right.$$

x^0: $-2d_1 - 3d_0 = 0$ so $d_1 = -3d_0/2$

x^1: $-2d_2 - 3d_1 = 0$ so $d_2 = -\frac{3}{2}d_1 = \frac{9}{4}d_0$

x^2: $\cancel{2d_3} - \cancel{2d_3} - 3d_2 = (-4+1)K$ so $K = d_2 = \frac{9}{4}d_0$, $d_3 =$ arb.

x^3: $6d_4 - 2d_4 - 3d_3 = (-\frac{9}{2} + \frac{3}{4})K$, so $d_4 = \frac{3}{4}d_3 - \frac{15}{16}$

x^4: $12d_5 - 2d_5 - 3d_4 = (-\frac{9}{5} + \frac{9}{40})K$, so $d_5 = \frac{9}{40}d_3 - \frac{351}{800}K$

etc.

so $y_2(x) = Ky_1\ln x + \frac{d_0}{x} - \frac{3}{2}d_0 + \frac{9}{4}d_0x + d_3x^2 + (\frac{3}{4}d_3 - \frac{15}{16}K)x^3 + (\frac{9}{40}d_3 - \frac{351}{800}K)x^4 + \cdots$

$= K[y_1\ln x + \frac{4}{9}\frac{1}{x} - \frac{2}{3} + x - \frac{15}{16}x^3 - \frac{351}{800}x^4 + \cdots] + d_3[x^2 + \frac{3}{4}x^3 + \frac{9}{40}x^4 + \cdots]$

where K and d_3 are arbitrary.

If we let $K=$"C_1" and $d_3=$"C_2", then the foregoing result is the general solution of the ODE. Or, if we merely seek a second LI solution we can set $d_3=0$ (because the d_3 series merely reproduces the solution y_1 found above) and set $K=1$, say, so

$$y_2(x) = y_1(x)\ln x + \frac{4}{9}\frac{1}{x} - \frac{2}{3} + x - \frac{15}{16}x^3 - \frac{351}{800}x^4 + \cdots .$$

7. (f) The maple command `dsolve(x^2*diff(y(x),x,x)-x^2*diff(y(x),x)-2*y(x)=0, y(x), type=series);` gives

$$y(x) = C_1x^2(1 + \tfrac{1}{2}x + \tfrac{3}{20}x^2 + \tfrac{1}{30}x^3 + \tfrac{1}{168}x^4 + \tfrac{1}{1120}x^5 + O(x^6))$$
$$+ C_2\left[\frac{(\ln x)(O(x^6))}{x} + \frac{12 + 6x - x^3 - \frac{1}{2}x^4 - \frac{3}{20}x^5 + O(x^6)}{x}\right]$$

Is this result consistent with our result obtained by hand in Exercise 6f? Yes. The $\ln x$ term never actually shows up, even as we go to higher orders. For example, if we precede the dsolve command with Order := 9 then the output is $\cdots + C_2[\frac{(\ln x)(O(x^9))}{x} + \cdots]$. Further, the rest of the C_2 terms can be arranged as

$$12C_2\underbrace{(\tfrac{1}{x} + \tfrac{1}{2})}_{\text{this agrees with our second solution, in 6f}} - C_2\underbrace{(x^2 + \tfrac{1}{2}x^3 + \tfrac{3}{20}x^4 + \cdots)}_{\text{This reproduces our first solution and can therefore be discarded.}}$$

9.(b) We seek $p(x)$ and $q(x)$ so that $y'' + p(x)y' + q(x)y = 0$ has LI solutions $F(x), G(x)$. We need merely solve

$$\begin{array}{lcl} F'' + pF' + qF = 0 & \text{or} & (F')p + (F)q = -F'' \\ G'' + pG' + qG = 0 & & (G')p + (G)q = -G'' \end{array}$$

by Cramer's rule, say. That such solution is possible follows from the assumed linear independence of F and G, for then the determinant $\begin{vmatrix} F' & F \\ G' & G \end{vmatrix} \neq 0$ since that determinant is the Wronskian (actually, the negative of the Wronskian) of F and G.

10. (a) If $y(x) = x^{\alpha+i\beta}\sum_0^\infty a_n x^n$ is a solution of $y''+py'+qy=0$, then $0 = \overline{y''+py'+qy} = \bar{y}''+p\bar{y}'+q\bar{y}$ shows that $\bar{y}$ is too. If we call $\overline{x^{\alpha+i\beta}\sum_0^\infty a_n x^n}$ $= x^{\alpha-i\beta}\sum_0^\infty \bar{a}_n x^n \equiv x^{\alpha+i\beta}\sum_0^\infty \beta x^n$, then we see that $\beta_n = \bar{a}_n$.

(b) Putting $a_n = c_n + id_n$, $\bar{a}_n = b_n = c_n - id_n$, and (10.2) into (10.1), obtain

$$\begin{aligned} y(x) &= Ax^\alpha[\cos(\beta\ln x)+i\sin(\beta\ln x)]\sum(c_n+id_n)x^n \\ &\quad + Bx^\alpha[\quad " \quad - \quad " \quad]\sum(c_n-id_n)x^n \\ &= x^\alpha\{A\cos()\sum c_n x^n + Ai\cos()\sum d_n x^n \\ &\qquad -A\sin()\sum d_n x^n + Ai\sin()\sum c_n x^n \\ &\qquad +B\cos()\sum c_n x^n - Bi\cos()\sum d_n x^n \\ &\qquad -B\sin()\sum d_n x^n - Bi\sin()\sum c_n x^n\} \\ &= x^\alpha\{\underbrace{(A+B)}_{\text{“C”}}[\cos()\sum c_n x^n - \sin()\sum d_n x^n] \\ &\qquad + \underbrace{i(A-B)}_{\text{“D”}}[\cos()\sum d_n x^n + \sin()\sum c_n x^n]\} \end{aligned}$$

(c) $x^2y''+x(1+x)y'+y=0$. $r^2-r+r+1=0$ gives $r=\pm i$ so $\alpha=0$ and $\beta=1$. This will be messy, so let us seek the $Cx^\alpha[\;]$ part of the solution first; for brevity, we will write $\cos(\ln x)\equiv c$ and $\sin(\beta\ln x)\equiv s$. Seek

$$y(x) = \sum_0^\infty (cc_n - sd_n)x^n.$$

$$y'(x) = \sum(-sc_n - cd_n + ncc_n - nsd_n)x^{n-1}$$

$$y''(x) = \sum[-cc_n + sd_n - nsc_n - ncd_n + (n-1)(-sc_n - cd_n + ncc_n - nsd_n)]x^{n-2}$$

Putting these in the ODE gives

$$\begin{aligned} x^2y'' &\to \sum_0^\infty\{c[(n^2-\cancel{n}-\cancel{1})c_n-(2n-\cancel{1})d_n]+s[-(2n-\cancel{1})c_n-(n^2-\cancel{n}-\cancel{1})d_n] \\ +xy' &\to \qquad + c(\cancel{n}\cancel{c}_n-\cancel{d}_n)-s(\cancel{c}_n+\cancel{nd}_n) \\ +y &\to \qquad\quad + c(\cancel{c}_n)-s(\cancel{d}_n)\}x^n \\ +x^2y' &\to \quad + \sum_0^\infty[c(nc_n-d_n)-s(nd_n+c_n)]x^{n+1} = 0 \end{aligned}$$

or,

$$\sum_0^\infty\{c(n^2c_n-2nd_n)+s(-2nc_n-n^2d_n)\}x^n + \sum_0^\infty[c(nc_n-d_n)-s(nd_n+c_n)]x^{n+1}=0$$

or,

$$\sum_1^\infty\{\quad " \quad\}x^n + \sum_1^\infty\{c[(n-1)c_{n-1}-d_{n-1}]-s[(n-1)d_{n-1}+c_{n-1}]\}x^n = 0$$

or,

$$\sum_1^\infty\{c\underbrace{[n^2c_n-2nd_n+(n-1)c_{n-1}-d_{n-1}]}_{*}+s\underbrace{[-2nc_n-n^2d_n-(n-1)d_{n-1}-c_{n-1}]}_{☆}\}x^n=0$$

The linearly independent terms are the $\cos(\ln x)\,x^n$ and $\sin(\ln x)\,x^n$ combinations, so we must set $*=0$ and ☆$=0$:

$$n^2c_n - 2nd_n + (n-1)c_{n-1} - d_{n-1} = 0$$

$$-2nc_n - n^2d_n - (n-1)d_{n-1} - c_{n-1} = 0$$

for $n=1,2,\dots$, where c_0 and d_0 remain arbitrary.

$n=1$ gives $c_1 = (-2c_0+d_0)/5$ and $d_1 = -(c_0+2d_0)/5$

$n=2$ gives $c_2 = (2c_0-d_0)/20$ and $d_2 = (c_0+2d_0)/20$

$n=3$ gives $c_3 = -(17c_0-6d_0)/780$ and $d_3 = -(6c_0+17d_0)/780$

and so on.

Right now the outcome is looking unclear since (10.3) seems to contain the $\underline{\underline{4}}$ arbitrary constants C, D, c_0, d_0. Let's see... We have computed the c_n's and d_n's, so, with $C=1$ and $D=0$, say, (10.3) becomes

$$\begin{aligned} y(x) = {} & \cos(\ln x)\left[c_0 + \tfrac{1}{5}(-2c_0+d_0)x + \tfrac{1}{20}(2c_0-d_0)x^2 - \tfrac{1}{780}(17c_0-6d_0)x^3 + \cdots\right] \\ & -\sin(\ln x)\left[d_0 - \tfrac{1}{5}(c_0+2d_0)x + \tfrac{1}{20}(c_0+2d_0)x^2 - \tfrac{1}{780}(6c_0+17d_0)x^3 + \cdots\right] \end{aligned}$$

$$\begin{aligned} = {} & c_0\Big\{\cos(\ln x)\left[1 - \tfrac{2}{5}x + \tfrac{2}{20}x^2 - \tfrac{17}{780}x^3 + \cdots\right] \\ & \quad -\sin(\ln x)\left[-\tfrac{1}{5}x + \tfrac{1}{20}x^2 - \tfrac{6}{780}x^3 + \cdots\right]\Big\} \\ & + d_0\Big\{\cos(\ln x)\left[\tfrac{1}{5}x - \tfrac{1}{20}x^2 + \tfrac{6}{780}x^3 - \cdots\right] \\ & \quad -\sin(\ln x)\left[1 - \tfrac{2}{5}x + \tfrac{2}{20}x^2 - \tfrac{17}{780}x^3 + \cdots\right]\Big\} \end{aligned}$$

Now we can see that the latter is, in fact, of the form (10.3), where c_0, d_0 play the roles of C and D.

Section 4.4

4. Denoting the left-hand side of (14) as $f(r)$,

$$f(r) = (1-2xr+r^2)^{-1/2}, \quad f'(r) = -\tfrac{1}{2}(-2x+2r)(1-2xr+r^2)^{-3/2} = (x-r)(1-2xr+r^2)^{-3/2}$$

$$f''(r) = -(\ )^{-3/2} - \tfrac{3}{2}(x-r)(-2x+2r)(\ )^{-5/2}$$

etc., so $f(0)=1$, $f'(0)=x$, $f''(0) = -1-\frac{3}{2}x(-2x) = -1+3x^2, \ldots$

so $f(r) = \underset{P_0(x)}{\underset{\uparrow}{1}} + [\underset{P_1(x)}{\underset{\uparrow}{x}}]r + \left[\underset{P_2(x)}{\underset{\uparrow}{\frac{-1+3x^2}{2}}}\right]r^2 + \cdots$

5. Denote the left-hand side of (14) as f. Changing the sign of x in f gives the same result as changing the sign of r. Thus

$$f(r,x) = \sum_0^\infty P_n(x) r^n, \quad f(r,-x) = \sum_0^\infty P_n(-x) r^n \qquad *$$

but also

$$f(r,-x) = f(-r,x) = \sum_0^\infty P_n(x)(-r)^n = \sum_0^\infty (-1)^n P_n(x) r^n \qquad ✫$$

Comparison of * and ✫ reveal that $P_n(-x) = (-1)^n P_n(x)$; i.e., $P_0(x), P_2(x), P_4(x), \ldots$ are even functions of x and $P_1(x), P_3(x), \ldots$ are odd functions of x.

6. (a) $\partial/\partial r$ gives $-\frac{1}{2}\frac{-2x+2r}{(1-2rx+r^2)^{3/2}} = \sum_1^\infty n r^{n-1} P_n(x)$

Thus, $$\frac{x-r}{(1-2rx+r^2)^{1/2}} = \sum_1^\infty n r^{n-1}(1-2rx+r^2) P_n(x)$$

$$\sum_0^\infty [r^n x P_n(x) - r^{n+1} P_n(x)] = \sum_1^\infty [n r^{n-1} P_n(x) - 2r^n n x P_n(x) + n r^{n+1} P_n(x)]$$

$$\begin{aligned} \sum_0^\infty r^n x P_n(x) - \sum_1^\infty r^n P_{n-1}(x) = {} & \sum_0^\infty (n+1) r^n P_{n+1}(x) \\ & - \sum_1^\infty 2r^n n x P_n(x) + \sum_{1\cancel{2}}^\infty (n-1) r^n P_{n-1}(x) \end{aligned}$$

Equating coefficients of powers of r,

$$x P_n(x) - \cancel{P_{n-1}(x)} = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-\cancel{1})P_{n-1}(x) \qquad (n=1,2,\ldots)$$

or $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \qquad (n=1,2,\dots)$

or $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \qquad (n=2,3,\dots)$

7.(a) $\partial/\partial x$ gives $-\frac{1}{2}\frac{-2r}{(1-2xr+r^2)^{3/2}} = \sum_0^\infty P_n'(x)r^n$

$$\frac{r}{(1-2xr+r^2)^{3/2}} = \sum_0^\infty (1-2xr+r^2)P_n'(x)r^n$$

$$\cancel{r}\sum_0^\infty P_n(x)r^{\overset{n+1}{\cancel{n}}} = \sum_0^\infty P_n'(x)r^n - 2x\sum_0^\infty P_n'(x)r^{n+1} + \sum_0^\infty P_n'(x)r^{n+2}$$

$$\sum_1^\infty P_{n-1}(x)r^n = \sum_0^\infty P_n'(x)r^n - 2x\sum_1^\infty P_{n-1}'(x)r^n + \sum_2^\infty P_{n-2}'(x)r^n$$

so, for $n=2,3,\dots$

$$P_{n-1}(x) = P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x).$$

8. (a) the last step is to expand

$$\frac{1}{r}\ln\left(\frac{1+r}{1-r}\right) = \frac{1}{r}2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \frac{r^7}{7} + \cdots\right) \qquad \text{for } |r|<1$$

$$= 2\sum_0^\infty \frac{r^{2n}}{2n+1}, \quad \text{so} \quad \frac{2}{2n+1} = \int_{-1}^1 [P_n(x)]^2dx$$

10.(a) $(1-x^2)y'' - 2xy' + \overset{0}{\cancel{\lambda}}y = 0$. To expand about $x=1$ it is convenient to set $x-1\equiv t$. Then the ODE becomes $t(2+t)Y'' + 2(1+t)Y' = 0$ on $Y(t)$. The indicial equation is $r^2=0$ so $r=0,0$. With $r=0$, seek $Y(t) = \sum_0^\infty a_nt^n$ and obtain

$$(2t+t^2)(2a_2+6a_3t+\cdots) + (2+2t)(a_1+2a_2t+3a_3t^2+\cdots) = 0$$

t^0: $2a_1 = 0$ so $a_1=0$

t^1: $8a_2+2a_1=0$ so $a_2=0$

t^2: $18a_3+6a_2=0$ so $a_3=0$

and so on. Thus, $Y(t) = a_0+0+0+\cdots = a_0$ so $Y_1(t) = 1$. Of course – that is the bounded solution $P_0(t)$. According to (41b) in Theorem 4.3.1, seek the other solution as $Y_2(t) = (1)\ln t + \sum_1^\infty c_nt^n$. Putting this in the ODE gives

$$(2t+t^2)\left(-\frac{1}{t^2}+2c_2+6c_3t+\cdots\right) + (2+2t)\left(\frac{1}{t}+c_1+2c_2t+3c_3t^2+\cdots\right) = 0$$

t^{-1}: $-2+2=0$

t^0: $-1+2c_1+2=0$ so $c_1=-1/2$

t: $4c_2+4c_2+2c_1=0$ so $c_2=1/8$

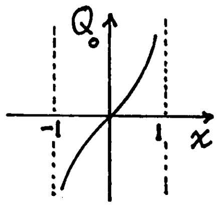
and so on, so

$$Y_2(t) = \ln t - \frac{1}{2}t + \frac{1}{8}t^2 - \cdots \quad \text{or,} \quad y_2(x) = \ln|x-1| - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \cdots$$

11. (a) $y'=p$ gives $(1-x^2)p' - 2xp = 0$

$\frac{dp}{p} - \frac{2x}{1-x^2}dx = 0$, $\ln p + \ln(1-x^2) = \ln C$. Then $p = y' = \frac{C}{1-x^2} = \frac{C}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$

so $y(x) = \frac{C}{2}[\ln(1+x) - \ln(1-x)] = \frac{C}{2}\ln\left(\frac{1+x}{1-x}\right) = CQ_0(x)$



(b) $y(x) = A(x)P_n(x)$, $y' = AP_n' + A'P_n$,

$y'' = AP_n'' + 2A'P_n' + A''P_n$

$$(1-x^2)(\cancel{AP_n''} + 2A'P_n' + A''P_n) - 2x(\cancel{AP_n'} + A'P_n) + n(n+1)\cancel{AP_n} = 0.$$

With $A'=q$, we can reduce the order and obtain

$$2\frac{dP_n}{P_n} + \frac{dq}{q} - \frac{2x\,dx}{1-x^2} = 0, \quad q = A' = \frac{\text{constant}}{(1-x^2)[P_n(x)]^2} \quad \text{so } A(x) = C_n\int^x \frac{dt}{(1-t^2)[P_n(t)]^2},$$

from which (11.3) follows.

12.(a) For $\rho > a$, write
$$\Phi(\rho,\phi) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{\rho[1-2\cos\phi\,(\frac{a}{\rho})+(\frac{a}{\rho})^2]^{1/2}} - \frac{1}{\rho[1+2\cos\phi\,(\frac{a}{\rho})+(\frac{a}{\rho})^2]^{1/2}}\right\}$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{\rho}\left\{\sum_0^\infty\left(\frac{a}{\rho}\right)^n P_n(\cos\phi) - \sum_0^\infty\left(\frac{a}{\rho}\right)^n \underbrace{P_n(-\cos\phi)}_{=(-1)^n P_n(\cos\phi) \text{ by Exercise 5}}\right\} \text{ by (14)}$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{2}{\rho}\sum_{1,3,\ldots}^\infty\left(\frac{a}{\rho}\right)^n P_n(\cos\phi)$$

For $\rho < a$, write
$$\Phi(\rho,\phi) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{a[1-2\cos\phi\,(\frac{\rho}{a})+(\frac{\rho}{a})^2]^{1/2}} - \frac{1}{a[1+2\cos\phi\,(\frac{\rho}{a})+(\frac{\rho}{a})^2]^{1/2}}\right\}$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{a}\left\{\sum_0^\infty\left(\frac{\rho}{a}\right)^n P_n(\cos\phi) - \sum_0^\infty\left(\frac{\rho}{a}\right)^n \underbrace{P_n(-\cos\phi)}_{=(-1)^n P_n(\cos\phi)}\right.$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{2}{a}\sum_{1,3,\ldots}^\infty\left(\frac{\rho}{a}\right)^n P_n(\cos\phi).$$

(b) If a is arbitrarily small, then $\rho > a$ and (12.2) gives
$$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\frac{2Q}{\rho}\sum_{1,3,\ldots}\left(\frac{a}{\rho}\right)^n P_n(\cos\phi) = \frac{1}{4\pi\epsilon_0}\frac{2Q}{\rho}\left[\frac{a}{\rho}(\cos\phi) + \cdots\right] \sim \frac{2Qa}{4\pi\epsilon_0}\frac{\cos\phi}{\rho^2}$$
as $a \to 0$.

(c) If a is arbitrarily large, then $\rho < a$ and (12.2) gives
$$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\frac{2Q}{a}\sum_{1,3,\ldots}\left(\frac{\rho}{a}\right)^n P_n(\cos\phi) = \frac{1}{4\pi\epsilon_0}\frac{2Q}{a}\frac{\rho\cos\phi}{a} + \cdots \sim \frac{1}{4\pi\epsilon_0}\frac{2Q}{a^2}z$$
as $a \to \infty$.

Section 4.5

1.(a) $x^\alpha e^{-\beta x} \to (\infty)(0)$, which is indeterminate. To apply l'Hôpital's rule we need it to be in either of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so re-express it as $\frac{x^\alpha}{e^{\beta x}}$. Depending on how large α is, a sufficient number of derivatives of the numerator will produce either 0 or a negative power of x. That same number of differentiations of the denominator merely give β to a power times $e^{\beta x}$, so l'Hôpital gives the limit $0/\infty$ as $x \to \infty$, i.e., 0. Alternatively, notice in
$$x^\alpha/e^{\beta x} = x^\alpha/(1+\beta x+\beta^2x^2/2!+\cdots)$$
that for any $N > \alpha$ the term $\beta^N x^N/N!$ in the denominator dominates the numerator so the ratio must $\to 0$ as $x \to \infty$.

2.(a) l'Hôpital gives $\frac{\ln x}{x^\alpha} \sim \frac{1/x}{\alpha x^{\alpha-1}} = \frac{1}{\alpha x^\alpha} \to 0$ as $x \to \infty$, no matter how small α is.

(b) l'Hôpital gives $\frac{x^\alpha}{\ln x} \sim \frac{\alpha x^{\alpha-1}}{1/x} = \alpha x^\alpha \to 0$ as $x \to 0$, no matter how small α is.

3. (b) $\int_0^\infty \frac{x^3\,dx}{x^5+2} = \int_0^1 \frac{x^3\,dx}{x^5+2} + \int_1^\infty \frac{x^3\,dx}{x^5+2} \equiv I_1 + I_2$. I_1 converges because the integrand is continuous on $0 \le x \le 1$, and I_2 is convergent because $x^3/(x^5+2) \sim 1/x^2$ gives a convergent integral — a p-integral with $p = 2 > 1$.

(c) $x^3/(x^4+2) \sim 1/x$ as $x \to \infty$, which is a borderline divergent p-integral. Thus, the integral is divergent.

(e) $I = \int_0^1 dx/x^2 + \int_1^\infty dx/x^2 \equiv I_1 + I_2$. I_2 is a <u>con</u>vergent horizontal p-integral since $p=2>1$, but I_1 is a <u>diver</u>gent vertical p-integral since $p=2>1$. Thus, I is divergent.

(f) $I = \int_0^1 dx/x^{1/2} + \int_1^\infty dx/x^{1/2} \equiv I_1 + I_2$. I_1 is a <u>con</u>vergent vertical p-integral since $p=1/2<1$, but I_2 is a <u>diver</u>gent horizontal p-integral since $p=1/2<1$. Thus, I is divergent.

(h) $\left|\frac{\cos x}{x(x-1)}\right| \leq \frac{1}{x(x-1)} \sim \frac{1}{x^2}$ as $x \to \infty$. Since $\int_5^\infty dx/x^2$ is a convergent p-integral, the given integral is absolutely convergent and hence convergent.

NOTE: Whereas $1/[x(x-1)] \sim 1/x^2$, we would <u>not</u> say that "$\int_5^\infty dx/[x(x-1)] \sim \int_5^\infty dx/x^2$ as $x \to \infty$." Indeed, $\int_5^\infty dx/x^2$ is not even a function of x; it is a number.

(i) $I = \int_1^2 e^x dx/\sqrt{x-1} = \int_0^1 e^{\xi+1} d\xi/\sqrt{\xi} = e\int_0^1 (e^\xi/\xi^{1/2})\,d\xi$. As $\xi \to 0$, $e^\xi/\xi^{1/2} \sim 1/\xi^{1/2}$ so the vertical p-integral converges because $p=1/2<1$.

4\. (a) Can show by l'Hôpital's rule that $\ln x/x^{1/4} \to 0$ as $x \to \infty$, so $\ln x < x^{1/4}$ for all sufficiently large x (really, $\ln x < x^\alpha$ for <u>any</u> $\alpha > 0$; we merely use 1/4 for definiteness). Thus, $\ln x/x^2 < x^{1/4}/x^2 = 1/x^{1.75}$ which gives a convergent p-integral (since $p=1.75>1$), hence the original integral converges.

(b) Alternative to the hint, we could proceed directly, as follows. As $x \to 0$, $|\ln x| < x^{-0.14}$, say, since, by l'Hôpital, $\frac{\ln x}{x^{-0.14}} \sim \frac{1/x}{-0.14x^{-1.14}} = -\frac{1}{0.14}x^{0.14} \to 0$ as $x \to 0$.

Since, for sufficiently small x, $|\ln x/\sqrt{x}| < x^{-0.14}/\sqrt{x} = 1/x^{0.64}$, which gives a convergent (since $0.64<1$) vertical p-integral. Thus, the given integral converges.

5\. None, for we need $p>1$ for "convergence at ∞" and we need $p<1$ for "convergence at 0".

6\. (b) $x = 1/\xi$ gives $I = \int_\infty^{1/3} -\frac{\sqrt{\xi}}{\xi^2}d\xi = \int_{1/3}^\infty d\xi/\xi^{3/2}$ is singular because of the ∞ limit.

7\. (b) $-1 < \alpha < \infty$

(d) $x^\alpha \sin x \sim x^{\alpha+1}$ as $x \to 0$, so we need $\alpha+1>-1$ for convergence; i.e., $-2<\alpha<\infty$

(f) $x^\alpha/\sqrt{x^2-1} \sim x^{\alpha-1}$ as $x \to \infty$, so we need $\alpha-1<-1$ for convergence; i.e., $\alpha<0$.

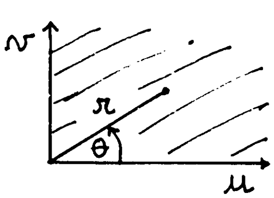
NOTE: The singularity at $x=1$ is integrable (for <u>any</u> α) because, with $x-1 \equiv \xi$, $x^\alpha/\sqrt{x^2-1} = x^\alpha/\sqrt{(x-1)(x+1)} \sim 1/\sqrt{2\xi} = \frac{1}{\sqrt{2}}\frac{1}{\xi^{1/2}}$ and the exponent 1/2 is <1.

8\. (a) $\Gamma(3.5) = 2.5\,\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.5)\Gamma(0.5) = 1.875\sqrt{\pi} = 3.32335$.

What is the Maple name for the gamma function? The command ?gamma is of no help, nor ?lib, but ?factorial leads us to the name GAMMA. Then, the command GAMMA(3.5); gives 332335.... .

9\. There are a number of ways of evaluating the integral $I = \int_0^\infty e^{-u^2}du$, the following method being especially pretty. With

$$I^2 = \int_0^\infty e^{-u^2}du \int_0^\infty e^{-v^2}dv = \int_0^\infty\int_0^\infty e^{-(u^2+v^2)}du\,dv,$$



it is useful to imagine the latter as a double integral in a u,v plane, rather than merely as a product of two single integrals. The u^2+v^2 suggests changing from u,v to polar coordinates: $I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2} \underbrace{r\,dr\,d\theta}_{d(\text{area})} = \frac{\pi}{2}\int_0^\infty e^{-r^2}\underline{\underline{r}}\,dr$

What this step buys for us is the r (underlined), which permits the successful use, now, of the change of variables $r^2=t$: $I^2 = \frac{\pi}{2}\int_0^\infty e^{-t}\frac{dt}{2} = \frac{\pi}{4}$, $I = \frac{\sqrt{\pi}}{2}$.

10. (b) $\int_0^1 x^m(\ln x)^n dx \overset{\ln x = t,\ x=e^t}{=} \int_{-\infty}^0 e^{mt}t^n e^t dt = -\int_0^{-\infty} e^{(m+1)t}t^n dt \overset{t=-u}{=} -\int_0^\infty e^{-(m+1)u}(-u)^n(-du)$

$$= (-1)^n\int_0^\infty e^{-(m+1)u}u^n du = (-1)^n\int_0^\infty e^{-v}\left(\frac{v}{m+1}\right)^n\frac{dv}{m+1} = (-1)^n\frac{\Gamma(n+1)}{(m+1)^2} = \frac{(-1)^n n!}{(m+1)^2}$$

(d) $\int_0^\infty xe^{-\sqrt{x}}dx \overset{\sqrt{x}=t}{=} \int_0^\infty t^2e^{-t}\,2t\,dt = 2\int_0^\infty e^{-t}t^3dt = 2\Gamma(4) = (2)(3!) = 12$

(e) $\int_0^\infty (x-1)^2e^{-x^3}dx = \int_0^\infty (x^2-2x+1)e^{-x^3}dx \overset{x^3=t}{=} \int_0^\infty (t^{2/3}-2t^{1/3}+1)e^{-t}\frac{1}{3}t^{-2/3}dt$

$$= \frac{1}{3}\int_0^\infty (1-2t^{-1/3}+t^{-2/3})e^{-t}dt = \frac{1}{3}\left[\overset{1}{\cancel{\Gamma(1)}}-2\Gamma(\tfrac{2}{3})+\Gamma(\tfrac{1}{3})\right].$$

12. `int(t^(2/3)*exp(-sqrt(t)), t=0..infinity);` gives $\frac{112}{81}\frac{\pi\sqrt{3}}{\Gamma(2/3)}$.
To show that this agrees with (24) in Example 6, we begin by using the recursion formula (16):
$$2\Gamma(\tfrac{10}{3}) = 2\tfrac{7}{3}\Gamma(\tfrac{7}{3}) = \tfrac{14}{3}\tfrac{4}{3}\Gamma(\tfrac{4}{3}) = \tfrac{56}{9}\tfrac{1}{3}\Gamma(\tfrac{1}{3}).$$

Now use (17.2) in Exercise 17: $\frac{56}{9}\frac{1}{3}\Gamma(\frac{1}{3}) = \frac{56}{27}\frac{\pi}{\sin\frac{\pi}{3}\,\Gamma(\frac{2}{3})} = \frac{56}{27}\frac{\pi}{\frac{\sqrt{3}}{2}\Gamma(\frac{2}{3})}\times\frac{\sqrt{3}}{\sqrt{3}} = \frac{112}{81}\frac{\pi\sqrt{3}}{\Gamma(2/3)}$.

14. $\Gamma(p) = \int_0^\infty u^{2p-2}e^{-u^2}2u\,du = 2\int_0^\infty u^{2p-1}e^{-u^2}du$, so
$$\begin{aligned}
\Gamma(p)\Gamma(q) &= 4\int_0^\infty u^{2p-1}e^{-u^2}du\int_0^\infty v^{2q-1}e^{-v^2}dv\\
&= 4\int_0^\infty\int_0^\infty u^{2p-1}v^{2q-1}e^{-(u^2+v^2)}du\,dv\\
&= 4\int_0^{\pi/2}\int_0^\infty r^{2(p+q)-2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\,e^{-r^2}r\,dr\,d\theta\\
&= 4\int_0^{\pi/2}\underset{\text{set }\cos^2\theta=t}{\underset{\uparrow}{\cos^{2p-1}\theta}}\sin^{2q-1}\theta\,d\theta\int_0^\infty \underset{\text{set }r^2=t}{\underset{\uparrow}{r^{2(p+q)-1}}}e^{-r^2}dr\\
&= 4\int_1^0 t^{p-1/2}(1-t)^{q-1/2}\left(\frac{dt}{-2\sqrt{1-t}\sqrt{t}}\right)\int_0^\infty t^{p+q-1/2}e^{-t}\frac{1}{2}t^{-1/2}dt\\
&= \int_0^1 t^{p-1}(1-t)^{q-1}dt\int_0^\infty t^{p+q-1}e^{-t}dt = B(p,q)\Gamma(p+q).
\end{aligned}$$

15. (a) With $x=t/(1+t)$, (14.1) gives
$$B(p,q) = \int_0^\infty\left(\frac{t}{1+t}\right)^{p-1}\left(\frac{1}{1+t}\right)^{q-1}\frac{dt}{(1+t)^2} = \int_0^\infty\frac{t^{p-1}}{(1+t)^{p+q}}dt$$
(b) With $t=\cos^2\theta$,
$$\begin{aligned}
B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}dt = \int_{\pi/2}^0 \cos^{2p-2}\theta\,(1-\cos^2\theta)^{q-1}\,2(-\sin\theta)\cos\theta\,d\theta\\
&= 2\int_0^{\pi/2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta
\end{aligned}$$

16. (a) Let $q = 2Q-1$ so $Q = (q+1)/2$, and let $p = 2P-1$ so $P = (p+1)/2$. Then

$$\int_0^{\pi/2} \cos^q\theta \sin^p\theta \, d\theta = \int_0^{\pi/2} \cos^{2Q-1}\theta \sin^{2P-1}\theta \, d\theta \underset{\uparrow \text{ by } (15.2)}{=} \tfrac{1}{2} B(P,Q) = \tfrac{1}{2} B\left(\tfrac{p+1}{2}, \tfrac{q+1}{2}\right)$$

To use (15.2) we need $P>0$ and $Q>0$. These follow from the given conditions that $p>-1$ and $q>-1$.

(b) $\int_0^{\pi/2} \tan^p\theta \, d\theta = \int_0^{\pi/2} \cos^{-p}\theta \sin^p\theta \, d\theta$. Let $-p = 2Q-1$ so $Q = (1-p)/2$, and let $p = 2P-1$ so $P = (p+1)/2$. Then the given integral

$$\begin{aligned}
&= \int_0^{\pi/2} \cos^{2Q-1}\theta \sin^{2P-1}\theta \, d\theta = \tfrac{1}{2} B(P,Q) \text{ by } (15.2) \\
&= \tfrac{1}{2} B\left(\tfrac{1+p}{2}, \tfrac{1-p}{2}\right) \\
&= \tfrac{1}{2} \int_0^\infty \frac{t^{(1+p)/2-1}}{(1+t)^{(1+p)/2+(1-p)/2}} \, dt \quad \text{from } (15.1) \\
&= \tfrac{1}{2} \int_0^\infty \frac{t^{(p-1)/2}}{1+t} \, dt \\
&= \tfrac{1}{2} \frac{\pi}{\sin\left(\frac{p+1}{2}\pi\right)}, \text{ by } (17.1), = \frac{\pi}{2} \frac{1}{\cos\frac{p\pi}{2}}.
\end{aligned}$$

Finally, observe that $p \to -p$ sends $\int_0^{\pi/2} \tan^p\theta \, d\theta \to \int_0^{\pi/2} \cot^p\theta \, d\theta$. Thus,

$$\int_0^{\pi/2} \cot^p\theta \, d\theta = \frac{\pi}{2} \frac{1}{\cos\left(-\frac{p\pi}{2}\right)} = \frac{\pi}{2} \frac{1}{\cos\left(\frac{p\pi}{2}\right)} \quad \text{since } \cos(-x) = \cos x.$$

(c) $$\int_0^\infty \frac{x^a \, dx}{(1+x^b)^c} \overset{x^b=t}{\overset{\downarrow}{=}} \int_0^\infty \frac{t^{a/b} \frac{1}{b} t^{\frac{1}{b}-1}}{(1+t)^c} \, dt = \frac{1}{b} \int_0^\infty \frac{t^{(a-b+1)/b}}{(1+t)^c} \, dt.$$ In (15.1), $p-1 = (a-b+1)/b$ and $p+q = c$, so $p = (a+1)/b$ and $q = (cb-a-1)/b$, so the integral $= \frac{1}{b} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{1}{b} \frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(\frac{cb-a-1}{b}\right)}{\Gamma(c)}$.

In (15.1) we required that $p = (a+1)/b > 0$ and $q = (cb-a-1)/b > 0$. That is, $a > -1$ and (if $b>0$) $cb-a-1 > 0$. There is no loss in assuming that $b>0$, for if $b<0$ we can still get the integral into the form $\int_0^\infty x^a dx/(1+x^b)^c$ where b ___is___ >0.

17. $$\int_0^\infty \frac{x^{\alpha-1}}{1+x} \, dx \underset{\uparrow \, (15.1) \text{ with } p=\alpha, \, q=1-\alpha}{=} B(p,q) = B(\alpha, 1-\alpha) \underset{\uparrow \, (14.1)}{=} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = \Gamma(\alpha)\Gamma(1-\alpha)$$

and this, together with (17.1) gives $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin\alpha\pi$.

18. (a) $\dot\theta = 0$ in (18.1) gives $\theta = \pm\theta_0$, so a swing from $+\theta_0$ to $-\theta_0$ to $+\theta_0$ gives one period, and $-\theta_0$ to $+\theta_0$ gives a half period, $T/2$. From (18.1), $\sqrt{l/2g}\,\dot\theta = \sqrt{\cos\theta - \cos\theta_0}$ so

$$\sqrt{\frac{l}{2g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{T/2} dt.$$

The θ integrand is _even_, so we can change the limits to $\int_0^{\theta_0}$ and $\int_0^{T/4}$, respectively. With $\theta_0 = \pi/2$,

$$\begin{aligned}
T(\pi/2) &= 4\sqrt{l/2g} \int_0^{\pi/2} d\theta/\sqrt{\cos\theta} = 4\sqrt{l/2g}\, \tfrac{1}{2} B\left(\tfrac{1}{2}, \tfrac{1}{4}\right) \\
&= \sqrt{2l/g}\; \Gamma(1/2)\Gamma(1/4)/\Gamma(3/4) = \sqrt{2\pi l/g}\; \Gamma(1/4)/\Gamma(3/4).
\end{aligned}$$

(b) $T(\theta_0) = 4\sqrt{l/2g} \int_0^{\theta_0} d\theta/\sqrt{\cos\theta - \cos\theta_0} \to 0$ as the upper limit $\theta_0 \to 0$ _if_ the integrand is bounded; but it is NOT bounded. Proceeding formally, if $\theta_0 \to 0$ and $0 \le \theta \le \theta_0$ then $\cos\theta - \cos\theta_0 = (1-\theta^2/2+\cdots) - (1-\theta_0^2/2+\cdots) = (\theta_0^2-\theta^2)/2 + \cdots \sim (\theta_0^2-\theta^2)/2$, so

$T(\theta_0) \sim 4\sqrt{l/2g} \int_0^{\theta_0} d\theta/\sqrt{(\theta_0^2-\theta^2)/2} = 4\sqrt{l/g} \int_0^1 d\psi/\sqrt{1-\psi^2} = 4\sqrt{l/g}\, \sin^{-1}\psi\big|_0^1 = 2\pi\sqrt{l/g}$ as

$\theta_0 \to 0$. Thus, $T(\theta_0) \to 2\pi\sqrt{l/g}$, not 0, as $\theta_0 \to 0$. NOTE: Proceeding heuristically along different lines, note that d/dt of (18.1) gives the nonlinear ODE

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \qquad *$$

For small motions (i.e., $|\theta| << 1$ radian), $\sin\theta \sim \theta$ and the linearized version of $*$ is the familiar (Section 3.5) harmonic oscillator equation $\ddot{\theta} + (g/l)\theta = 0$, with general solution $\theta(t) = A\sin(\sqrt{g/l}\,t + \phi)$ with frequency $\sqrt{g/l}$ radians/sec and period T given by $\sqrt{g/l}\,T = 2\pi$ or, $T = 2\pi\sqrt{l/g}$, once again.

19. (b) For $F(x) = 4 + O(x^2)$ we want to show that $F(x) - 4 \sim Cx^2$ (as $x \to 0$).
Well, $F(x) - 4 = -4x^2 + 4x^4 - \cdots \sim -4x^2$ as $x \to 0$. ✓

(f) For $H(x) = O(x)$ as $x \to \infty$ we want to show that $H(x) \sim Cx$ as $x \to \infty$. Well,
$H(x) = \frac{7x^3 - x + 1}{x^2 + 4} \sim \frac{7x^3}{x^2} = 7x$ as $x \to \infty$. ✓

Section 4.6

2. Actually, the solution steps are given by (2)-(10). Let us verify that (9) and (10) do give (14a,b), respectively. (9) says

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} + \frac{(-1)^1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\Gamma(7/2)}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{1}{\frac{3}{2}(\frac{1}{2}\sqrt{\pi})}\frac{x^2}{4} + \frac{1}{2(\frac{5}{2})(\frac{3}{2})(\frac{1}{2}\sqrt{\pi})}\frac{x^4}{16} - \cdots\right] = \sqrt{\frac{x}{2\pi}}\left(2 - \frac{1}{3}x^2 + \frac{1}{60}x^4 - \cdots\right)$$

$$= \sqrt{\frac{2x}{\pi}}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right) = \sqrt{\frac{2}{\pi x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \sqrt{\frac{2}{\pi x}}\sin x.$$

Similarly for $J_{-1/2}(x)$.

3. $$Y_0(x) = \frac{2}{\pi}[\underset{\sim}{Y}_0(x) + (\gamma - \ln 2)J_0(x)] \sim \frac{2}{\pi}\left\{\frac{1}{\sqrt{\pi x}}\left[\left(\frac{\pi}{2} - \gamma + \ln 2\right)\sin x - \left(\frac{\pi}{2} + \gamma - \ln 2\right)\cos x\right] + (\gamma - \ln 2)\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2\sqrt{\pi x}}\sin x - \cancel{\frac{\gamma}{\sqrt{\pi x}}\sin x} + \cancel{\frac{\ln 2}{\sqrt{\pi x}}\sin x} - \frac{\pi}{2\sqrt{\pi x}}\cos x - \cancel{\frac{\gamma}{\sqrt{\pi x}}\cos x} + \cancel{\frac{\ln 2}{\sqrt{\pi x}}\cos x} + (\cancel{\gamma} - \cancel{\ln} 2)\sqrt{\frac{2}{\pi x}}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)\right\}$$

$$= \frac{1}{\sqrt{\pi x}}(\sin x - \cos x) = \sqrt{\frac{2}{\pi x}}\sin(x - \pi/4),$$ where we've used $\cos(A-B) = \cos A\cos B + \sin A\sin B$ and $\sin(A-B) = \sin A\cos B - \sin B\cos A$

4. (a) Let's verify (4.1) for the case where Z is J. That is, we seek to show that

$$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$$

Using (9),

$$x^\nu J_\nu(x) = \sum_0^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\frac{x^{2k+2\nu}}{2^{2k+\nu}}$$

so $$[x^\nu J_\nu(x)]' = \sum_0^\infty \frac{2(k+\nu)(-1)^k}{k!\,\Gamma(\nu+k+1)}\frac{x^{2k+2\nu-1}}{2^{2k+\nu}} = \sum_0^\infty \frac{(k+\nu)(-1)^k}{k!\,\Gamma(\nu+k+1)}\frac{x^{2k+2\nu-1}}{2^{2k+\nu-1}} \qquad *$$

Next, (9) gives $x^{\nu}J_{\nu-1}(x) = x^{\nu}\frac{x^{\nu-1}}{2^{\nu-1}}\sum_{0}^{\infty}\frac{(-1)^k}{k!\Gamma(\nu+k)}\frac{x^{2k}}{2^{2k}}$

$$= \sum_{0}^{\infty}\frac{(-1)^k}{k!\Gamma(\nu+k)}\frac{x^{2k+2\nu-1}}{2^{2k+\nu-1}}. \qquad ★$$

But $* = ★$ because, by the recursion formula (16) in Section 4.5, $\Gamma(\nu+k+1) = (\nu+k)\Gamma(\nu+k)$.

(b) First, derive (4.4). Since (4.4) is for $J, Y, H^{(1)}, H^{(2)}$, use the upper formulas in (4.1) and (4.2):

(4.1) gives $\nu x^{\nu-1}Z_\nu + x^{\nu}Z_\nu' = x^{\nu}Z_{\nu-1} \to Z_\nu' = Z_{\nu-1} - \nu x^{-1}Z_\nu \quad *$

(4.2) " $-\nu x^{-\nu-1}Z_\nu + x^{-\nu}Z_\nu' = -x^{-\nu}Z_{\nu+1} \to Z_\nu' = -Z_{\nu+1} + \nu x^{-1}Z_\nu \quad ★$

Equating the right-hand sides of $*$ and $★$ gives $Z_{\nu-1} - \nu x^{-1}Z_\nu = -Z_{\nu+1} + \nu x^{-1}Z_\nu$ or, $Z_{\nu+1} = 2\nu x^{-1} - Z_{\nu-1}$ ✓

To derive (4.5), use the upper formula in (4.1) and the lower formula in (4.2):

(4.1) gives $\nu x^{\nu-1}Z_\nu + x^{\nu}Z_\nu' = x^{\nu}Z_{\nu-1} \to Z_\nu' = Z_{\nu-1} - \nu x^{-1}Z_\nu \quad *$

(4.2) " $-\nu x^{-\nu-1}Z_\nu + x^{-\nu}Z_\nu' = x^{-\nu}Z_{\nu+1} \to Z_\nu' = Z_{\nu+1} + \nu x^{-1}Z_\nu \quad ★$

Equate the right-hand sides of $*$ and $★$, and change the Z's to I's, so $I_{\nu-1} - \nu x^{-1}I_\nu = I_{\nu+1} + \nu x^{-1}I_\nu$, so $I_{\nu+1} = I_{\nu-1} - 2\nu x^{-1}I_\nu$. ✓

To derive (4.6), use the lower formula in (4.1) and the upper formula in (4.2) and proceed as above.

(c) First, we need the Maple names for the various Bessel functions. ?bessel tells us that the names are BesselJ(ν,x), BesselY(ν,x), BesselI(ν,x), and BesselK(ν,x) for $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$, and $K_\nu(x)$. (It didn't mention the Hankel functions, nor did I find the Hankel functions by using the commands ?hankel or ?lib.) As representative, the command

```
diff(x^3*BesselJ(3,x), x);
```

gives the result $3x^2\,\mathrm{BesselJ}(3,x) + x^3\left(\mathrm{BesselJ}(2,x) - 3\,\frac{\mathrm{BesselJ}(3,x)}{x}\right)$ which simplifies (cancelling the first and last terms) to $x^3J_2(x)$, in agreement with (4.1).

5. (a) See solution to Exercise 2.

(b) Merely set $\nu = n - 1/2$, and let Z be J, in (4.4).

(c) $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ by setting $n=1$ in (5.3).

$J_{-3/2}(x) = -\frac{1}{x}J_{-1/2}(x) - J_{1/2}(x)$ by setting $n=0$ in (5.3)

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)$$

6. (a) $y = \sigma v$, $y' = \sigma v' + \sigma' v$, $y'' = \sigma v'' + 2\sigma' v' + \sigma'' v$, so the ODE becomes

$(\sigma v'' + 2\sigma' v' + \sigma'' v) + p(\sigma v' + \sigma' v) + q(\sigma v) = 0$

or $v'' + \underbrace{\left(\frac{2\sigma' + p\sigma}{\sigma}\right)}_{\text{set this}=0}v' + \left(\frac{\sigma'' + p\sigma' + q\sigma}{\sigma}\right)v = 0$, so $\sigma(x) = e^{-\frac{1}{2}\int^x p(t)\,dt}$.

(b) For the Bessel equation (1), $p(x) = 1/x$ and $q(x) = 1 - \frac{v^2}{x^2}$ so $\sigma(x) = e^{-\frac{1}{2}\int^x dt/t}$ $= e^{-\frac{1}{2}\ln x} = x^{-1/2}$. Then (6.1) becomes

$$v'' + \left\{\left[-\tfrac{1}{2}(-\tfrac{3}{2})x^{-5/2} + \tfrac{1}{x}(-\tfrac{1}{2}x^{-3/2}) + (1-\tfrac{v^2}{x^2})x^{-1/2}\right]/x^{-1/2}\right\} v = 0$$

or, simplifying, $v'' + (1 - \frac{v^2 - 1/4}{x^2})v = 0.$

7.(a) $\int_{x=0}^{x=c} 2(xy')\frac{d(xy')}{\cancel{dx}}\cancel{dx} + 2\int_0^c (\kappa^2x^2 - n^2)\frac{dy}{\cancel{dx}}y\,\cancel{dx} = 0$ gives (7.3).

(b) In this solution, keep in mind that, according to standard mathematical notation f' means $df/d(\text{argument})$, so $J_n'(\kappa x)$ means $dJ_n/d(\kappa x)$, not $dJ_n(\kappa x)/dx$. Now, with $y(x) = J_n(\kappa x)$, $xy'(x) = x\frac{d}{dx}J_n(\kappa x) = \kappa x J_n'(\kappa x)$. Further, with $v = n$ and $Z = J$ (4.2) gives

$$(x^{-n}J_n(x))' = -x^{-n}J_{n+1}(x), \text{ or, } -nx^{-n-1}J_n(x) + x^{-n}J_n'(x) = -x^{-n}J_{n+1}(x),$$

or, $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$, so (changing x to κx)

$$\kappa x J_n'(\kappa x) = \underline{\underline{n}}\, J_n(\kappa x) - \kappa x J_{n+1}(\kappa x). \qquad *$$

Because of the $\underline{\underline{n}}$ in $*$, and the fact that $J_1(0) = J_2(0) = \cdots = 0$, $*$ gives $\kappa x J_n'(\kappa x) = 0$ at $x = 0$ for $n = 0, 1, 2, \ldots$. Recalling that $J_n(\kappa c) = 0$, it also follows from $*$ that, at $x = c$, $(xy'(x))^2$ is $= \kappa^2c^2[J_{n+1}(\kappa c)]^2$.

(c) Thus, (7.3) gives (7.4).

(d) $n^2y^2\Big|_{x=0}^{x=c} = n^2J_n(\kappa x)^2\Big|_0^c = n^2\overset{0}{\cancel{J_n(\kappa c)}}{}^2 - \underset{\substack{\uparrow\\ 0 \text{ for } n=0}}{n^2}\underset{\substack{\uparrow\\ 0 \text{ for } n\ge 1}}{J_n(0)^2} = 0$ for all $n = 0, 1, 2, \ldots$

Also, $\int_{x=0}^{x=c} \underbrace{x^2}_{u}\underbrace{y\,dy}_{dv} = x^2\frac{y^2}{2}\Big|_{x=0}^{x=c} - \int_0^c \frac{y^2}{2}2x\,dx$

$$= \underset{0 \text{ at } x=0 \nearrow}{\tfrac{1}{2}x^2}\underset{\llcorner 0 \text{ at } x=c}{J_n(\kappa x)^2}\Big|_0^c - \int_0^c [J_n(\kappa x)]^2x\,dx = -\int_0^c [J_n(\kappa x)]^2 x\,dx$$

so (7.4) becomes $c^2\kappa^2[J_{n+1}(\kappa c)]^2 + 2\kappa^2\left\{-\int_0^c [J_n(\kappa x)]^2x\,dx\right\} - 0 = 0$ which, in turn, gives (7.2).

8. (a) $e^{xt/2}e^{-x/2t} = \sum_{j=0}^\infty \frac{1}{j!}(\frac{xt}{2})^j \sum_{k=0}^\infty \frac{(-1)^k}{k!}(\frac{x}{2t})^k = \sum_{j=0}^\infty\sum_{k=0}^\infty \frac{(-1)^k}{j!k!}(\frac{x}{2})^{j+k}t^{j-k}.$

The coefficient of t^0 (obtained by setting $j = k$) is $= \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}(\frac{x}{2})^{2k} = J_0(x)$. ✓

(b) $\frac{1}{2}(t - \frac{1}{t})e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{-\infty}^\infty J_n'(x)t^n. \qquad *$

On the other hand,

$$\begin{aligned}\tfrac{1}{2}(t - \tfrac{1}{t})e^{\frac{x}{2}(t-\frac{1}{t})} &= \tfrac{1}{2}\sum_{n=-\infty}^\infty [J_n(x)t^{n+1} - J_n(x)t^{n-1}]\\ &= \tfrac{1}{2}\sum_{m=-\infty}^\infty [J_{m-1}(x)t^m - J_{m+1}(x)t^m]\\ &= \sum_{n=-\infty}^\infty \tfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]t^n. \qquad ☆\end{aligned}$$

Equating coefficients of like powers of t in $*$ and ☆ gives

$$J_n'(x) = \tfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(c) $\frac{x}{2}(1 + \frac{1}{t^2})e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{-\infty}^\infty nJ_n(x)t^{n-1},$

$$\sum_{-\infty}^\infty \tfrac{x}{2}J_n(x)t^n + \sum_{-\infty}^\infty \tfrac{x}{2}J_n(x)t^{n-2} = \sum_{-\infty}^\infty nJ_n(x)t^{n-1},$$

or, shifting indicies, $\sum_{-\infty}^{\infty} \frac{x}{2} J_n(x) t^n + \sum_{-\infty}^{\infty} \frac{x}{2} J_{n+2}(x) t^n = \sum_{-\infty}^{\infty} (n+1) J_{n+1}(x) t^n$ and equating coefficients of like powers of t gives

$$\frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x) = (n+1) J_{n+1}(x)$$

or, equivalently, (8.3).

(d) Actually, (8.2) does not hold for $n=0$ so it does not give $J_0'(x)$, but it does hold for $n=1$, giving $J_1'(x) = \frac{1}{2}[J_0(x) - J_2(x)]$. Using Maple,

diff (BesselJ(1, x), x);

gives $J_1'(x) = J_0(x) - \frac{1}{x} J_1(x)$, which is not the same form. However, with $n=0$ (8.3) gives $J_2(x) = \frac{2J_1(x)}{x} - J_0(x)$ and putting this into our result gives

$$J_1'(x) = \frac{1}{2}\left[J_0(x) - \left(\frac{2J_1(x)}{x} - J_0(x)\right)\right] = J_0(x) - \frac{J_1(x)}{x}. \checkmark$$

9. $\frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta = \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \underbrace{\int_0^{\pi} \sin^{2j}\theta\, d\theta}_{\leftarrow I}$

$I = 2\int_0^{\pi/2} \sin^{2j}\theta\, d\theta$ (because the integrand is symmetric about $\theta = \pi/2$)

$= 2\left(\frac{1}{2}\right) B\left(\frac{2j+1}{2}, \frac{0+1}{2}\right)$ by (16.1) in Exercise 16 of Section 4.5

$= \frac{\Gamma\left(\frac{2j+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(j+1)}$ by (14.1) in Exercise 14 of Section 4.5

$= \frac{\sqrt{\pi}}{j!}\Gamma\left(j+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{j!}\left(j-\frac{1}{2}\right)\left(j-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} = \frac{\pi}{j!}\frac{(2j-1)(2j-3)\cdots(1)}{2^j}$

$= \frac{\pi}{j!}\frac{1}{2^j}\frac{(2j-1)(2j-2)(2j-3)\cdots(2)(1)}{(2j-2)(2j-4)\cdots 2} = \frac{\pi}{j!\,2^j}\frac{(2j-1)!}{2^{j-1}(j-1)!}$

so $\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta = \frac{1}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!} x^{2j} \frac{\pi\,(2j-1)!}{j!\,2^j 2^{j-1}(j-1)!} = \frac{1}{\cancel{\pi}}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j} x^{2j}\frac{\cancel{\pi}}{j!\,2^j 2^{j-1}(j-1)!}$

$= \sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{x}{2}\right)^{2j} = J_0(x) \checkmark$

10. Simply write out (4a,b,c), with $\nu = 3$, say, and the choice $r = -\nu = -3$.

$k=0$: $0a_0 = 0$ so $a_0 =$ arb.

$k=1$: $(4-9)a_1 = 0$ so $a_1 = 0$

$k=2$: $(1-9)a_2 + a_0 = 0$ so $a_2 = -a_0/8$

$k=3$: $(0-9)a_3 + a_1 = 0$ so $a_3 = 0$

$k=4$: $(1-9)a_4 + a_2 = 0$ so $a_4 = -a_0/64$

$k=5$: $(4-9)a_5 + a_3 = 0$ so $a_5 = 0$

$k=6$: $0a_6 + a_4 = 0$ so $a_6 =$ arb. and $a_4 = 0$. But $a_4 = 0$ now implies that $a_0 = 0$ and hence $a_2 = 0$ too. This backward 0 propagation is the key.

$k=7$: $(16-9)a_7 + a_5 = 0$ so $a_7 = 0$

$k=8$: $(25-9)a_8 + a_6 = 0$ so $a_8 = -a_6/16$

etc.

so $y(x) = \sum_0^{\infty} a_k x^{k+r} = \sum_0^{\infty} a_k x^{k-3} = 0x^{-3} + 0x^{-2} + 0x^{-1} + 0x^0 + 0x + 0x^2 + a_0 x^3 + 0x^4 - \frac{a_0}{16}x^5 + \cdots$

$= a_0\left(x^3 - \frac{1}{16}x^5 + \cdots\right)$

Meanwhile, (17) (with $\nu = n = 3$) gives $J_{-3}(x) = \frac{(-1)^3}{3!0!}\left(\frac{x}{2}\right)^3 + \frac{(-1)^4}{4!1!}\left(\frac{x}{2}\right)^5 + \cdots = -\frac{x^3}{48} + \frac{x^5}{768} - \cdots$ which $= -\frac{1}{48}\left(x^3 - \frac{x^5}{16} + \cdots\right)$ and which agrees with our foregoing result (if we set $a_0 = -1/48$). Anyway, the point is to observe how the terms in (17) are indeed absent if $k = 0, 1, 2, \dots, n-1$; it is, because of the above-noted backward propagation of 0's.

11. Let us denote the right-hand side of (26) as $AJ_\nu(x) + BJ_{-\nu}(x)$. Write

$$\alpha Y_\nu(x) + \beta J_\nu(x) = 0. \qquad (\nu \neq \text{integer}) \quad \text{——} \ *$$

Then $\alpha(AJ_\nu(x) + BJ_{-\nu}(x)) + \beta J_\nu(x) = 0$, or, $(\alpha A + \beta)J_\nu(x) + \alpha BJ_{-\nu}(x) = 0$. We already know that $J_\nu, J_{-\nu}$ are LI [as can be seen from (12) and (13), for neither one is a scalar multiple of the other] so it must be true that $\alpha A + \beta = 0$ and $\alpha B = 0$. Since $B = -1/\sin\nu\pi \neq 0$, it follows that $\alpha = 0$, and then that $\beta = 0$. Thus, it follows from $*$ that Y_ν, J_ν are LI.

12. Recall from the NOTE on page 240 that Z_ν in (50) should be interpreted as $Z_{|\nu|}$. Thus, $(x^a y')' + bx^c y = 0$ gives $y(x) = x^{\nu/\alpha} Z_{|\nu|}(\alpha\sqrt{|b|}\, x^{1/\alpha})$

$$\alpha = 2/(c-a+2), \ \nu = (1-a)/(c-a+2)$$

where $Z_{|\nu|}$ denotes $J_{|\nu|}$ and $Y_{|\nu|}$ if $b > 0$, and $I_{|\nu|}$ and $K_{|\nu|}$ if $b < 0$. Thus, implementation is very straightforward once we identify a, b, c.

(b) $xy'' - 2y' - x^2 y = 0$ gives $y'' - \frac{2}{x}y' - xy = 0$
$(x^a y')' + bx^c y = 0$ gives $y'' + \frac{a}{x}y' + bx^{c-a}y = 0$ $>$ so $a = -2, b = -1, c = -1$
$\alpha = 2/3, \nu = 1$
so $y(x) = x^{3/2} Z_{|1|}(\frac{2}{3}x^{3/2})$. Hence, $y(x) = x^{3/2}[AI_1(\frac{2}{3}x^{3/2}) + BK_1(\frac{2}{3}x^{3/2})]$

(c) $xy'' - 2y' + xy = 0$ gives $y'' - \frac{2}{x}y' + y = 0$
$(x^a y')' + bx^c y = 0$ gives $y'' + \frac{a}{x}y' + bx^{c-a}y = 0$ $>$ so $a = -2, b = 1, c = -2$
$\alpha = 1, \nu = 3/2$

so $y(x) = x^{3/2} Z_{3/2}(x)$. Hence, $y(x) = x^{3/2}[AJ_{3/2}(x) + BY_{3/2}(x)]$

(e) $y'' + \sqrt[3]{x}\, y = 0$
$y'' + \frac{a}{x}y' + bx^{c-a}y = 0$ $>$ $a = 0, b = 1, c = 1/3$
$\alpha = 6/7, \nu = 3/7$ so $y(x) = x^{3/6} Z_{3/7}(\frac{6}{7}x^{7/6})$
Hence, $y(x) = \sqrt{x}\,(AJ_{3/7}(\frac{6}{7}x^{7/6}) + BY_{3/7}(\frac{6}{7}x^{7/6}))$

13. (b) dsolve(x*diff(y(x),x,x)-2*diff(y(x),x)-x^2*y(x)=0, y(x));
gives $y(x) = C_1 x^{3/2}\,\text{BesselI}(1, \frac{2}{3}x^{3/2}) + C_2 x^{3/2}\,\text{BesselK}(1, \frac{2}{3}x^{3/2})$

(c) The dsolve command gives $y(x) = C_1[-\sin x + x\cos x] + C_2[\cos x + x\sin x]$.
The latter *seems* not to agree with our result in 12(c), but notice from (26) and Exercise 5(c)'s solution that

$$y(x) = x^{3/2}[AJ_{3/2}(x) + BY_{3/2}(x)] = x^{3/2}\left[AJ_{3/2}(x) + B\,\frac{0J_{3/2}(x) - J_{-3/2}(x)}{-1}\right] \text{ by (26)}$$
$$= x^{3/2}[AJ_{3/2}(x) + BJ_{-3/2}(x)]$$
$$= x^{3/2}\left[A\sqrt{\tfrac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) + B\sqrt{\tfrac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)\right]$$
$$= C_1(\sin x - x\cos x) + C_2(x\sin x + \cos x). \ \checkmark$$

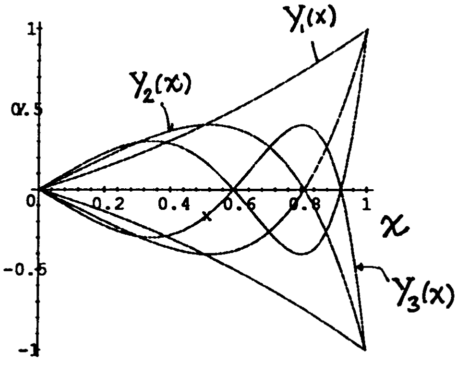
15. $y''+4y=0$. $a=0, b=4, c=0, \alpha=1, \nu=1/2$ so $y(x)=x^{1/2}Z_{1/2}(2x)$. Thus,
$y(x)=\sqrt{x}\,[AJ_{1/2}(2x)+BJ_{-1/2}(2x)]$ $\{$or, equivalently, $\sqrt{x}\,[AJ_{1/2}(2x)+BY_{1/2}(2x)]\}$
$= A\sqrt{\frac{1}{\pi}}\sin 2x + B\sqrt{\frac{1}{\pi}}\cos 2x$ (by Exercise 5)
$= C_1 \sin 2x + C_2 \cos 2x$. ✓

16. In his book "Bessel Functions for Engineers", 2nd ed., N.W. McLachlan writes "This problem was studied by Daniel Bernoulli in 1732, and later by Euler in 1781. It is the first instance of (a) a differential equation whose solutions are now called Bessel functions, (b) the determination of the normal vibrational modes of a continuous system."‡

(a) $[\rho g(l-x)Y']'+\rho\omega^2 Y=0$. Letting $l-x=\xi$, obtain $(\xi y')'+\frac{\omega^2}{g}y=0$ so, in (50), $a=1, b=\omega^2/g, c=0, \alpha=2, \nu=0$ so $y(\xi)=\xi^0 Z_0(\frac{2\omega}{\sqrt{g}}\xi^{1/2})$
so $y(\xi)=AJ_0(\frac{2\omega}{\sqrt{g}}\sqrt{\xi})+BY_0(\frac{2\omega}{\sqrt{g}}\sqrt{\xi})$
or $Y(x)=AJ_0(\frac{2\omega}{\sqrt{g}}\sqrt{l-x})+BY_0(\frac{2\omega}{\sqrt{g}}\sqrt{l-x})$. Boundedness at $x=l$ implies $B=0$.

(b) $Y(0)=0=AJ_0(2\omega\sqrt{l/g})$.
If $A=0$ then $Y(x)\equiv 0$, which corresponds to no motion at all. Thus, $A\neq 0$ and $J_0(2\omega\sqrt{l/g})=0$. Let z_n be the zeros of $J_0(x)$; i.e., $J_0(z_n)=0$ for $n=1,2,\dots$. Then the allowable vibrational frequencies ω are given by $2\omega_n\sqrt{l/g}=z_n$, or,
$\omega_n=\frac{1}{2}\sqrt{\frac{g}{l}}\,z_n$ $(n=1,2,\dots)$ where $z_1\approx 2.405$, $z_2\approx 5.520$, $z_3\approx 8.654$, etc.
If we call the corresponding mode shapes $Y_n(x)$ (not to be confused with the Bessel function Y), then
$Y_1(x)=AJ_0(2.405\sqrt{1-x/l})$, $Y_2(x)=AJ_0(5.520\sqrt{1-x/l})$, $Y_3(x)=AJ_0(8.654\sqrt{1-x/l})$, etc.
See part (c) for plots, for $A=1$ and $l=1$.

(c)
```
> with(plots):
> implicitplot({y=BesselJ(0,2.405*sqrt(1-x)),y=-BesselJ(0,2.405*sqrt
  (1-x)),y=BesselJ(0,5.520*sqrt(1-x)),y=-BesselJ(0,5.520*sqrt(1-x)),
  y=BesselJ(0,8.654*sqrt(1-x)),y=-BesselJ(0,8.654*sqrt(1-x))},x=0..1
  ,y=-2..2,numpoints=2000);
```



To obtain the zeros z_n, fsolve(BesselJ(0,x)=0, x=0..3); gives 2.404825558, fsolve(BesselJ(0,x)=0, x=3..6); gives 5.520078110, and fsolve(BesselJ(0,x), x=6..9); gives 8.653727913.

---

‡ London: Oxford University Press, 1955. See pages 106-107.

CHAPTER 5

Section 5.2

1. Need $|f(t)| \le Ke^{ct}$ for all $t > T$

(b) Can take $K=10$, $c=-5$, $T=0$, for example. Yes, exponential order.

(c) $f(t) = \sinh 2t = (e^{2t}-e^{-2t})/2 \sim e^{2t}/2$ as $t\to\infty$. Can take $K=1, c=2, T=0$, say. Yes, exponential order.

(e) No, $\sinh t^2 = (e^{t^2}-e^{-t^2})/2 \sim e^{t^2}/2$ grows faster than any exponential. That is, $\frac{e^{t^2}/2}{Ke^{ct}} = \frac{1}{2K}e^{t^2-ct} \to \infty$ as $t\to\infty$ no matter how large we make c.

(f) Yes. Can take $K=5$, $c=4$, $T=0$, for example.

2. No. For example, $f(t) = \sin(e^{t^2})$ is of exponential order ($K=1, c=0, T=0$, for example) but $df/dt = 2t\underline{e^{t^2}}\cos(e^{t^2})$ is not — because of the $\underline{e^{t^2}}$ factor.

3. No. For example both $f(t)=e^t$ and $g(t)=t^2$ are, but $f(g(t)) = e^{t^2}$ is not.

4. If $s\le 0$ then $\int_0^\infty t^{-1/2}e^{-st}\,dt$ is divergent. We can see the difficulty in (12) since $\sqrt{\pi/s}\to\infty$ as $s\to 0$ through positive values.

5. $L\{t^{-3/2}\} = \int_0^\infty t^{-3/2}e^{-st}\,dt$ does not exist because the integral is divergent: $t^{-3/2}e^{-st} \sim t^{-3/2}$ (as $t\to 0$) gives a divergent vertical p-integral ($p=3/2>1$).

6. $L\{t^{-2/3}\} = \int_0^\infty t^{-2/3}e^{-st}\,dt$ does exist: this time $t^{-2/3}e^{-st}\sim t^{-2/3}$ (as $t\to 0$) gives a convergent vertical p-integral ($p=2/3<1$).

7. $$\int_0^\infty \underbrace{\cos at}_{u}\ \underbrace{e^{-st}\,dt}_{dv,\ v=-e^{-st}/s} = -\frac{1}{s}\cos at\, e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}(-a\sin at)\,dt$$
$$= \frac{1}{s} - \frac{a}{s}\int_0^\infty \underbrace{\sin at}_{u}\ \underbrace{e^{-st}\,dt}_{dv,\ v=-e^{-st}/s}$$
$$= \frac{1}{s} - \frac{a}{s}\left(-\frac{1}{s}\cancel{\sin at}\, e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}(a\cos at)\,dt\right) = \frac{1}{s} - \frac{a^2}{s^2}\int_0^\infty \cos at\, e^{-st}\,dt,$$

so $(1+\frac{a^2}{s^2})L\{\cos at\} = \frac{1}{s}$, $L\{\cos at\} = s/(s^2+a^2)$. NOTE: In place of the choice $u=\cos at$ and $dv = e^{-st}\,dt$, in the first step, we could have chosen $u=e^{-st}$ and $dv = \cos at\,dt$.

Alternatively, $\int_0^\infty \cos at\, e^{-st}\,dt = \int_0^\infty \mathrm{Re}\, e^{iat}\, e^{-st}\,dt = \mathrm{Re}\int_0^\infty e^{-(s-ia)t}\,dt$
$$= \mathrm{Re}\,\frac{e^{-(s-ia)t}}{-(s-ia)}\Big|_0^\infty = \mathrm{Re}\left(0+\frac{1}{s-ia}\right) = \mathrm{Re}\left(\frac{1}{s-ia}\,\frac{s+ia}{s+ia}\right) = \frac{s}{s^2+a^2}.$$

10. (a) $$L\{\sinh at\} = \int_0^\infty \frac{e^{at}-e^{-at}}{2}\,e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{-(s-a)t}\,dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t}\,dt$$
$$= \frac{1}{2}\,\frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty - \frac{1}{2}\,\frac{e^{-(s+a)t}}{-(s+a)}\Big|_0^\infty = \frac{1}{2}\left(0+\frac{1}{s-a}\right) - \frac{1}{2}\left(0+\frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

11. (a) $$L\{t\sin at\} = \int_0^\infty t\sin at\, e^{-st}\,dt = \frac{1}{2i}\int_0^\infty te^{-(s-ia)t}\,dt - \frac{1}{2i}\int_0^\infty te^{-(s+ia)t}\,dt \quad -*$$

Now, $\int_0^\infty \underbrace{t}_{u}\underbrace{e^{-bt}\,dt}_{dv} = -\frac{te^{-bt}}{b}\Big|_0^\infty - \int_0^\infty -\frac{e^{-bt}}{b}\,dt = 0 + \frac{1}{b}\left(-\frac{1}{b}e^{-bt}\right)\Big|_0^\infty = \frac{1}{b^2}$, so * gives

$$L\{t\sin at\} = \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] = \frac{1}{2i}\,\frac{\cancel{s^2}+i2as-\cancel{a^2}-(\cancel{s^2}-i2as-\cancel{a^2})}{(s-ia)^2(s+ia)^2} = \frac{2as}{(s^2+a^2)^2}$$

(b) $\frac{d}{ds}\int_0^\infty \sin at\, e^{-st}\,dt = \frac{d}{ds}\frac{a}{s^2+a^2}$ gives $-\int_0^\infty t \sin at\, e^{-st}\,dt = \frac{a(-1)(2s)}{(s^2+a^2)^2}$

gives $L\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$ again.

12.(a) $\frac{d}{ds}\int_0^\infty \cos at\, e^{-st}\,dt = \frac{d}{ds}\frac{s}{s^2+a^2}$ gives $-\int_0^\infty t \cos at\, e^{-st}\,dt = \frac{1}{s^2+a^2} + \frac{s(-1)(2s)}{(s^2+a^2)^2}$

gives $L\{t \cos at\} = (s^2-a^2)/(s^2+a^2)^2$.

(b) $\int_0^\infty e^{-st}\,dt = \frac{1}{s}$, $\frac{d}{ds}\int_0^\infty e^{-st}\,dt = \frac{d}{ds}\frac{1}{s}$ gives $\int_0^\infty te^{-st}\,dt = \frac{1}{s^2}$. Again, $\frac{d}{ds}\int_0^\infty te^{-st}\,dt = \frac{d}{ds}\frac{1}{s^2}$ gives $\int_0^\infty t^2e^{-st}\,dt = \frac{2}{s^3}$, and so on.

13. $L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$. That is,

$$L\{e^{at}\} = \lim_{T\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^T = \frac{1}{s-a} - \frac{1}{s-a}\lim_{T\to\infty}\left(e^{-(\mathrm{Re}\,s-a)T}e^{-i(\mathrm{Im}\,s)T}\right) = \frac{1}{s-a} \text{ if } \mathrm{Re}\,s-a>0.$$

Proof: $|e^{-(\mathrm{Re}\,s-a)T}e^{-i(\mathrm{Im}\,s)T}| = |e^{-(\mathrm{Re}\,s-a)T}||e^{-i(\mathrm{Im}\,s)T}| = e^{-(\mathrm{Re}\,s-a)T} \to 0$ as $T\to\infty$ if $\mathrm{Re}\,s>a$. We've used the fact that $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$ for all θ.

14. NOTE: Maple seems to have trouble with entries 18 and 19 if $a \neq 0$. For example,

```
readlib(laplace):
laplace(Delta(t),t,s);
```

gives 1 and

```
readlib(laplace):
invlaplace(1,s,t);
```

gives Dirac(t), but Maple does not give the transform of D(t-a) or the inverse of e^{-as} (namely, e^{-as} and $\delta(t-a)$, respectively).

Section 5.3

1.(c) Partial fractions: $\frac{1}{s^2-a^2} = \frac{1}{(s-a)(s+a)} = \frac{A}{s-a} + \frac{B}{s+a} = \frac{(A+B)s+(A-B)a}{s^2-a^2}$

$$\left.\begin{array}{l} 1:\ 1=(A-B)a \\ s:\ 0=A+B \end{array}\right\} \to \begin{array}{l} A=1/2a \\ B=-1/2a \end{array} \quad \text{so } L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = L^{-1}\left\{\frac{1}{2a}\frac{1}{s-a} - \frac{1}{2a}\frac{1}{s+a}\right\}$$

$$= \frac{1}{2a}L^{-1}\{1/s-a\} - \frac{1}{2a}L^{-1}\{1/s+a\} \quad \text{by linearity of } L^{-1}$$

$$= \frac{1}{2a}e^{at} - \frac{1}{2a}e^{-at} = \frac{1}{a}\sinh at$$

Convolution: $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = L^{-1}\left\{\left(\frac{1}{s-a}\right)\left(\frac{1}{s+a}\right)\right\} = e^{at} * e^{-at}$

$$= \int_0^t e^{a\tau}e^{-a(t-\tau)}\,d\tau = e^{-at}\int_0^t e^{a\tau}e^{a\tau}\,d\tau = e^{-at}\left.e^{2a\tau}/2a\right|_0^t = \frac{1}{a}\sinh at. ✓$$

(e) Partial fractions: $\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$, so $L^{-1}\left\{\frac{1}{s^2+s}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$

$= L^{-1}\{\frac{1}{s}\} - L^{-1}\{\frac{1}{s+1}\}$ (by linearity of L^{-1}) $= 1-e^{-t}$.

Convolution: $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = L^{-1}\{\frac{1}{s}\} * L^{-1}\{\frac{1}{s+1}\} = 1 * e^{-t} = \int_0^t e^{-\tau}\,d\tau = -e^{-\tau}\Big|_0^t = 1-e^{-t}$. ✓

2. (e)

```
readlib(laplace):
invlaplace(1/(s*(s+1)),s,t);
```

gives $1-e^{-t}$. ✓

3. (b) $1/(s^2-3s+3) = \dfrac{1}{(s-3/2)^2+3/4} = \dfrac{1}{\sqrt{3/4}}\,\dfrac{\sqrt{3/4}}{(s-3/2)^2+3/4} \to \dfrac{1}{\sqrt{3/4}}\,e^{3t/2}\sin\frac{\sqrt{3}}{2}t$ by entry 9, with $a=3/2$ and $b=\sqrt{3}/2$.

(c) $1/(s^2-s) = \dfrac{1}{(s-1/2)^2-1/4} = \dfrac{1}{i/2}\,\dfrac{i/2}{(s-\frac12)^2-\frac14} \to \frac{2}{i}\,e^{t/2}\sin(it/2)$ (entry 9; $a=\frac12, b=\frac{i}{2}$)

$$= \frac{2}{i}e^{t/2}\,\frac{e^{iit/2}-e^{-iit/2}}{2i} = 2e^{t/2}\,\frac{e^{t/2}-e^{-t/2}}{2}$$
$$= e^{t}-1$$

(g) $\dfrac{s+1}{s^2-s} = \dfrac{s+1}{(s-1/2)^2-1/4} = \dfrac{(s-1/2)+3/2}{(s-1/2)^2-1/4} = \dfrac{s-1/2}{(s-1/2)^2-1/4} + \dfrac{3}{2}\dfrac{2}{i}\,\dfrac{i/2}{(s-1/2)^2-1/4}$

$$\to e^{t/2}\cos\frac{it}{2} + \frac{3}{2}\frac{2}{i}\,e^{t/2}\sin\frac{it}{2} = e^{t/2}\left[\frac{e^{-t/2}+e^{t/2}}{2} + \frac{3}{i}\,\frac{e^{-t/2}-e^{+t/2}}{2i}\right] = 2e^{t}-1$$

(We do not claim this is the simplest method in this case.)

4. By (8), the proposition is true for $n=1$. According to the method of induction it remains to show that if it holds for $n=k$ then it holds for $n=k+1$.

$n=k$: $L\{f^{(k)}\} = s^kL\{f\} - s^{k-1}f(0) - s^{k-2}f'(0) - \cdots - s^0f^{(k-1)}(0)$ by assumption.

$n=k+1$: $L\{f^{(k+1)}\} = L\{g'\}$ where $g \equiv f^{(k)}$

$$= sL\{g\} - g(0)$$
$$= s\left[s^kL\{f\} - s^{k-1}f(0) - \cdots - f^{(k-1)}(0)\right] - f^{(k)}(0)$$
$$= s^{k+1}L\{f\} - s^kf(0) - \cdots - f^{(k)}(0) \quad ✓$$

5. (a) $f*g = \int_0^t f(\tau)g(t-\tau)\,d\tau$ (Let $t-\tau=\mu$)

$$= \int_t^0 f(t-\mu)g(\mu)(-d\mu) = \int_0^t g(\mu)f(t-\mu)\,d\mu = g*f.$$

(b) $f*(g*h) = \int_0^t f(\tau)\left[\int_0^t g(\mu)h(t-\mu)\,d\mu\right]\Big|_{t=t-\tau}d\tau = \int_0^t f(\tau)\left[\int_0^{t-\tau} g(\mu)h(t-\tau-\mu)\,d\mu\right]d\tau$

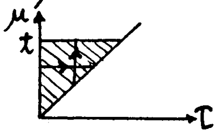
$$= \int_0^t\int_0^{t-\tau} f(\tau)g(\mu)h(t-\tau-\mu)\,d\mu\,d\tau. \quad ♀$$

$(f*g)*h = \int_0^t\left[\int_0^t f(\mu)g(t-\mu)\,d\mu\right]\Big|_{t=\tau}h(t-\tau)\,d\tau$

Change $\tau \rightleftarrows \mu$

$$= \int_0^t\int_0^\tau f(\mu)g(\tau-\mu)h(t-\tau)\,d\mu\,d\tau \overset{\downarrow}{=} \int_0^t\int_0^\mu f(\tau)g(\mu-\tau)h(t-\mu)\,d\tau\,d\mu$$

Now switch the order of integration $= \displaystyle\int_0^t\int_\tau^t f(\tau)g(\mu-\tau)h(t-\mu)\,d\mu\,d\tau$



Finally, let $\mu-\tau=\nu$. Then $= \displaystyle\int_0^t\int_0^{t-\tau} f(\tau)g(\nu)h(t-\tau-\nu)\,d\nu\,d\tau$, which is the same as ♀. ✓

6. $L\{f*g*h\} = L\{f*(g*h)\} = L\{f\}L\{g*h\} = F(s)[G(s)H(s)] = F(s)G(s)H(s).$

7. From entry 7, $1/s^3 \to \frac{1}{2!}t^2 = t^2/2.$

Alternatively, $L^{-1}\{1/s^3\} = L^{-1}\{\frac{1}{s}\frac{1}{s}\frac{1}{s}\} = 1*1*1 = 1*\int_0^t d\tau = 1*t = \int_0^t \tau\,d\tau = t^2/2.$

9. (a) $L\{3e^{3t}\} = sL\{e^{3t}\} - 1$?

$$\frac{3}{s-3} = \frac{s}{s-3} - 1 = \frac{s-(s-3)}{s-3} = \frac{3}{s-3} \quad ✓$$

$L\{9e^{3t}\} = s^2L\{e^{3t}\} - s - 3$?

$$\frac{9}{s-3} = \frac{s^2}{s-3} - s - 3 = \frac{s^2-(s+3)(s-3)}{s-3} = \frac{9}{s-3} \quad ✓$$

(c) $L\{2t+5\} = sL\{t^2+5t-1\} - (-1)$?

$$\frac{2}{s^2} + \frac{5}{s} = s\left(\frac{2}{s^3} + \frac{5}{s^2} - \frac{1}{s}\right) + 1 = \frac{2}{s^2} + \frac{5}{s} - 1 + 1 \quad ✓$$

$L\{2\} = s^2L\{t^2+5t-1\} - s(-1) - 5$?

$$\cancel{\frac{2}{s}} = s^2\left(\cancel{\frac{2}{s^3}} + \cancel{\frac{5}{s^2}} - \cancel{\frac{1}{s}}\right) + \cancel{s} - \cancel{5} \quad ✓$$

10. (b) $L\{\int_0^t \cos 3(t-\tau)\,d\tau\} = L\{1 * \cos 3t\} = L\{1\}L\{\cos 3t\} = \frac{1}{s}\frac{s}{s^2+9} = \frac{1}{s^2+9}$.

(c) $L\{\int_0^t (t-\tau)^8 e^{-3\tau}\,d\tau\} = L\{t^8 * e^{-3t}\} = L\{t^8\}L\{e^{-3t}\}$

$= \frac{8!}{s^9}\frac{1}{s+3}$.

11. (c) $L^{-1}\{F(s)G(s)\} = L^{-1}\{\frac{1}{s^2}\frac{2}{s^3}\} = L^{-1}\{\frac{2}{s^5}\} = 2\frac{1}{4!}t^4 = \frac{1}{12}t^4$ whereas $f(t)g(t) = t^3$

Section 5.4

1. (b) $3x'+x = 6e^{2t}$. $3[s\bar{x}(s) - \overset{0}{\cancel{x(0)}}] + \bar{x}(s) = \frac{6}{s-2}$ gives $\bar{x}(s) = \frac{2}{(s-2)(s+1/3)}$
$= \frac{6}{7}\left(\frac{1}{s-2} - \frac{1}{s+1/3}\right)$ so $x(t) = \frac{6}{7}(e^{2t} - e^{-t/3})$.

(e) $x''+5x' = 10$. $s^2\bar{x} - s\overbrace{x(0)}^{\text{“A”}} - \overbrace{x'(0)}^{\text{“B”}} + 5[s\bar{x} - x(0)] = \frac{10}{s}$ gives
$\bar{x} = \frac{10}{s^2(s+5)} + \frac{B+5A}{s(s+5)} + \frac{A}{s+5}$, $x = 10t * e^{-5t} + (B+5A)1 * e^{-5t} + Ae^{-5t}$
$= 10\int_0^t \tau e^{-5(t-\tau)}d\tau + (B+5A)\int_0^t e^{-5(t-\tau)}d\tau + Ae^{-5t}$
$= 2t + \frac{2}{5} + \frac{2}{5}e^{-5t} + (B+5A)(1-e^{-5t})/5 + Ae^{-5t}$
or, equivalently and more simply, $x(t) = 2t + C + De^{-5t}$

(f) $x''-x' = 1+t+t^2$. $s^2\bar{x} - s\underbrace{x(0)}_{\text{“A”}} - \underbrace{x'(0)}_{\text{“B”}} - s\bar{x} + \underbrace{x(0)}_{A} = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}$

so $\bar{x} = \frac{B-A}{s^2-s} + \frac{A}{s-1} + \frac{1}{s^2(s-1)} + \frac{1}{s^3(s-1)} + \frac{2}{s^4(s-1)}$

$x(t) = (B-A)1 * e^t + Ae^t + t * e^t + \frac{1}{2}t^2 * e^t + \frac{2}{6}t^3 * e^t$
$= (B-A)\int_0^t e^\tau d\tau + Ae^t + \int_0^t(\tau + \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3)e^{t-\tau}d\tau$
$= (B-A)(e^t-1) + Ae^t - 4t - \frac{3}{2}t^2 - 4 - \frac{t^3}{3} + 4e^t$ or, more simply,
$= -4t - \frac{3}{2}t^2 - \frac{1}{3}t^3 + C + De^t$

(i) $x''-x'-12x = t$, $x(0)=-1$, $x'(0)=0$. $s^2\bar{x} - s(-1) - 0 - (s\bar{x}+1) - 12\bar{x} = 1/s^2$ gives
$\bar{x} = (1+s^2-s^3)/(s^2(s^2-s-12))$. To expand in partial fractions, let's use Maple.
convert((1+s^2-s^3)/(s^2*(s^2-s-12)), parfrac, s);
gives $\bar{x} = -\frac{1}{12}\frac{1}{s^2} + \frac{1}{144}\frac{1}{s} - \frac{37}{63}\frac{1}{s+3} - \frac{47}{112}\frac{1}{s-4}$,

so $x(t) = 1/144 - \frac{1}{12}t - \frac{37}{63}e^{-3t} - \frac{47}{112}e^{4t}$

(o) Of course, the simplest approach is to integrate twice first:
$x''(t) - x''(0) + 5x'(t) - 5x'(0) = t^5/5$ or $x''+5x' = \frac{t^5}{5}+1$. Again,
$x'(t) - x'(0) + 5x(t) - 5x(0) = \frac{t^6}{30} + t$ or $x'+5x = \frac{t^6}{30} + t$, and now apply
the Laplace transform. Nevertheless, let us transform immediately:
$[s^3\bar{x} - \cancel{s^2(0)} - \cancel{s(0)} - 1] + 5[s^2\bar{x} - \cancel{s(0)} - \cancel{0}] = \frac{24}{s^5}$ so $(s^3+5s^2)\bar{x} = \frac{24}{s^5} + 1$,
$\bar{x} = \frac{24+s^5}{s^7(s+5)} = \frac{24}{5}\frac{1}{s^7} - \frac{24}{25}\frac{1}{s^6} + \frac{24}{125}\frac{1}{s^5} - \frac{24}{625}\frac{1}{s^4} + \frac{24}{3125}\frac{1}{s^3} + \frac{3101}{15625}\frac{1}{s^2} - \frac{3101}{78125}\frac{1}{s} + \frac{3101}{78125}\frac{1}{s+5}$,

$x(t) = \frac{1}{150}t^6 - \frac{1}{125}t^5 + \frac{1}{125}t^4 - \frac{4}{625}t^3 + \frac{12}{3125}t^2 + \frac{3101}{15625}t + \frac{3101}{78125}(e^{-5t}-1)$

(u) $x'''' - x = 1$, $x(0) = x'(0) = x''(0) = 0$, $x'''(0) = 4$. $s^4\bar{x} - s^3(0) - s^2(0) - s(0) - 4 - \bar{x} = 1/s$,
$\bar{x} = \frac{1}{(s^4-1)}\left(\frac{1}{s}+4\right)$. Let us expand $1/(s^4-1)$ in partial fractions as $\frac{1}{s^4-1} = \frac{1}{2}\frac{1}{s^2-1} - \frac{1}{2}\frac{1}{s^2+1}$
(NOTE: We could break it down further in the form
$\frac{1}{s^4-1} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+i} + \frac{D}{s-i}$, but the former will suffice, and is even more convenient because $1/(s^2-1)$ and $1/(s^2+1)$ are easily invertible.) $L^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$ and $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, so

$$x(t) = 2(\sinh t - \sin t) + 1 * \left(\tfrac{1}{2}\sinh t - \tfrac{1}{2}\sin t\right)$$
$$= 2(\sinh t - \sin t) + \frac{1}{2}\int_0^t (\sinh \tau - \sin \tau)\,d\tau = 2(\sinh t - \sin t) + \frac{1}{2}(\cosh t + \cos t - 2)$$

Or, using Maple,

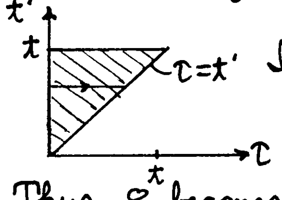
```
dsolve({diff(x(t),t,t,t,t)-x(t)=1, x(0)=0, D(x)(0)=0, D(D(x))=0, D(D(D(x)))=4}, x(t));
```

we obtain the (equivalent) form $x(t) = -1 + \frac{5}{4}e^{t} + \frac{1}{2}\cos t - 2\sin t - \frac{3}{4}e^{-t}$

2. (a) The left-hand side of the transformed ODE will be an algebraic function of s times $\bar{x}(s)$ plus an algebraic function of s, because $L\{x^{(n)}(t)\} = s^n\bar{x} - \underbrace{s^{n-1}x(0) - \cdots - x^{(n-1)}(0)}_{A(s)}$ is algebraic in s. Since $A(s)$ is of lower degree than the s^n multiplying $\bar{x}$, it follows that $P(s)$ in (2.1) will necessarily be of at least one degree less than $Q(s)$.

(b) By partial fractions, $X(s) = \frac{P(s)}{Q(s)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}$ where (see Appendix A, page 1264) $a_j = P(s_j)/Q'(s_j)$.
Inverting gives (2.2). NOTE: Here, $\bar{x}(s)$ and $X(s)$ are the same thing, $L\{x(t)\}$.

3. (a) $mx'' + kx = f(t)$; $x(0) = x_0$, $x'(0) = x_0'$. Integrating once gives
$mx'(t) - mx'(0) + k\int_0^t x(\tau)\,d\tau = \int_0^t f(\tau)\,d\tau$, and then again gives
$mx(t) - mx_0 - mx_0't + k\int_0^t\int_0^{t'} x(\tau)\,d\tau\,dt' = \int_0^t\int_0^{t'} f(\tau)\,d\tau\,dt'$. — ♀



Interchanging the order of integration, $\int_0^t\int_0^{t'} F(\tau)\,d\tau\,dt' = \int_0^t\int_\tau^t F(\tau)\,dt'\,d\tau$
$= \int_0^t F(\tau)(t-\tau)\,d\tau$

Thus, ♀ becomes

$$mx(t) - mx_0 - mx_0't + k\underbrace{\int_0^t x(\tau)(t-\tau)\,d\tau}_{x(t)*t} = \underbrace{\int_0^t f(\tau)(t-\tau)\,d\tau}_{f(t)*t}$$

(b) $m\bar{x} - mx_0\frac{1}{s} - mx_0'\frac{1}{s^2} + k(\bar{x})\left(\frac{1}{s^2}\right) = (\bar{f})\left(\frac{1}{s^2}\right)$, so $\bar{x} = \frac{mx_0 s + mx_0'}{ms^2+k} + \frac{\bar{f}}{ms^2+k}$ ✓

4. $mx'' + cx' + kx = f(t)$

$$mx' - mx_0' + cx - cx_0 + k\int_0^t x(\tau)\,d\tau = \int_0^t f(\tau)\,d\tau$$
$$mx - mx_0 - mx_0't + c\int_0^t x(\tau)\,d\tau - cx_0t + k\int_0^t\int_0^{t'} x(\tau)\,d\tau\,dt' = \int_0^t\int_0^{t'} f(\tau)\,d\tau\,dt'$$

$$mx(t) - mx_0 - (mx_0' + cx_0)t + \underbrace{\int_0^t [k(t-\tau) + c]\,x(\tau)\,d\tau}_{k\,t*x(t) + c*x(t)} = \underbrace{\int_0^t (t-\tau)\,f(\tau)\,d\tau}_{t*f(t)}$$

and so on.

5. $tx'' + x' + tx = 0$, $x(0) = 1$, $x'(0) = 0$. Laplace transforming gives

(a) $\int_0^\infty t x'' e^{-st}\,dt + s\bar{x} - x(0) + \int_0^\infty t x e^{-st}\,dt = 0$

or, $-\frac{d}{ds}\int_0^\infty x'' e^{-st}\,dt + s\bar{x} - 1 - \frac{d}{ds}\int_0^\infty x e^{-st}\,dt = 0$

$-\frac{d}{ds}[s^2\bar{x} - s\underbrace{x(0)}_{1} - \cancel{x'(0)}] + s\bar{x} - 1 - \frac{d}{ds}\bar{x} = 0$, $-s^2\frac{d\bar{x}}{ds} - 2s\bar{x} + 1 + s\bar{x} - 1 - \frac{d\bar{x}}{ds} = 0$,

$(s^2+1)\frac{d\bar{x}}{ds} + s\bar{x} = 0$, $\frac{d\bar{x}}{\bar{x}} + \frac{s\,ds}{s^2+1} = 0$, $\frac{d\bar{x}}{\bar{x}} + \frac{1}{2}\frac{2s\,ds}{s^2+1} = 0$, $\ln\bar{x} + \frac{1}{2}\ln(s^2+1) = \text{constant}$

$= \ln C$, say. Then $\ln\left(\bar{x}\sqrt{s^2+1}\right) = \ln C$ gives $\bar{x} = \frac{C}{\sqrt{s^2+1}}$

(b) $\bar{x} = \frac{C}{s(1+1/s^2)^{1/2}} = \frac{C}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$. Let $1/s^2 = r$. Taylor series gives

$$(1+r)^{-1/2} = 1 - \frac{1}{2}r + \frac{3}{8}r^2 - \frac{15}{48}r^3 + \frac{105}{384}r^4 - \cdots = 1 - \frac{1}{2}r + \frac{3}{8}r^2 - \frac{5}{16}r^3 + \frac{35}{128}r^4 - \cdots$$

so $\bar{x} = \frac{C}{s}\left(1 - \frac{1}{2}\frac{1}{s^2} + \frac{3}{8}\frac{1}{s^4} - \frac{5}{16}\frac{1}{s^6} + \frac{35}{128}\frac{1}{s^8} - \cdots\right) = C\left(\frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3}{8}\frac{1}{s^5} - \frac{5}{16}\frac{1}{s^7} + \frac{35}{128}\frac{1}{s^9} - \cdots\right)$

so $x(t) = C\left(1 - \frac{1}{2}\frac{1}{2!}t^2 + \frac{3}{8}\frac{1}{4!}t^4 - \frac{5}{16}\frac{1}{6!}t^6 + \cdots\right) = C\left(1 - \frac{1}{4}t^2 + \frac{1}{64}t^4 - \frac{1}{2304}t^6 + \cdots\right)$

and then $x(0) = 1 = C$ gives $C = 1$.

NOTE: You might wish to discuss the foregoing exercise in class. One important feature is the complication caused by the nonconstant coefficients in the ODE, factors of t causing d/ds's and hence leading to an ODE on X(s) rather than an algebraic one. The differentiations under the integral sign (with respect to s) actually involve the Leibniz rule, which is introduced in Chapter 13. Also, the idea of inverting a transform by expanding it in inverse powers of s and then inverting term by term is important. It is easy to come up with other such examples, such as:

ex) As a simple example, we know that $L^{-1}\{\frac{1}{s-a}\} = e^{at}$. Alternatively,

$$\frac{1}{s-a} = \frac{1}{s}\frac{1}{1-a/s} = \frac{1}{s}\left(1 + \frac{a}{s} + \left(\frac{a}{s}\right)^2 + \left(\frac{a}{s}\right)^3 + \cdots\right) = \frac{1}{s} + \frac{a}{s^2} + \frac{a^2}{s^3} + \cdots \rightarrow 1 + at + a^2t^2/2! + \cdots$$

ex) $L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$. Alternatively,

$$\frac{s}{s^2+a^2} = \frac{1}{s^2}\frac{s}{1+(a/s)^2} = \frac{1}{s}\left(1 - \left(\frac{a}{s}\right)^2 + \left(\frac{a}{s}\right)^4 - \cdots\right) = \frac{1}{s} - \frac{a^2}{s^3} + \frac{a^4}{s^5} - \cdots \rightarrow 1 - a^2t^2/2! + a^4t^4/4! - \cdots$$

ex) $L^{-1}\{\frac{1}{s+\sqrt{s}}\} = ?$ Maple is unable to determine this inverse. Yet, we can proceed as above:

$$\frac{1}{s+\sqrt{s}} = \frac{1}{s}\frac{1}{1+1/\sqrt{s}} = \frac{1}{s}\left(1 - \frac{1}{\sqrt{s}} + \frac{1}{s} - \frac{1}{s^{3/2}} + \cdots\right) = \frac{1}{s} - \frac{1}{s^{3/2}} + \frac{1}{s^2} - \cdots \rightarrow 1 - \frac{2}{\sqrt{\pi}}t^{1/2} + t - \cdots$$

ex) Finally, consider $L^{-1}\{\frac{1}{\sqrt{s-a}}\}$. This inverse can be obtained directly from entry 16 in Appendix C, or by using the entry $L\{\frac{1}{\sqrt{s}}\} = \frac{1}{\sqrt{\pi t}}$ and then the s-shift entry 29, obtaining $L^{-1}\{\frac{1}{\sqrt{s-a}}\} = \frac{1}{\sqrt{\pi t}}e^{at}$. Alternatively,

$$\frac{1}{\sqrt{s-a}} = \frac{1}{\sqrt{s}}\left(1 - \frac{a}{s}\right)^{-1/2} = \frac{1}{\sqrt{s}}\left(1 + \frac{1}{2}\frac{a}{s} + \frac{1\cdot 3}{2\cdot 4}\frac{a^2}{s^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{a^3}{s^3} + \cdots\right)$$

$$= \frac{1}{\sqrt{s}} + \frac{1}{2}\frac{a}{s^{3/2}} + \frac{1\cdot 3}{2\cdot 4}\frac{a^2}{s^{5/2}} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{a^3}{s^{7/2}} + \cdots$$

$$\rightarrow \frac{t^{-1/2}}{\Gamma(1/2)} + \frac{a}{2}\frac{t^{1/2}}{\Gamma(3/2)} + \frac{3a^2}{8}\frac{t^{3/2}}{\Gamma(5/2)} + \frac{5a^3}{16}\frac{t^{5/2}}{\Gamma(7/2)} + \cdots = \frac{1}{\sqrt{\pi t}}\left(1 + at + \frac{a^2t^2}{2!} + \cdots\right) = \frac{1}{\sqrt{\pi t}}e^{at}.$$

6. $C(T)=\int_0^\infty e^{-0.0744v^2/T^2}\rho(v)\,dv$. Set $0.0744=a$, for brevity, and set $av^2=t$ and $1/T^2=s$, so $C(\frac{1}{\sqrt{s}})=\int_0^\infty e^{-st}\rho(\sqrt{t/a})\frac{1}{2\sqrt{a}}t^{-1/2}\,dt$.

(a) If $C(T)=T$, then $C(\frac{1}{\sqrt{s}})=\frac{1}{\sqrt{s}}$ so $L\{\frac{1}{2\sqrt{at}}\rho(\sqrt{t/a})\}=\frac{1}{\sqrt{s}}$ so $\frac{1}{2\sqrt{at}}\rho(\sqrt{t/a})=L^{-1}\{\frac{1}{\sqrt{s}}\}=\frac{1}{\sqrt{\pi t}}$. Solving the latter for ρ, $\rho(t)=\text{const.}=2\sqrt{\frac{a}{\pi}}$

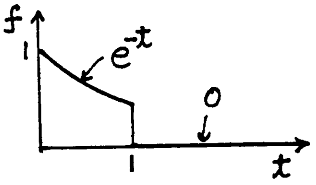
(b) If $C(T)=Te^{-1/T}$, then $C(\frac{1}{\sqrt{s}})=\frac{1}{\sqrt{s}}e^{-\sqrt{s}}$ so $\frac{1}{2\sqrt{at}}\rho(\sqrt{t/a})=L^{-1}\{\frac{1}{\sqrt{s}}e^{-\sqrt{s}}\}=\frac{1}{\sqrt{\pi}}\frac{e^{-1/(4t)}}{\sqrt{t}}$.
Thus, $\rho(\sqrt{t/a})=2\sqrt{a/\pi}\,e^{-1/(4t)}$. Setting $\sqrt{t/a}=\tau$,
$\rho(\tau)=2\sqrt{a/\pi}\,e^{-1/(4a\tau^2)}$ or, $\rho(v)=2\sqrt{a/\pi}\,e^{-1/(4av^2)}$.

7. We want to choose the kernel $K(t,s)$ in $F(s)=\int_0^\infty K(t,s)f(t)\,dt$ so that $L\{f'(t)\}=a\bar{f}+b$ (i.e., a multiple of $\bar{f}$ plus a constant). Well,
$L\{f'(t)\}=\int_0^\infty \underbrace{K(t,s)}_{u}\underbrace{f'(t)dt}_{dv}=K(t,s)f(t)\Big|_0^\infty-\int_0^\infty\frac{\partial K}{\partial t}(t,s)f(t)\,dt$, so require of K that $-\frac{\partial K}{\partial t}(t,s)=$ a multiple of $K=sK$, say. Integrating $-\frac{\partial K}{\partial t}=sK$ gives $K(t,s)=e^{-st}$. In that case, $L\{f'(t)\}=e^{-st}f(t)\Big|_0^\infty+s\int_0^\infty K(t,s)f(t)\,dt$
$=-f(0)+s\bar{f}(s)$,

which is the desired form.

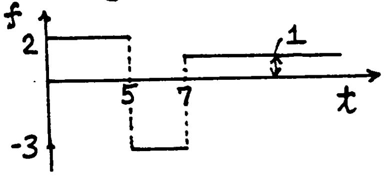
Section 5.5

1. (b) $f(t)=e^{-t}[\overset{\text{or } H(t)}{1}-H(t-1)]=e^{-t}-e^{-t}H(t-1)=e^{-t}-e^{-1}e^{-(t-1)}H(t-1)$
so $F(s)=\frac{1}{s+1}-e^{-1}L\{e^{-(t-1)}H(t-1)\}$
↑use entry 30, where $f(t)=e^{-t}$ and $a=1$
$=\frac{1}{s+1}-e^{-1}e^{-s}\frac{1}{s+1}=\frac{1-e^{-(s+1)}}{s+1}$

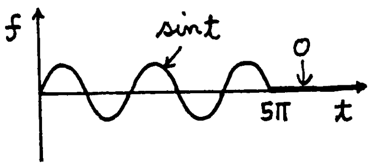


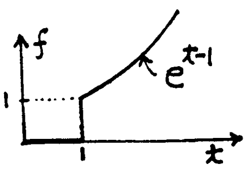
Of course we can evaluate $F(s)$ other ways too. For example, by definition
$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^1 e^{-t}e^{-st}dt=\int_0^1 e^{-(s+1)t}dt=[1-e^{-(s+1)}]/(s+1)$. ✓

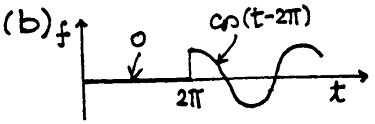
(c) $f(t)=2[H(t)-H(t-5)]-3[H(t-5)-H(t-7)]+1H(t-7)$
so $F(s)=2(\frac{1}{s}-\frac{e^{-5s}}{s})-3(\frac{e^{-5s}}{s}-\frac{e^{-7s}}{s})+\frac{e^{-7s}}{s}$
$=2\frac{1}{s}-5\frac{e^{-5s}}{s}+4\frac{e^{-7s}}{s}$

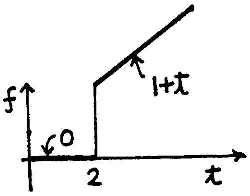


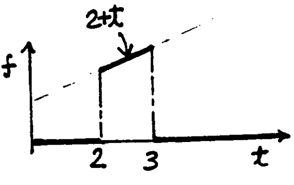
(g) $f(t)=\sin t\,[\underset{\uparrow\text{or }H(t)}{1}-H(t-5\pi)]=\sin t-\sin[(t-5\pi)+5\pi]H(t-5\pi)$
$=\sin t+\sin(t-5\pi)H(t-5\pi)$,
since $\sin(A+B)=\sin A\cos B+\sin B\cos A$. Thus, using entry 30 with $f(t)=\sin t$ and $a=5\pi$,
$F(s)=\frac{1}{s^2+1}+e^{-5\pi s}\frac{1}{s^2+1}=\frac{1+e^{-5\pi s}}{s^2+1}$



2. (a)


(b)


(c)


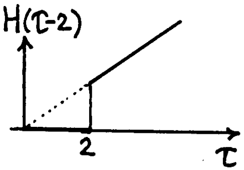
(d)


3. (b) First, observe that $\int_0^t \tau H(\tau)d\tau = \begin{cases} 0, & t<0 \\ t^2/2, & t>0 \end{cases} = \frac{t^2}{2}H(t)$, and also that $\int^t \tau H(\tau)d\tau = \frac{t^2}{2}H(t) + \text{constant}$. Now, letting $\tau - 2 = \mu$,

$$\int_0^t \tau H(\tau-2)d\tau = \int_{-2}^{t-2}(2+\mu)H(\mu)d\mu = (2\mu + \frac{\mu^2}{2})H(\mu)\Big|_{-2}^{t-2} =$$

$$[2(t-2)+\tfrac{1}{2}(t-2)^2]H(t-2) - (\text{etc})\overset{0}{\cancel{H(-2)}} = (\tfrac{1}{2}t^2-2)H(t-2).$$

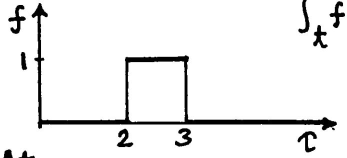
Or, proceeding more simply, referring to the graph of the integrand $\tau H(\tau-2)$ shown at the right,

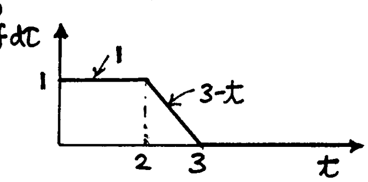


$$\int_0^t \tau H(\tau-2)d\tau = \begin{cases} 0, & t<2 \\ \int_2^t \tau d\tau, & t>2 \end{cases} = \begin{cases} 0, & t<2 \\ \frac{1}{2}t^2-2, & t>0 \end{cases} = (\tfrac{1}{2}t^2-2)H(t-2).$$

(c) $\int_0^t [1-H(\tau-5)]d\tau = t - \int_0^t H(\tau-5)d\tau = t - \int_{-5}^{t-5} H(\mu)d\mu = t - \mu H(\mu)\Big|_{\mu=-5}^{\mu=t-5} = t-(t-5)H(t-5)$

(e) The simplest way to do this one is by inspection of the graph of the integrand $f(\tau) = H(\tau-2) - H(\tau-3)$.





We have, from the graph at the right,

$$\int_t^\infty [H(\tau-2)-H(\tau-3)]d\tau = 1[1-H(t-2)] + (3-t)[H(t-2)-H(t-3)] = 1+(t-3)H(t-3)-(t-2)H(t-2)$$

(h) $t * H(t-1) = \int_0^t H(\tau-1)(t-\tau)d\tau = t\int_0^t H(\tau-1)d\tau - \int_0^t \tau H(\tau-1)d\tau$

$$= t\int_{-1}^{t-1} H(\mu)d\mu - \int_{-1}^{t-1}(1+\mu)H(\mu)d\mu = t\,\mu H(\mu)\Big|_{-1}^{t-1} - (\mu+\frac{\mu^2}{2})H(\mu)\Big|_{-1}^{t-1}$$

$$= t(t-1)H(t-1) - [t-1+\frac{(t-1)^2}{2}]H(t-1) = \frac{(t-1)^2}{2}H(t-1)$$

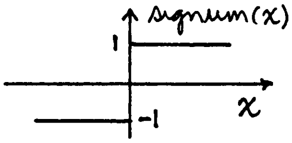
4. (b) int(x*Heaviside(x-2), x=0..t);

gives $\frac{1}{2}t^2-2$, which is INCORRECT; it should be that times $H(t-2)$. ?

(c) int(1- Heaviside(x-5), x=0..t);

gives $\frac{1}{2}t - \frac{1}{2}t\,\text{signum}(t-5) + \frac{5}{2}\text{signum}(t-5) + \frac{5}{2}$ (call this ♀)

Now, signum(x) means the "sign of the number" x: $\text{signum}(x) \equiv \begin{cases} +1, & x>0 \\ -1, & x<0 \end{cases}$, as sketched at the right. Thus we can see that



$$\boxed{\text{signum}(x) = 2H(x)-1}$$

Thus,

♀ $= \frac{1}{2}\{t+5-t[2H(t-5)-1]+5[2H(t-5)-1]\} = t-(t-5)H(t-5)$, as found in 3(c). ✓

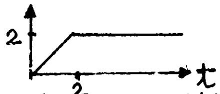
(e) int(Heaviside(x-2) - Heaviside(x-3), x=t..infinity);

gives $\frac{1}{2} - \frac{1}{2}t\,\text{signum}(t-2) - \frac{3}{2}\text{signum}(t-3) + \text{signum}(t-2) + \frac{1}{2}t\,\text{signum}(t-3)$
$= 1+(t-3)H(t-3)-(t-2)H(t-2)$, as in 3(e). ✓

5. (b) $x' - x = e^{-t}H(t-3)$, $x(0)=0$.

$$s\bar{x} - 0 - \bar{x} = \int_0^\infty e^{-st}e^{-t}H(t-3)dt = \int_3^\infty e^{-(s+1)t}dt = e^{-3(s+1)}/(s+1)$$

so $\bar{x} = e^{3}e^{-3s}/(s^2-1)$. $x(t) = e^{-3}H(t-3)\sinh(t-3)$, from entries 5 (with $a=1$) and 30 (with $f(t)=\sinh t$ and $a=3$).

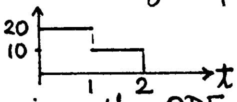
(c) $f(t) =$ 

$= t[1-H(t-2)]+2H(t-2) = t-(t-2)H(t-2)$

so $x'-x = t-(t-2)H(t-2)$. $s\bar{x}-\bar{x} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$, $\bar{x} = \frac{1}{s^2(s-1)} - \frac{e^{-2s}}{s^2(s-1)}$.

First, $\frac{1}{s^2(s-1)} = \frac{1}{s^2}\frac{1}{s-1} \to t * e^t = \int_0^t \tau e^{t-\tau}d\tau = e^t-t-1$. With this result and entry 30, $\frac{e^{-2s}}{s^2(s-1)} \to (e^{t-2}-(t-2)-1)H(t-2)$, so $x(t) = e^t-t-1-[e^{t-2}-t+1]H(t-2)$.

(e) $x'-x = f(t)$, $x(0)=0$. $s\bar{x}-\bar{x} = \int_0^\infty f(t)\,e^{-st}dt = 0$ because $f(t)=0$ everywhere except at a single point (where $t=200$). Thus, $\bar{x}=0$, $x(t)=0$.

(j) $f(t) =$ 

$= 20[1-H(t-1)]+10[H(t-1)-H(t-2)] = 20-10H(t-1)-10H(t-2)$

Transforming the ODE gives $s\bar{x}-\bar{x} = \frac{20}{s} - 10\frac{e^{-s}}{s} - 10\frac{e^{-2s}}{s}$

so $\bar{x} = 20\frac{1}{s(s-1)} - 10\frac{1}{s-1}\frac{e^{-s}}{s} - 10\frac{1}{s-1}\frac{e^{-2s}}{s}$,

$$\begin{aligned} x(t) &= 20\, 1 * e^t - 10\, e^t * H(t-1) - 10\, e^t * H(t-2) \\ &= 20\int_0^t e^\tau d\tau - 10\int_0^t H(\tau-1)e^{t-\tau}d\tau - 10\int_0^t H(\tau-2)e^{t-\tau}d\tau & ① \\ &= 20(e^t-1) - 10H(t-1)\int_1^t e^{t-\tau}d\tau - 10H(t-2)\int_2^t e^{t-\tau}d\tau & ② \\ &= 20(e^t-1) - 10H(t-1)(e^{t-1}-1) - 10H(t-2)(e^{t-2}-1) \end{aligned}$$

NOTE: Let us review some of the steps. Transforming the ODE gave

$$(s-1)\bar{x} = \bar{f}$$

so

$$\bar{x} = \frac{1}{s-1}\bar{f}(s)$$

and the convolution theorem gives $x(t) = e^t * f(t) = e^t * [20-10H(t-1)-10H(t-2)] =$ etc. The point that we wish to emphasize is that if we invert $\frac{1}{s-1}\bar{f}(s)$ by the convolution theorem, as we did, then it is wasteful to take the transform of $f(t)$ because we end up convolving the inverse of $1/(s-1)$ with the inverse of $\bar{f}(s)$, which is merely the original function $f(t)$. Thus, the above solution was indeed wasteful in that we started with the expression $20-10H(t-1)-10H(t-2)$ for $f(t)$, then transformed it, and then inverted the transform, ending up with the original function $f(t)$! This point was addressed in the last paragraph on page 263.

NOTE: How do we justify the step (going from line ① to line ②) $\int_0^t H(\tau-1)\,e^{t-\tau}d\tau = H(t-1)\int_1^t e^{t-\tau}d\tau$? More generally,

$$\boxed{\int_0^t H(\tau-a)f(\tau)d\tau = H(t-a)\int_a^t f(\tau)d\tau} \qquad (a>0, t>0)$$

because the $H(\tau-a)$ factor in the integrand is 0 until $\tau=a$. Thus, if $t<a$ then the integral gets "turned off" before the integrand gets "turned on", and if $t>a$ then we can change the lower limit to a because the integrand is 0 for $t<a$.

6. (b)

```
dsolve({diff(x(t),t)-x(t)= exp(-t)*Heaviside(t-3), x(0)=0}, x(t));
```

gives the result $x(t) = e^t\int_0^t \frac{1}{(e^u)^2}du\int_0^t \text{Heaviside}(u-3)du$.

That is, dsolve does not recognize the Heaviside function.

NOTE: For the Maple dsolve command to cope with the Heaviside function (and also with the Dirac delta function introduced in the next section), use the option **method=laplace.**

Thus, dsolve({diff(x(t),t) - x(t) = exp(-t) * Heaviside(t-3), x(0)=0},
x(t), method = laplace);
gives the same answer as obtained above in Exercise 5(b).

7. (a) $x'' - x = H(t-1)$, $x(0) = x'(0) = 0$. Call $H(t-1) = f(t)$

$$s^2\bar{x} - \bar{x} = \bar{f}, \quad \bar{x} = \underset{\underset{\sinh t}{\downarrow}}{\frac{1}{s^2-1}}\ \underset{\underset{f(t)}{\downarrow}}{\bar{f}(s)} \quad \text{so } x(t) = \sinh t * f(t) = \sinh t * H(t-1)$$

$$= \int_0^t H(\tau-1)\sinh(t-\tau)\,d\tau$$

$$= H(t-1)\int_1^t \sinh(t-\tau)\,d\tau \quad \text{(by boxed}$$

$$= H(t-1)\int_1^t -\sinh(\tau-t)\,d\tau \quad \text{formula, p66}$$

$$= -H(t-1)\int_{1-t}^0 \sinh\mu\,d\mu \quad \text{of these solutions)}$$

so $x(t) = H(t-1)[\cosh(t-1) - 1]$ NOTE: $\cosh(-x) = \cosh x$.

(b) $$s^2\bar{x} - \bar{x} = \bar{f}, \quad \bar{x} = \underset{\underset{\sinh t}{\downarrow}}{\frac{1}{s^2-1}}\ \underset{\underset{f(t)}{\downarrow}}{\bar{f}(s)} \quad \text{so } x(t) = \sinh t * f(t) = \sinh t * e^{-t}H(t-3)$$

$$= \int_0^t e^{-\tau}H(\tau-3)\sinh(t-\tau)\,d\tau$$

$$= H(t-3)\int_3^t e^{-\tau}\sinh(t-\tau)\,d\tau$$

$$= \tfrac{1}{2}H(t-3)\int_3^t \left(e^{t-2\tau} + e^{-t}\right)d\tau$$

so $x(t) = \frac{1}{4}H(t-3)\left[(5-2t)e^{-t} + e^{t-6}\right]$.

Section 5.6

1. (b) $x'' - 4x = 6\delta(t-1)$, $x(0) = 0$, $x'(0) = -3$. Call $6\delta(t-1) = f(t)$.

$$s^2\bar{x} + 3 - 4\bar{x} = \bar{f}(s), \quad \bar{x} = -\frac{3}{s^2-4} + \underset{\underset{\frac{1}{2}\sinh 2t}{\downarrow}}{\frac{1}{s^2-4}}\ \underset{\underset{f(t)=6\delta(t-1)}{\searrow}}{\bar{f}(s)}$$

$$\text{so } x(t) = -\tfrac{3}{2}\sinh 2t + \tfrac{1}{2}\sinh 2t * f(t) = -\tfrac{3}{2}\sinh 2t + \underbrace{\tfrac{1}{2}\int_0^t 6\delta(\tau-1)\sinh 2(t-\tau)\,d\tau}_{\text{Gives 0 if } t<1 \text{ and gives } 3\sinh 2(t-1) \text{ if } t>1}$$

Thus,

$$x(t) = -\tfrac{3}{2}\sinh 2t + 3H(t-1)\sinh 2(t-1)$$

Check by Maple:
dsolve({diff(x(t),t,t) - 4*x(t) = 6*Dirac(t-1), x(0)=0, D(x)(0)=-3},
x(t), method = laplace);
does give the same answer as above.

(c) $x'' - 3x' + 2x = 2 + \delta(t-5)$, $x(0) = x'(0) = 0$

$$(s^2-3s+2)\bar{x} = \bar{f}(s), \quad \bar{x} = \frac{1}{s^2-3s+2}\bar{f}(s) \qquad \frac{1}{s^2-3s+2} = \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\text{so } x(t) = (e^{2t} - e^t) * [2 + \delta(t-5)] = \int_0^t (e^{2\tau} - e^\tau)2\,d\tau + \int_0^t \delta(\tau-5)\left[e^{2(t-\tau)} - e^{t-\tau}\right]d\tau$$

$$= e^{2t} - 2e^t + 1 + \begin{cases} 0, & t<5 \\ e^{2(t-5)} - e^{t-5}, & t>5 \end{cases} \quad \text{[according to (13)]}$$

$$= e^{2t} - 2e^t + 1 + H(t-5)(e^{2t-10} - e^{t-5}).$$ Note that the first two terms

are from the homogeneous solution $C_1e^{2t}+C_2e^{t}$, the third is a particular solution corresponding to the forcing term 2, and the Heaviside term is a particular solution corresponding to the forcing term $\delta(t-5)$.

NOTE: In case the steps in the integration of the delta function are unclear, they follow from (13) on pg 278 and are like those in (16) in Example 2. Nevertheless, perhaps it would be best to state the result in general, for reference in the remainder of these exercises, namely,

$$\boxed{\int_0^t \delta(\tau-a)f(\tau)\,d\tau = H(t-a)f(a)} \qquad (a>0, t>0)$$

(f) $2x''-x' = \delta(t-1)-\delta(t-2)$, $x(0)=x'(0)=0$

$(2s^2-s)\bar{x} = \bar{f}$ so $\bar{x} = \frac{1}{2s^2-s}\bar{f}(s)$. $\frac{1}{2s^2-s} = -\frac{1}{s}+\frac{1}{s-1/2} \rightarrow -1+e^{t/2}$

so $x(t) = (-1+e^{t/2}) * [\delta(t-1)-\delta(t-2)] = \int_0^t [\delta(\tau-1)-\delta(\tau-2)](-1+e^{(t-\tau)/2})\,d\tau$

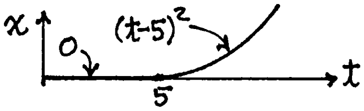
$= H(t-1)[-1+e^{(t-1)/2}]-H(t-2)[-1+e^{(t-2)/2}]$

(h) $x''' = 2\delta(t-5)$, $x(0)=x'(0)=x''(0)=0$. In this case we can solve by quadrature (i.e., by direct integration): $\int_0^t (\quad)\,dt$ gives

$x''(t)-\cancel{x''(0)} = 2\int_0^t \delta(\tau-5)\,d\tau = 2H(t-5)$

$x'(t)-\cancel{x'(0)} = 2\int_0^t H(\tau-5)\,d\tau = 2(t-5)H(t-5)$

$x(t)-\cancel{x(0)} = 2\int_0^t (\tau-5)H(\tau-5)\,d\tau = (t-5)^2H(t-5)$ so $x(t) = (t-5)^2H(t-5)$



(j) $x''''-4x'' = 3\delta(t-1)$, $x(0)=x'(0)=x''(0)=0$, $x'''(0)=1$

$(s^4\bar{x}-1)-4(s^2\bar{x}) = \bar{f}$, $\bar{x} = \frac{1}{s^4-4s^2}+\frac{1}{s^4-4s^2}\bar{f}$

First, $\frac{1}{s^4-4s^2} = \frac{1}{s^2}\frac{1}{s^2-4} \rightarrow t*\frac{1}{2}\sinh 2t = \frac{1}{2}\int_0^t (t-\tau)\sinh 2\tau\,d\tau = -\frac{t}{4}+\frac{\sinh 2t}{8}$

so $x(t) = -\frac{1}{4}t+\frac{1}{8}\sinh 2t + (-\frac{1}{4}t+\frac{1}{8}\sinh 2t)*[3\delta(t-1)]$

$= -\frac{1}{4}t+\frac{1}{8}\sinh 2t+\frac{3}{8}\int_0^t \delta(\tau-1)[\sinh 2(t-\tau)-2(t-\tau)]\,d\tau$

$= -\frac{1}{4}t+\frac{1}{8}\sinh 2t+\frac{3}{8}H(t-1)[\sinh 2(t-1)-2(t-1)]$

(l) $x''''-x = \delta(t-1)$, $x(0)=x'(0)=x''(0)=x'''(0)=0$

$(s^4-1)\bar{x} = \bar{f}$, $\bar{x} = \frac{1}{s^4-1}\bar{f}$.

First, $\frac{1}{s^4-1} = \frac{1}{s^2-1}\frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s^2-1}-\frac{1}{s^2+1}\right) \rightarrow \frac{1}{2}(\sinh t-\sin t)$, so

$x(t) = \frac{1}{2}(\sinh t-\sin t)*\delta(t-1) = \frac{1}{2}\int_0^t \delta(\tau-1)[\sinh(t-\tau)-\sin(t-\tau)]\,d\tau$

$= \frac{1}{2}H(t-1)[\sinh(t-1)-\sin(t-1)]$.

2. (a) Is $\int_{-\infty}^{\infty} g(t)\,\delta(-t)\,dt = \int_{-\infty}^{\infty} g(t)\,\delta(t)\,dt$?

$\int_{\infty}^{-\infty} g(-\tau)\,\delta(\tau)(-d\tau) = g(0)$?

$\int_{-\infty}^{\infty} g(-\tau)\,\delta(\tau)\,d\tau = g(0)$?

$g(0) = g(0)$ ✓

3. NOTE: You might consider discussing this exercise in class. The properties of the delta and Heaviside functions that we will use are these:

$$H'(t) = \delta(t) \quad or, \quad H'(t-a) = \delta(t-a),$$

and

$$f(t)\delta(t-a) = \begin{cases} f(a)\delta(t-a), & if \quad f(a) \neq 0 \\ 0, & if \quad f(a) = 0. \end{cases}$$

(a) Problem: Verify the solution $x(t) = H(t-2)\sinh(t-2)$ of $x'' - x = \delta(t-2)$, $x(0) = x'(0) = 0$.

$x(t) = H(t-2)\sinh(t-2)$ so $x(0) = H(-2)\sinh(-2) = 0$ because $H(-2) = 0$ ✓

$x'(t) = H'(t-2)\sinh(t-2) + H(t-2)\cosh(t-2) = \delta(t-2)\sinh(t-2) + H(t-2)\cosh(t-2)$

$= \cancel{0\,\delta(t-2)} + H(t-2)\cosh(t-2)$

$= H(t-2)\cosh(t-2)$ so $x'(0) = H(-2)\cosh(-2) = 0$ ✓

$x''(t) = H'(t-2)\cosh(t-2) + H(t-2)\sinh(t-2) = \delta(t-2)\cosh(t-2) + H(t-2)\sinh(t-2)$

$= \delta(t-2) + H(t-2)\sinh(t-2)$

so $x'' - x = \delta(t-2) + \cancel{H(t-2)\sinh}(t-2) - \cancel{H(t-2)\sinh}(t-2) = \delta(t-2)$ ✓

(d) Problem: Verify the solution $x(t) = 2 + t - 2e^{-t} + H(t-2)(1 - e^{-(t-2)})$ of $x'' + x' = 1 + \delta(t-2)$, $x(0) = 0$, $x'(0) = 3$.

$x(t) = 2 + t - 2e^{-t} + H(t-2)(1 - e^{-(t-2)})$ so $x(0) = 2 + 0 - 2 + 0 = 0$ ✓

$x'(t) = 1 + 2e^{-t} + \delta(t-2)(1 - e^{-(t-2)}) + H(t-2)e^{-(t-2)}$

$= 1 + 2e^{-t} + H(t-2)e^{-(t-2)}$ so $x'(0) = 1 + 2 + 0 = 3$ ✓

$x''(t) = -2e^{-t} + \delta(t-2)e^{-(t-2)} + H(t-2)(-e^{-(t-2)})$

$= -2e^{-t} + \delta(t-2) - H(t-2)e^{-(t-2)}$

so $x'' + x' = \cancel{-2e}^{-t} + \delta(t-2) - \cancel{H(t-2)e}^{-(t-2)} + 1 + \cancel{2e}^{-t} + \cancel{H(t-2)e}^{-(t-2)} = 1 + \delta(t-2)$ ✓

(g) Problem: Verify the solution $x(t) = 8e^{t} - 4e^{2t} + 100\,H(t-3)(e^{2t-6} - e^{t-3})$ of $x'' - 3x' + 2x = 100\,\delta(t-3)$, $x(0) = 4$, $x'(0) = 0$.

$x(t) = 8e^{t} - 4e^{2t} + 100H(t-3)(e^{2t-6} - e^{t-3})$ so $x(0) = 8 - 4 + 0 = 4$ ✓

$x'(t) = 8e^{t} - 8e^{2t} + 100\,\delta(t-3)(e^{2t-6} - e^{t-3}) + 100H(t-3)(2e^{2t-6} - e^{t-3})$

$= 8e^{t} - 8e^{2t} + 100H(t-3)(2e^{2t-6} - e^{t-3})$ so $x'(0) = 8 - 8 + 0 = 0$ ✓

$x''(t) = 8e^{t} - 16e^{2t} + 100\,\delta(t-3)(2e^{2t-6} - e^{t-3}) + 100H(t-3)(4e^{2t-6} - e^{t-3})$

$= 8e^{t} - 16e^{2t} + 100\,\delta(t-3) + 100H(t-3)(4e^{2t-6} - e^{t-3})$

so $x'' - 3x' + 2x = \cancel{8e^{t}} - \cancel{16e}^{2t} + 100\,\delta(t-3) + 100H(t-3)(4e^{2t-6} - e^{t-3})$

$\cancel{-24e^{t}} + \cancel{24e}^{2t} - 300H(t-3)(2e^{2t-6} - e^{t-3})$

$\cancel{+16e^{t}} - \cancel{8e}^{2t} + 200H(t-3)(e^{2t-6} - e^{t-3}) = 100\,\delta(t-3)$ ✓

Section 5.7

1. (b) $1/(s^2 - a^2) \rightarrow \frac{1}{a}\sinh at$ so $1/(s^2 - a^2)^2 \rightarrow \frac{1}{a}\sinh at * \frac{1}{a}\sinh at$

$= \frac{1}{a^2}\int_0^t \sinh a\tau \sinh a(t-\tau)\,d\tau = \frac{1}{2a^2} t\cosh at - \frac{1}{2a^3}\sinh at$. NOTE: As you have no doubt surmised, I use the simple → notation in these solutions, to denote either Laplace transform or Laplace inverse.

There is often more than one way to work out these inversions and you may very well have a better way in mind. Another method that comes to mind in the present example is to use the fact that, to within a factor of $-2s$, the given transform is the s-derivative of the simpler transform $1/(s^2-a^2)$. That is,

$$1/(s^2-a^2) \to \frac{1}{a}\sinh at$$

so (Theorem 5.7.4) $\frac{d}{ds}\left(\frac{1}{s^2-a^2}\right) = -\frac{2s}{(s^2-a^2)^2} \to -t\frac{1}{a}\sinh at,$

$$\frac{1}{(s^2-a^2)^2} = -\frac{1}{2s}\,\frac{-2s}{(s^2-a^2)^2} \to -\frac{1}{2} * \left(-\frac{t}{a}\sinh at\right)$$

$$= \frac{1}{2a}\int_0^t \tau \sinh a\tau \, d\tau = -\frac{1}{2a^3}\sinh at + \frac{t}{2a^2}\cosh at, \text{ as before.}$$

(c) $\frac{s}{(s-2)^2} = \frac{(s-2)+2}{(s-2)^2} = \frac{1}{s-2} + \frac{2}{(s-2)^2} \to e^{2t} + 2e^{2t}t = (1+2t)e^{2t}$

That is, $\frac{2}{s^2} \to 2t$ so, by s-shift, $\frac{2}{(s-2)^2} \to 2e^{2t}t$.

(e) $1/\sqrt{s} \to 1/\sqrt{\pi t}$ so, by s-shift, $1/\sqrt{s+1} \to e^{-t}/\sqrt{\pi t}$

(f) $\frac{s}{(s-a)^{3/2}} = \frac{(s-a)+a}{(s-a)^{3/2}} = \frac{1}{\sqrt{s-a}} + \frac{a}{(s-a)^{3/2}} \to e^{at}\left(\frac{1}{\sqrt{\pi t}} + 2\sqrt{\frac{t}{\pi}}\right)$

because $1/\sqrt{s} \to 1/\sqrt{\pi t}$ so (by s-shift) $1/\sqrt{s-a} \to e^{at}/\sqrt{\pi t}$
and $1/s^{3/2} \to 2\sqrt{t/\pi}$ so (" ") $1/(s-a)^{3/2} \to 2e^{at}\sqrt{t/\pi}$

(i) $1/s^2 \to t$ so (s-shift) $1/(s+1)^2 \to te^{-t}$.
Then (by t-shift) $e^{-2s}/(s+1)^2 \to H(t-2)(t-2)e^{-(t-2)}$.

(ℓ) $L\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt = \ln\left(1-\frac{a^2}{s^2}\right)$ and d/ds gives

$$\int_0^\infty -t f(t) e^{-st} dt = \frac{2a^2}{s}\frac{1}{s^2-a^2} \to 2a^2 1 * \frac{1}{a}\sinh at = 2a\int_0^t \sinh a\tau \, d\tau$$

so $-t\, f(t) = 2(\cosh at - 1)$

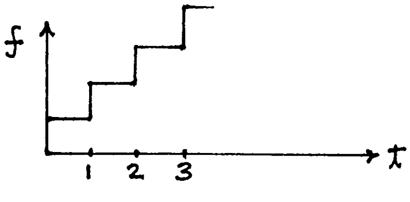
hence, $f(t) = \frac{2}{t}(1-\cosh at)$

(n) $s^2+2s-4=0$ gives $s=-1\pm\sqrt{5} \equiv s_\pm$. Then $\frac{1}{(s-s_+)(s-s_-)} = \frac{1}{2\sqrt{5}}\left(\frac{1}{s-s_+} - \frac{1}{s-s_-}\right)$

$$\to \frac{1}{2\sqrt{5}}\left(e^{s_+t} - e^{+s_-t}\right) = \frac{1}{2\sqrt{5}}\left(e^{(-1+\sqrt{5})t} - e^{(-1-\sqrt{5})t}\right) = \frac{1}{\sqrt{5}}e^{-t}\sinh\sqrt{5}t.$$

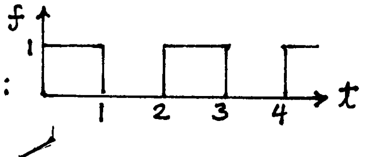
Then $\frac{e^{-3s}}{(s-s_+)(s-s_-)} \to \frac{1}{\sqrt{5}}H(t-3)\, e^{-(t-3)}\sinh\sqrt{5}(t-3)$.

(q) $\frac{1}{s(1-e^{-s})} = \frac{1}{s}(1+e^{-s}+e^{-2s}+e^{-3s}+\cdots)$
$= \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots$
$\to 1 + H(t-1) + H(t-2) + H(t-3) + \cdots$

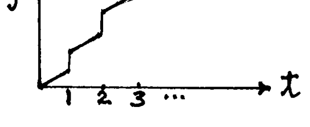


(r) $\frac{1}{s(1+e^{-s})} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \cdots$

$\to 1 - H(t-1) + H(t-2) - H(t-3) + \cdots$ is a "square wave":



(t) $\frac{1}{s^2(1-e^{-s})} = \frac{1}{s^2} + \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \cdots \to 1*1 + 1*H(t-1) + 1*H(t-2) + \cdots$
$= t + H(t-1)(t-1) + H(t-2)(t-2) + \cdots$



(u) $\frac{1}{(s+1)(1-e^{-2s})} = \frac{1}{s+1} + \frac{e^{-2s}}{s+1} + \frac{e^{-4s}}{s+1} + \cdots \to e^{-t} + H(t-2)e^{-(t-2)} + H(t-4)e^{-(t-4)} + \cdots$

(v) $e^{-s}/(s-8) \to H(t-1)e^{8(t-1)}$

(w) $L\{J_0(t)\} = \int_0^\infty J_0(t)\, e^{-st}\, dt = 1/\sqrt{s^2+1}$
d/ds gives $\int_0^\infty -tJ_0(t)\, e^{-st}\, dt = -s/(s^2+1)^{3/2}$ so $s/(s^2+1)^{3/2} \to tJ_0(t)$

(x) $\frac{1}{s(1-e^{s})} = -\frac{e^{-s}}{s(1-e^{-s})} = -\frac{e^{-s}}{s}(1+e^{-s}+e^{-2s}+\cdots) = -\frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \cdots$

$\to -H(t-1) - H(t-2) - H(t-3) - \cdots$

2. The invlaplace command gives the inverse in most, but not all, cases. For example, it does not give any result in cases (q), (r), (t), (u), (x).

3. (a) sin t is periodic with period 2π, so (37) gives

$$L\{\sin t\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} \sin t\, e^{-st}\, dt = \frac{1}{1-e^{-2\pi s}}\,\frac{1-e^{-2\pi s}}{s^2+1} = \frac{1}{s^2+1} \quad \checkmark$$

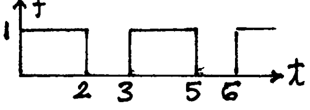
4. (a) $L\{f(at)\} = \int_0^\infty f(at)\, e^{-st}\, dt \overset{at=\mu}{=} \int_0^\infty f(\mu)\, e^{-s\mu/a}\, \frac{d\mu}{a} = \frac{1}{a}F(\frac{s}{a})$

(b) $L^{-1}\{F(as)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(as)\, e^{st}\, ds \overset{as=z}{=} \frac{1}{2\pi i}\int_{a\gamma-i\infty}^{a\gamma+i\infty} F(z)\, e^{zt/a}\, \frac{dz}{a} = \frac{1}{a}f(\frac{t}{a}).$

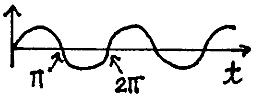
The foregoing may be too hard for the student. Equivalently, we can show that $L\{\frac{1}{a}f(\frac{t}{a})\} \overset{?}{=} F(as)$: $L\{\frac{1}{a}f(\frac{t}{a})\} = \int_0^\infty \frac{1}{a}f(\frac{t}{a})\, e^{-st}\, dt \underset{t/a=\mu}{=} \int_0^\infty \frac{1}{a}f(\mu)\, e^{-sa\mu}\, a\, d\mu = F(as). \checkmark$

This approach stays away from the more sophisticated inversion formula.

5. (b) $\bar{f}(s) = \frac{1}{1-e^{-3s}}\int_0^3 f(t)e^{-st}\, dt = \frac{1}{1-e^{-3s}}\int_0^2 e^{-st}\, dt = \frac{1}{s}\,\frac{1-e^{-2s}}{1-e^{-3s}}$



(c) $\bar{f}(s) = \frac{1}{1-e^{-\pi s}}\int_0^\pi \sin 2t\, e^{-st}\, dt = \frac{2}{s^2+4}$. Of course! The periodic function is simply



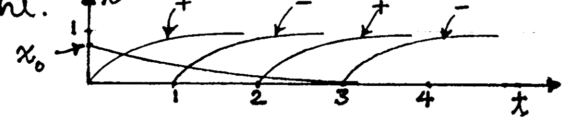
(d) $\bar{f}(s) = \frac{1}{1-e^{-2s}}\int_0^2 e^{-t}\, e^{-st}\, dt = \frac{1}{s+1}\,\frac{1-e^{-2(s+1)}}{1-e^{-2s}}$

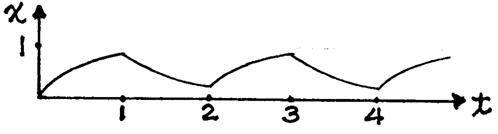
6. (a) $x'+x = f(t)$, $s\bar{x}+\bar{x}-x_0 = \bar{f}$, $\bar{x} = \frac{x_0}{s+1} + \frac{1}{s+1}\bar{f}$ so $x(t) = x_0e^{-t} + e^{-t} * f(t)$

Now, $f(t) = 1 - H(t-1) + H(t-2) - H(t-3) + \cdots$, so

$$x(t) = x_0e^{-t} + \int_0^t e^{-(t-\tau)}[1 - H(\tau-1) + H(\tau-2) - H(\tau-3) + \cdots]\, d\tau$$
$$= x_0e^{-t} + e^{-t}\int_0^t e^{\tau}[\qquad \text{"} \qquad]\, d\tau$$
$$= x_0e^{-t} + e^{-t}\{\int_0^t e^{\tau}d\tau - H(t-1)\int_1^t e^{\tau}d\tau + H(t-2)\int_2^t e^{\tau}d\tau - H(t-3)\int_3^t e^{\tau}d\tau + \cdots\}$$
$$= x_0e^{-t} + e^{-t}[(e^{t}-1) - H(t-1)(e^{t}-e) + H(t-2)(e^{t}-e^{2}) - H(t-3)(e^{t}-e^{3}) + \cdots]$$
$$= x_0e^{-t} + \quad (1-e^{-t}) - H(t-1)(1-e^{1-t}) + H(t-2)(1-e^{2-t}) - H(t-3)(1-e^{3-t}) + \cdots$$

(b) The graph of x(t) is the sum of the graphs shown at left. If $x_0=0$, for ex., x(t) is somewhat as shown at right.





It is interesting to inquire if $x(t)$ is periodic. From the right-hand figure it looks like it is (with period 2), but closer inspection reveals that it is not quite periodic. For example, $x(0)=0$ but $x(2)\neq 0$. If we realize that the $x(t)$ graph (shown at the right for the special case where $x_0=0$) is obtained by summing the graphs in the left-hand figure (with the curves labeled + counted positive and those labeled − counted negative) we can see that the sum $x(t)$ tends to a periodic function as $t\to\infty$.

To investigate this point analytically, fall back on the definition of a periodic function of period T, namely, $f(t+T)=f(t)$ for all t. [This was (32) in the text.] Well,

$$x(t)=x_0e^{-t}+\overset{\longrightarrow \text{①}}{[1-e^{-(t-0)}]H(t-0)}-[1-e^{-(t-1)}]H(t-1)+[1-e^{-(t-2)}]H(t-2)-[1-e^{-(t-3)}]H(t-3)+\cdots$$
$$x(t+2)=x_0e^{-(t+2)}+[1-e^{-(t+2)}]H(t+2)-[1-e^{-(t+1)}]H(t+1)+\underset{\longrightarrow \text{②}}{[1-e^{-t}]H(t)}-[1-e^{-(t-1)}]H(t-1)+\cdots$$

Observe that ① is identical to ②, so the question is: Can we choose x_0 so that the earlier terms match? That is, for periodicity we need

$$x_0e^{-t}=x_0e^{-(t+2)}+1-e^{-(t+2)}-1+e^{-(t+1)},$$

where we have used the fact that $H(t+2)=H(t+1)=1$ on $t\geq 0$. Canceling e^{-t} gives $x_0=x_0e^{-2}-e^{-2}+e^{-1}$, so we get $x_0=1/(e+1)$. That is, if $x_0=1/(e+1)$ we obtain a periodic solution.

7. (a) As in Exercise 6(a), $x(t)=x_0e^{-t}+e^{-t}*f(t)$.

This time $f(t)=\frac{3}{2}t-3H(t-2)-3H(t-4)-\cdots$, so

$$x(t)=x_0e^{-t}+\int_0^t e^{-(t-\tau)}[\tfrac{3}{2}\tau-3H(\tau-2)-3H(\tau-4)-\cdots]\,d\tau$$
$$=x_0e^{-t}+e^{-t}\int_0^t e^{\tau}[\tfrac{3}{2}\tau-3H(\tau-2)-3H(\tau-4)-\cdots]\,d\tau$$
$$=x_0e^{-t}+e^{-t}[\tfrac{3}{2}\int_0^t \tau e^{\tau}d\tau-3H(t-2)\int_2^t e^{\tau}d\tau-3H(t-4)\int_4^t e^{\tau}d\tau-\cdots]$$
$$=x_0e^{-t}+e^{-t}[\tfrac{3}{2}e^{t}(t-1)+\tfrac{3}{2}-3H(t-2)(e^{t}-e^{2})-3H(t-4)(e^{t}-e^{4})-\cdots]$$
$$=x_0e^{-t}+\tfrac{3}{2}(t-1)+\tfrac{3}{2}e^{-t}-3H(t-2)(1-e^{-(t-2)})-3H(t-4)(1-e^{-(t-4)})-\cdots$$

(b) As above, $x(t)=x_0e^{-t}+e^{-t}*f(t)$. This time, $f(t)=2-4H(t-2)+4H(t-4)-4H(t-6)+\cdots$

CHAPTER 6

Section 6.2

2. (b) $y' = 2xy$, $y(0) = 0$. $x_0 = 0, h = 0.2$

$y_1 = y_0 + f(x_0, y_0)h = 0 + 0(0.2) = 0$

$y_2 = y_1 + f(x_1, y_1)h = 0 + 0(0.2) = 0$, and so on. Indeed, the exact solution is $y(x) = 0$.

(c) As in (b), $y(x) = 0$.

(e) $y' = 2xe^{-y}$, $y(1) = -1$. $x_0 = 1, y_0 = -1, h = 0.2$

$y_1 = y_0 + f(x_0, y_0)h = y_0 + 2x_0 e^{-y_0} h = -1 + 2(2.718)(0.2) = 0.0873$

$y_2 = y_1 + f(x_1, y_1)h = y_1 + 2x_1 e^{-y_1} h = 0.0873 + 2(1.2)(0.916)(0.2) = 0.527$

$y_3 = y_2 + f(x_2, y_2)h = y_2 + x_2 e^{-y_2} h = 0.527 + 2(1.4)(0.590)(0.2) = 0.858$

3. (a) Exact solution is $y(x) = x^2 + 1$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Euler y_n	1	1	1.02	1.06	1.12	1.20	1.30	1.42	1.56	1.72	1.90
Exact $y(x)$	1	1.01	1.04	1.09	1.16	1.25	1.36	1.49	1.64	1.81	2.00

$Y_{10} = y(x_{10}) - y_{10} = 2.00 - 1.90 = 0.10.$

4. (d) $y' = y \sin x$, $y(0) = 1$ has exact solution $y(x) = e^{1 - \cos x}$.

Fortran code:

```
program sec6_2prob3d

      real yold,ynew,xold,h, analytic
      integer count
      print *,'Section 6.2, problem #3(d)'
      print *,'Solving the problem y_prime=ysin(x) with y(0)=1, h=0.1'
      print *,'---------------------------------------------------'
      yold=1.0
      xold=0.0
      h=0.1
      ynew=0.0
      count=0
      analytic=exp(1-cos(xold))
      print *,'x     n      Eulers  Analytical     Acc. Trunc. Err.'
      print *,'---------------------------------------------------'
      print '(F5.2, 3X, I2,   3X, F9.6, 3X,  F9.6, 7X, F9.6)',xold,count,yold$

      do count= 1,10,1
          ynew=yold+h*yold*sin(xold)
         yold=ynew
         xold=xold+h
         analytic=exp(1-cos(xold))
         print '(F5.2, 3X, I2,   3X, F9.6, 3X,  F9.6, 7X, F9.6)',xold,co$

      end do
      end
```

Printout:

```
Section 6.2, problem #3(d)
 Solving the problem y_prime=ysin(x) with y(0)=1, h=0.1
 ---------------------------------------------------
 x       n     Eulers     Analytical     Acc. Trunc. Err.
 ---------------------------------------------------
 0.00    0   1.000000   1.000000        0.000000
 0.10    1   1.000000   1.005008        0.005008
 0.20    2   1.009983   1.020133        0.010150
 0.30    3   1.030049   1.045676        0.015627
 0.40    4   1.060489   1.082138        0.021650
 0.50    5   1.101786   1.130226        0.028440
 0.60    6   1.154608   1.190846        0.036238
 0.70    7   1.219802   1.265108        0.045306
 0.80    8   1.298384   1.354312        0.055927
 0.90    9   1.391525   1.459932        0.068408
 1.00   10   1.500527   1.583595        0.083069
```

Euler's method is a first-order method so at a fixed x point ($x = 0.5$ in this exercise) the accumulated truncation error E should diminish as $E \sim Ch$ as $h \to 0$. In this exercise the Fortran code gives these values:

	$h=0.1$	$h=0.05$	$h=0.001$	$h=0.0005$	$h=0.0001$
Exact y at $x=0.5$	1.130226	1.130226	1.130226	1.130226	1.130226
Euler y at $x=0.5$	1.101786	1.115792	1.127303	1.128761	1.129932
Acc. Trunc. Error at $x=0.5$	0.02844	0.014434	0.002923	0.001465	0.000294
	↑a	↑b	↑c	↑d	↑e

These results are indeed consistent with the Euler method being a first order method because $E \sim Ch$ should diminish proportional to h, as $h \to 0$, and b is indeed around half of a, c is indeed around a fifth of b, and so on. In fact, the proportionality becomes more exact as $h \to 0$, as it should. For instance, $b = 0.508a$ and $d = 0.501c$.

5. No problem with $h<0$, i.e., with making steps to the left.

6. (a) $y_{n+1} = (1+Ah)y_n$ is a very simple difference equation (see Sec. 6.5.3), that is easily solved as follows.

$$y_1 = (1+Ah)y_0$$
$$y_2 = (1+Ah)y_1 = (1+Ah)^2 y_0$$
$$\vdots$$
$$y_n = (1+Ah)^n y_0 = \text{“}C\text{”}(1+Ah)^n$$

(b)
$$y_n = Ce^{\ln(1+Ah)^n}$$
$$= Ce^{n\ln(1+Ah)} \sim Ce^{n(Ah+\cdots)} \sim Ce^{Anh} = Ce^{Ax_n}$$

so $y_n \sim Ce^{Ax_n}$ as $h \to 0$ or, $y(x) = Ce^{Ax}$.

(c) rsolve(y(n+1) = (1+A*h)*y(n), y(n));

gives $y(0)(1+Ah)^n$

7. No. Using Euler's method the step size h can be changed from step to step. The motivation for such change would be efficiency and economics. For example the exact solution of $y'(x) = -20y(x)$; $y(0)=1$ is $y(x) = e^{-20x}$. Solving by Euler's method we need a very small step size h over $0 < x < 0.5$, say, because of the rapid plummet and leveling off of the solution; but once it begins to level off we no longer need such a small h.

Section 6.3

1. 2nd-order R-K:

$$\boxed{\begin{gathered} y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2), \\ k_1 = hf(x_n, y_n), \qquad k_2 = hf(x_{n+1}, y_n + k_1). \end{gathered}}$$

4th-order R-K:

$$\boxed{\begin{gathered} y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 = hf(x_n, y_n), \qquad k_2 = hf\left(x_n + \tfrac{h}{2}, y_n + \tfrac{1}{2}k_1\right), \\ k_3 = hf\left(x_n + \tfrac{h}{2}, y_n + \tfrac{1}{2}k_2\right), \quad k_4 = hf(x_{n+1}, y_n + k_3), \end{gathered}}$$

(a) $y' = 3000xy^{-2}$, $y(0) = 2$. Exact solution (by separation of variables) is $y = \sqrt[3]{4500x^2 + 8}$.

2nd-order R-K: $h = 0.02$

$n=0$: $k_1 = hf(x_0, y_0) = 0.02(3000)(0)2^{-2} = 0$

$k_2 = hf(x_1, y_0 + k_1) = 0.02(3000)(0.02)2^{-2} = 0.3$

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0 + 0.3) = \underline{2.15}$. $y(0.02) = \sqrt[3]{4500(0.02)^2 + 8} = \underline{2.13997}$.

$n=1$: $k_1 = hf(x_1, y_1) = 0.02(3000)(0.02)(2.15)^{-2} = 0.25960$

$k_2 = hf(x_1, y_1 + k_1) = 0.02(3000)(0.04)(2.15 + 0.25960)^{-2} = 0.41335$

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 2.15 + \frac{1}{2}(0.25960 + 0.41335) = \underline{2.4865}$. $y(0.04) = \underline{2.47712}$

4th-order R-K: $h=0.02$

n=0: $k_1 = h f(x_0, y_0) = 0.02(3000)(0)2^{-2} = 0$
$k_2 = h f(x_0 + h/2, y_0 + k_1/2) = 0.02(3000)(0.01)(2+0)^{-2} = 0.15$
$k_3 = h f(x_0 + h/2, y_0 + k_2/2) = 0.02(3000)(0.01)(2+0.075)^{-2} = 0.13935$
$k_4 = h f(x_1, y_0 + k_3) = 0.02(3000)(0.02)(2+0.13935)^{-2} = 0.26219$
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \underline{2.14015}$, $y(0.02) = \underline{2.13997}$

n=1: $k_1 = h f(x_1, y_1) = 0.02(3000)(0.02)(2.14015)^{-2} = 0.26199$
$k_2 = h f(x_1 + h/2, y_1 + k_1/2) = 0.02(3000)(0.03)(2.14015 + 0.13100)^{-2} = 0.34896$
$k_3 = h f(x_1 + h/2, y_1 + k_2/2) = 0.02(3000)(0.03)(2.14015 + 0.17448)^{-2} = 0.33598$
$k_4 = h f(x_2, y_1 + k_3) = 0.02(3000)(0.04)(2.14015 + 0.33598)^{-2} = 0.39144$
$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \underline{2.47737}$, $y(0.04) = \underline{2.47712}$

(b) $y' = 40xe^{-y}$, $y(0) = 3$. Exact solution (by separation of variables) is $y = \ln(20x^2 + e^3)$

2nd-order R-K: $h = 0.02$

n=0: $k_1 = h f(x_0, y_0) = 0.02(40)(0)e^{-3} = 0$
$k_2 = h f(x_1, y_0 + k_1) = 0.02(40)(0.02)e^{-(3+0)} = 0.000797$
$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = \underline{3.00040}$. $y(0.02) = \underline{3.00040}$

n=1: $k_1 = h f(x_1, y_1) = 0.02(40)(0.02)e^{-3.00040} = 0.000796$
$k_2 = h f(x_1, y_1 + k_1) = 0.02(40)(0.02)e^{-(3.00040 + 0.000796)} = 0.000796$
$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 3.00040 + 0.000796 = \underline{3.00120}$, $y(0.04) = \underline{3.00159}$

4th-order R-K: $h = 0.02$

n=0: $k_1 = h f(x_0, y_0) = 0.02(40)(0)e^{-3} = 0$
$k_2 = h f(x_0 + h/2, y_0 + k_1/2) = 0.02(40)(0.01)e^{-(3+0)} = 0.000398$
$k_3 = h f(x_0 + h/2, y_0 + k_2/2) = 0.02(40)(0.01)e^{-(3+0.000199)} = 0.000398$
$k_4 = h f(x_1, y_0 + k_3) = 0.02(40)(0.02)e^{-(3+.000398)} = 0.000796$
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \underline{3.000398}$, $y(0.02) = \underline{3.00040}$

n=1: $k_1 = h f(x_1, y_1) = 0.02(40)(0.02)e^{-3.000398} = 0.000796$
$k_2 = h f(x_1 + h/2, y_1 + k_1/2) = 0.02(40)(0.03)e^{-(3.000398+.000398)} = 0.00119$
$k_3 = h f(x_1 + h/2, y_1 + k_2/2) = 0.02(40)(0.03)e^{-(3.000398+.000597)} = 0.00119$
$k_4 = h f(x_2, y_1 + k_3) = 0.02(40)(0.04)e^{-(3.000398+.00119)} = 0.00159$
$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3.000398 + 0.001191 = \underline{3.00159}$, $y(0.04) = \underline{3.00159}$

2.(d) $y' = -y\tan x$, $y(0)=1$. Exact solution (by separation of variables) is $y(x)=\cos x$.
Fortran code for solution by 4th-order R-K:

```
real yold,ynew,xold,xnew,h,k1,k2,k3,k4
integer steps,i
OPEN(12,FILE='ques2')
h=0.05
steps=20
xold=0.0
yold=1.0
write(12,*) '           n         y'
do i= 1,steps,1
    k1=-h*yold*tan(xold)
    k2=-h*(yold+k1)*tan(xold+0.5*h)
    k3=-h*(yold+k2)*tan(xold+0.5*h)
    k4=-h*(yold+k3)*tan(xold+h)
    ynew=yold+(k1+2*k2+2*k3+k4)/6
    write(12, *)  i,  ynew
    yold=ynew
    xold=xold+h
enddo
write(12,*) 'Exact solution is' , cos(1.0)
end
```

x	n	y (R-K)	$y(x)=\cos x$ (Exact)
.05	1	9.987505E-01	0.9987503
.1	2	9.950083E-01	0.9950042
.15	3	9.887869E-01	0.9887711
.2	4	9.801059E-01	0.9800666
.25	5	9.689912E-01	0.9689124
.3	6	9.554746E-01	0.9553365
.35	7	9.395937E-01	0.9393727
.4	8	9.213923E-01	0.9210610
.45	9	9.009197E-01	0.9004471
.5	10	8.782308E-01	0.8775826

6. (a) Using maple:

```
> with(DEtools):
> de1:=diff(x(t),t)=0.02-0.01*sqrt(x(t));
```

$$de1 := \frac{\partial}{\partial t}\mathrm{x}(t) = .02 - .01\sqrt{\mathrm{x}(t)}$$

```
> dsolve({de1,x(0)=0},x(t),type=numeric,value=array([0,600,1200,1800,2400,3000,
  3600]),abserr=Float(1,-5));
Error, (in dsolve/numeric/rkf45) cannot evaluate boolean
```

```
> dsolve({de1,x(0)=0},x(t),type=numeric,method=dverk78,value=array([0,600,120
  0,1800,2400,3000,3600]),abserr=Float(1,-5));
Error, (in dsolve/numeric/dverk78) keyword was, abserr,
optional keyword must be one of, control, initial, number, output
, procedure, start, tolerance, value
```

so let's omit the abserr option

```
> dsolve({de1,x(0)=0},x(t),type=numeric,method=dverk78,value=array([0,600,120
  0,1800,2400,3000,3600]));
```

I don't know why the rkf45 default method failed (which it does, I find, for any $x(0)$ less than the steady-state value 4), so let's call for the dverk78 method (a 7th-8th order R-K method) instead.

$$\left[\begin{array}{c} [t \quad \mathrm{x}(t)] \\ \left[\begin{array}{cc} 0 & 0 \\ 600. & 3.313877493498139 \\ 1200. & 3.852107843603627 \\ 1800. & 3.967238747167962 \\ 2400. & 3.992701615124026 \\ 3000. & 3.998372087416714 \\ 3600. & 3.999636792316914 \end{array}\right] \end{array}\right]$$

← Here are the desired results. Observe the approach to the steady-state solution $x_s = 4$.

7\. (a) $x' = Q - 0.01\sqrt{x}$, $\frac{dx}{0.01\sqrt{x} - Q} = -dt$, $\frac{100\,dx}{\sqrt{x} - 100Q} \overset{\sqrt{x} = u}{=} -dt$, $\frac{200\,u\,du}{u - 100Q} = -dt$,

$$200\,\frac{(u - 100Q) + 100Q}{u - 100Q}\,du = -dt, \quad 200u + 20000Q\ln(u - 100Q) = -t + C$$

$$200\sqrt{x} + 20{,}000Q\ln(\sqrt{x} - 100Q) = -t + C$$

(b) Then $x(0) = 0$ gives $C = 20{,}000Q\ln(100Q)$. Also, let us rewrite $\ln(\sqrt{x} - 100Q)$ as $\ln(100Q - \sqrt{x})$ because $\int du/u = \ln|u|$ so we need to understand the $\ln(\;)$'s as $\ln|\;|$'s. Since $x(0) = 0$, we approach $x_s = 4$ from below; hence, $100Q - \sqrt{x}$ will be > 0, consistent with our $\ln|\;|$ interpretation. Thus, with $Q = 0.02$,

$$200\sqrt{x} + 400\ln(2 - \sqrt{x}) = -t + 400\ln 2.$$

Then the Maple command

```
fsolve(200*x^(1/2)+400*ln(2-x^(1/2))=-600+400*ln(2),x);
```

gives, for $x(600)$, 3.313877493. Changing this to 1200, 1800, …, 3600 gives

$x(1200) =$ 3.852107844
$x(1800) =$ 3.967238747
$x(2400) =$ 3.992701615
$x(3000) =$ 3.998372087
$x(3600) =$ 3.999636792.

Comparing the dverk78 results, in Exercise 6(a), with these, we see that the results agree to all 10 decimal places.

8\. $E_n \sim Ch^p$ is much more sensitive to p, as $h \to 0$, than to C. To illustrate that claim let $C = 1$ and $p = 2$. Then, for diminishing values of h we have

		$h = 0.1$	$h = 0.01$	$h = 0.001$	
	$Ch^p = 1h^2 =$	0.01	0.0001	0.000001	← original
Halving C,	$0.5Ch^p = 0.5h^2 =$	0.005	0.00005	0.0000005	← better
Doubling p,	$Ch^{2p} = 1h^4 =$	0.0001	0.00000001	0.000000000001	← <u>much</u> better

9\. We need to expand $f[x_n + \alpha h,\, y(x_n) + \beta f[x_n, y(x_n)]h]$ in h. Call it $F(h)$. Then, by chain differentiation,

$$F'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \alpha\frac{\partial f}{\partial x} + \beta f[x_n, y(x_n)]\frac{\partial f}{\partial y}$$
$$= \alpha f_x + \beta f f_y$$

so RHS of (18) $= y + \{af + b[f + (\alpha f_x + \beta f f_y)h + \cdots]\}h$

$= y + (a+b)fh + (\alpha f_x + \beta f f_y)bh^2 + \cdots$ ✓

10. (a)

```
#include<stdio.h>
#include<math.h>
FILE *fpt;
main()

{

/* Defining the variables now */

char c;
int  i, steps;
float xold, yold, xnew, ynew, h, x_n, y_n, k1, k2,k3,k4;
fpt=fopen("answer","a");
h=0.05;
xold=0.0;
yold=1.0;
steps=20;
fprintf(fpt, "\n");

/* Starting the loop now */

for (i=0;i< (steps);i++)
   {
      k1=h*(yold+2*xold-xold*xold);
      k2=h*((yold+0.5*k1)+2*(xold+0.5*h)-(xold+0.5*h)*(xold+0.5*h));
      k3=h*((yold+0.5*k2)+2*(xold+0.5*h)-(xold+0.5*h)*(xold+0.5*h));
      k4=h*((yold+k3)+2*(xold+h)-(xold+h)*(xold+h));
      ynew= yold+(k1+2*k2+2*k3+k4)/6.0;
      xold=xold+h;
      yold=ynew;
   }

/* Writing out the results to the file now */

fprintf(fpt,"The solution y(1) using h=0.05 is %2.9f \n", ynew);
fprintf(fpt,"The exact solution y(1) is %2.9f \n",1+exp(1));
fclose(fpt);

/* Result obtained- closing the file */

}
```

Output:

```
The solution y(1) using h=0.02 is 3.718281819
The exact solution y(1) is 3.718281828
y(exact)-y(estimate)= 0.000000009

The solution y(1) using h=0.05 is 3.718281474
The exact solution y(1) is 3.718281828
y(exact)-y(estimate)= 0.000000354

Thus, computed order of the method from Eqn. (28)≅ 4.0075
```

(b) Output:

```
The output after (x_n+h/2) was changed to x_n.....


The solution y(1) using h=0.05 is 3.701259429
The exact solution y(1) is 3.718281828
y(exact)-y(estimate)= 0.017022400

The solution y(1) using h=0.02 is 3.711558039
The exact solution y(1) is 3.718281828
y(exact)-y(estimate)= 0.006723790

The new order of the method ~ 1.01

It is no longer a fourth order method.
```

(c) Recall the comment, following (25), that the weights 1/6, 2/6, 2/6, 1/6 sum to 1 so that the effective slope is a weighted average slope. However, for the modified scheme $y_{n+1} = y_n + \frac{1}{6}(k_1 + 3k_2 + 2k_3 + k_4)$ the weights do not sum to 1, so we do not expect the scheme to be convergent. As a simpler analogy, we know that the Euler scheme $y_{n+1} = y_n + f(x_n, y_n)h$ gives convergence to the solution of $y' = f(x,y)$. In this case the weight is 1. If we change the scheme to $y_{n+1} = y_n + 3f(x_n, y_n)h$, say, then the scheme will, evidently, converge to the solution of the wrong problem, $y' = 3f(x,y)$; hence, it will <u>not</u> converge (i.e., to the solution of the <u>correct</u> problem $y' = f(x,y)$).

11. (a) The hint is self-explanatory.

(b) Express $f(x) = A + Bx + Cx^2$, such that

$$\begin{aligned} f(x_n) &= A + Bx_n + Cx_n^2 \\ f(x_n + h/2) &= A + B(x_n + \tfrac{h}{2}) + C(x_n + h/2)^2 \\ f(x_{n+1}) &= A + Bx_{n+1} + Cx_{n+1}^2 \end{aligned}$$

Solving the latter for A, B, C gives

$$\begin{aligned} A &= (2x_n^2 f_n + 2x_n^2 f_{n+1} - 4x_n^2 f_{n+1/2} + 3x_n f_n h + x_n f_{n+1} h - 4x_n f_{n+1/2} h + f_n h^2)/h^2, \\ B &= -(4x_n f_n + 4x_n f_{n+1} - 8x_n f_{n+1/2} + 3f_n h + f_{n+1} h - 4f_{n+1/2} h)/h^2 \\ C &= 2(f_n - 2f_{n+1/2} + f_{n+1}) \end{aligned}$$

(which we obtained by solving ★ by the Maple commands

```
with(linalg):
linsolve(array([[1,x,x^2],[1,x+h/2,(x+h/2)^2],[1,x,x^2]]),
         array([F,G,H]));
```

where x, F, G, H are shorthand for $x_n, f_n, f_{n+1/2}, f_{n+1}$.).

Then,

$$\int_{x_n}^{x_{n+1}} f(x)\,dx = \int_{x_n}^{x_n+h} (A + Bx + Cx^2)\,dx = \left(Ax + B\frac{x^2}{2} + C\frac{x^3}{3}\right)\Big|_{x_n}^{x_n+h}$$

$$= Ah + Bh(x_n + \tfrac{h}{2}) + \tfrac{1}{3}C(3x_n^2 h + 3x_n h^2 + h^3)$$

and putting the above expressions for A, B, C into this gives, after much cancellation and simplification,

$$\int_{x_n}^{x_{n+1}} f(x)\,dx = \frac{1}{6}\left[f(x_n) + 4f(x_n + h/2) + f(x_{n+1})\right]h.$$

Putting the latter into (11.1) gives the desired result (11.3). The approximations

$$\int_{x_n}^{x_{n+1}} f(x)\,dx \approx \begin{cases} f(x_n)h, & \text{(rectangular rule)} \\ \frac{1}{2}[f(x_n) + f(x_{n+1})]h, & \text{(trapezoidal rule)} \\ \frac{1}{6}[f(x_n) + 4f(x_n + h/2) + f(x_{n+1})] & \text{(Simpson's rule)} \end{cases}$$

are known as the rectangular, trapezoidal, and Simpson's rule, respectively, for a "single-step" integration. We can generalize to $\int_a^b f(x)dx$ by breaking the a, b interval into the steps a, a+h, a+2h, …, b and summing the individual steps,

2. (b) $\left.\begin{array}{l} y'=4z\,;\ y(2)=5 \\ z'=-y,\ z(2)=0 \end{array}\right\}$ $y''=4z'=-4y$ gives $y=A\sin 2x+B\cos 2x$.
Then $z=y'/4=(A\cos 2x-B\sin 2x)/2$

Then $\left.\begin{array}{l} y(2)=5=(\sin 4)A+(\cos 4)B \\ \text{and } z(2)=0=(\frac{\cos 4}{2})A-(\frac{\sin 4}{2})B \end{array}\right\}$ gives $\begin{array}{l} A=5\sin 4 \\ B=5\cos 4 \end{array}$

so exact solution is $y(x)=5(\sin 4\sin 2x+\cos 4\cos 2x)=5\cos(2x-4)$
$z(x)=\frac{5}{2}(\sin 4\cos 2x-\cos 4\sin 2x)=-\frac{5}{2}\sin(2x-4)$

Hence, exact $y(0.2)=\underline{4.60530497}$ and $z(0.2)=-\underline{0.973545856}$.

Euler: $y_1=y_0+f(x_0,y_0,z_0)h=y_0+4z_0h=5+0=\underline{5}$
$z_1=z_0+g(\quad \text{"} \quad)h=z_0-y_0h=0-5(0.2)=\underline{-1}$

2nd-Order R-K: The algorithm is $y_{n+1}=y_n+\frac{1}{2}(k_1+k_2)$
$k_1=hf(x_n,y_n)$, $k_2=hf(x_{n+1},y_n+k_1)$

so for a system $y'=f(x,y,z)$, $z'=g(x,y,z)$ it becomes

$$y_{n+1}=y_n+\tfrac{1}{2}(k_1+k_2)$$
$$z_{n+1}=z_n+\tfrac{1}{2}(l_1+l_2)$$
$$k_1=hf(x_n,y_n,z_n),\quad k_2=hf(x_{n+1},y_n+k_1,z_n+l_1)$$
$$l_1=hg(x_n,y_n,z_n),\quad l_2=hg(x_{n+1},y_n+k_1,z_n+l_1)$$

In the present example, for $n=0$,

$k_1=h(4z_0)=(0.2)(4)(0)=0$
$l_1=h(-y_0)=(0.2)(-5)=-1$
$k_2=h(4)(z_0+l_1)=(0.2)(4)(0-1)=-0.8$
$l_2=h(-(y_0+k_1))=(0.2)(-(5+0))=-1$
$y_1=y_0+\frac{1}{2}(k_1+k_2)=5+\frac{1}{2}(0-0.8)=\underline{4.6}$ (much better!)
$z_1=z_0+\frac{1}{2}(l_1+l_2)=0+\frac{1}{2}(-1-1)=\underline{-1}$

4-th Order R-K: The algorithm is given by (6):

$k_1=hf(x_0,y_0,z_0)=(0.2)4z_0=0$
$l_1=hg(\quad \text{"} \quad)=(0.2)(-y_0)=-1$
$k_2=hf(x_0+h/2,y_0+k_1/2,z_0+l_1/2)=(0.2)4(z_0+l_1/2)=-0.4$
$l_2=hg(\quad \text{"} \quad)=(0.2)[-(y_0+k_1/2)]=-1$
$k_3=hf(x_0+h/2,y_0+k_2/2,z_0+l_2/2)=(0.2)4(z_0+l_2/2)=-0.4$
$l_3=hg(\quad \text{"} \quad)=(0.2)[-(y_0+k_2/2)]=-0.96$
$k_4=hf(x_1,y_0+k_3,z_0+l_3)=(0.2)4(z_0+l_3)=-0.768$
$l_4=hg(\quad \text{"} \quad)=(0.2)[-(y_0+k_3)]=-0.92$
$y_1=y_0+(k_1+2k_2+2k_3+k_4)/6=\underline{4.60533333}$
$z_1=z_0+(l_1+2l_2+2l_3+l_4)/6=\underline{-0.9733333}$

3. (b)

	$x=3$	$x=5$	$x=10$
Exact: $y(x)=5\cos(2x-4)=$	-2.080734183	4.800851433	-4.788297402
$z(x)=-\frac{5}{2}\sin(2x-4)=$	-2.273243567	0.698538745	0.719758292

Maple: Using the default RKF45 numerical solver, the commands

```
with(DEtools):
dsolve({diff(y(x),x)=4*z(x), diff(z(x),x)=-y(x), y(2)=5, z(2)=0}, {y(x),z(x)]
type = numeric, value = array([3,5,10]));
```

gives these results:

```
        [x, y(x), z(x)]
[[ 3   -2.080734210   -2.273243577 ]]
[[ 5    4.800851531    0.6985387649]]
[[ 10  -4.788297661    0.7197583300]]
```

4.(a)
$$\begin{array}{lll} x'(t) = F(t,x,y,z) = y-1 & ; \quad x(0)=-3 & \\ y'(t) = G(\quad '' \quad) = z & ; \quad y(0)=0 & , h=0.3 \\ z'(t) = H(\quad '' \quad) = t+x+3(z-y+1) & ; \quad z(0)=2 & \end{array}$$

Euler:
$$\begin{aligned} x_{n+1} &= x_n + F(t_n, x_n, y_n, z_n)h \\ y_{n+1} &= y_n + G(\quad '' \quad)h \\ z_{n+1} &= z_n + H(\quad '' \quad)h \end{aligned}$$

2nd Order RK:
$$\begin{aligned} k_1 &= hF(t_n, x_n, y_n, z_n) \\ l_1 &= hG(\quad '' \quad) \\ m_1 &= hH(\quad '' \quad) \\ k_2 &= hF(t_{n+1}, x_n+k_1, y_n+l_1, z_n+m_1) \\ l_2 &= hG(\quad '' \quad) \\ m_2 &= hH(\quad '' \quad) \\ x_{n+1} &= x_n + \tfrac{1}{2}(k_1+k_2) \\ y_{n+1} &= y_n + \tfrac{1}{2}(l_1+l_2) \\ z_{n+1} &= z_n + \tfrac{1}{2}(m_1+m_2) \end{aligned}$$

4th Order RK:
$$\begin{aligned} k_1 &= hF(t_n, x_n, y_n, z_n) \\ l_1 &= hG(\quad '' \quad) \\ m_1 &= hH(\quad '' \quad) \\ k_2 &= hF(t_n+h/2, x_n+k_1/2, y_n+l_1/2, z_n+m_1/2) \\ l_2 &= hG(\quad '' \quad) \\ m_2 &= hH(\quad '' \quad) \\ k_3 &= hF(t_n+h/2, x_n+k_2/2, y_n+l_2/2, z_n+m_2/2) \\ l_3 &= hG(\quad '' \quad) \\ m_3 &= hH(\quad '' \quad) \\ k_4 &= hF(t_{n+1}, x_n+k_3, y_n+l_3, z_n+m_3) \\ l_4 &= hG(\quad '' \quad) \\ m_4 &= hH(\quad '' \quad) \\ x_{n+1} &= x_n + \tfrac{1}{6}(k_1+2k_2+2k_3+k_4) \\ y_{n+1} &= y_n + \tfrac{1}{6}(l_1+2l_2+2l_3+l_4) \\ z_{n+1} &= z_n + \tfrac{1}{6}(m_1+2m_2+2m_3+m_4) \end{aligned}$$

Now let us use the Euler and 2nd Order RK schemes, given above, to compute x_1, y_1, z_1 and x_2, y_2, z_2 for the present example.

Euler:

$$\begin{aligned}
x_1 &= x_0 + F(t_0, x_0, y_0, z_0)h = x_0 + (y_0 - 1)h = -3 + (0-1)(0.3) = -3.3 \\
y_1 &= y_0 + G(\quad '' \quad)h = y_0 + z_0 h = 0 + 2(0.3) = 0.6 \\
z_1 &= z_0 + H(\quad '' \quad)h = z_0 + [t_0 + x_0 + 3(z_0 - y_0 + 1)]h = 2 + [0 - 3 + 3(2 - 0 + 1)](0.3) = 3.8 \\
x_2 &= x_1 + F(t_1, x_1, y_1, z_1)h = x_1 + (y_1 - 1)h = -3.3 + (0.6 - 1)(0.3) = -3.42 \\
y_2 &= y_1 + G(\quad '' \quad)h = y_1 + z_1 h = 0.6 + (3.8)(0.3) = 1.74 \\
z_2 &= z_1 + H(\quad '' \quad)h = z_1 + [t_1 + x_1 + 3(z_1 - y_1 + 1)]h \\
&= 3.8 + [0.3 - 3.3 + 3(3.8 - 0.6 + 1)](0.3) = 6.68
\end{aligned}$$

2nd Order RK:

$$\begin{aligned}
k_1 &= h(y_0 - 1) = (0.3)(0-1) = -0.3 \\
l_1 &= h z_0 = (0.3)(2) = 0.6 \\
m_1 &= h[t_0 + x_0 + 3(z_0 - y_0 + 1)] = (0.3)[0 - 3 + 3(2 - 0 + 1)] = 1.8 \\
k_2 &= h(y_0 + l_1 - 1) = (0.3)(0 + 0.6 - 1) = -0.12 \\
l_2 &= h(z_0 + m_1) = (0.3)(2 + 1.8) = 1.14 \\
m_2 &= h[t_1 + x_0 + k_1 + 3(z_0 + m_1 - y_0 - l_1 + 1)] \\
&= (0.3)[0.3 - 3 - 0.3 + 3(2 + 1.8 - 0 - 0.6 + 1)] = 2.88 \\
x_1 &= x_0 + \tfrac{1}{2}(k_1 + k_2) = -3 + \tfrac{1}{2}(-0.3 - 0.12) = -3.21 \\
y_1 &= y_0 + \tfrac{1}{2}(l_1 + l_2) = 0 + \tfrac{1}{2}(0.6 + 1.14) = 0.87 \\
z_1 &= z_0 + \tfrac{1}{2}(m_1 + m_2) = 2 + \tfrac{1}{2}(1.8 + 2.88) = 4.34 \\
k_1 &= h(y_1 - 1) = 0.3(0.87 - 1) = -0.039 \\
l_1 &= h z_1 = 0.3(4.34) = 1.302 \\
m_1 &= h[t_1 + x_1 + 3(z_1 - y_1 + 1)] = 0.3[0.3 - 3.21 + 3(4.34 - 0.87 + 1)] = 3.15 \\
k_2 &= h(y_1 + l_1 - 1) = 0.3(0.87 + 1.302 - 1) = 0.3516 \\
l_2 &= h(z_1 + m_1) = 0.3(4.34 + 3.15) = 2.247 \\
m_2 &= h[t_2 + x_1 + k_1 + 3(z_1 + m_1 - y_1 - l_1 + 1)] \\
&= 0.3[0.6 - 3.21 - 0.039 + 3(4.34 + 3.15 - 0.87 - 1.302 + 1)] = 4.8915 \\
x_2 &= x_1 + \tfrac{1}{2}(k_1 + k_2) = -3.21 + \tfrac{1}{2}(-0.039 + 0.3516) = -3.0537 \\
y_2 &= y_1 + \tfrac{1}{2}(l_1 + l_2) = 0.87 + \tfrac{1}{2}(1.302 + 2.247) = 2.6445 \\
z_2 &= z_1 + \tfrac{1}{2}(m_1 + m_2) = 4.34 + \tfrac{1}{2}(3.15 + 4.8915) = 8.36075
\end{aligned}$$

We weren't asked to compare with the exact values, but let us do so anyway. The Maple command

```
dsolve({diff(x(t),t)=y(t)-1, diff(y(t),t)=z(t), diff(z(t),t)=t+x(t)+
  3(z(t)-y(t)+1), x(0)=-3, y(0)=0, z(0)=2}, {x(t),y(t),z(t)});
```

gives $x(t) = t^2 e^t - 3 - t$, $y(t) = (t^2 + 2t)e^t$, $z(t) = (t^2 + 4t + 2)e^t$

so $x(t_1) = x(0.3) = -3.178512707$, $x(t_2) = x(0.6) = -2.944037232$, $y(t_1) = y(0.3) = 0.931402577$, $y(t_2) = y(0.6) = 2.842505329$, $z(t_1) = 4.441035477$, $z(t_2) = 8.67328549$

The 2nd order RK results look too inaccurate, but I see no error. Perhaps it's that $h=0.3$ is too large

5. (b)
$$\begin{aligned} i' &= u & &; \ i(0) = i_0 \\ u' &= -\tfrac{R}{L}u - \tfrac{1}{LC}i + \tfrac{1}{L}E'(t) & &; \ u(0) = i_0' \end{aligned}$$

(e)
$$\begin{aligned} y' &= u & &; \ y(-2) = 7 \\ u' &= v & &; \ u(-2) = 4 \\ v' &= 2\sin u + 3x & &; \ v(-2) = 0 \end{aligned}$$

(g)
$$\begin{aligned} x' &= u & &; \ x(0) = 2 \\ u' &= -2x + 3y + 10\cos 3t & &; \ u(0) = -1 \\ y' &= v & &; \ y(0) = 4 \\ v' &= x - 5y & &; \ v(0) = 3 \end{aligned}$$

6. (b)
$$\begin{aligned} y' &= -3xz + x^2 & &; \ y(0) = 1 \\ z' &= u & &; \ z(0) = 2 \\ u' &= v & &; \ u(0) = 2 \\ v' &= y^2u - z & &; \ v(0) = -1 \end{aligned}$$

Then the Maple commands

```
with(DEtools):
dsolve({diff(y(x),x)=-3*x*z(x) + x^2,  diff(z(x),x)= u(x),
diff(u(x),x) = v(x), diff(v(x),x)=y(x)^2*u(x)-z(x), y(0)=1, z(0)= 2,
u(0)=2, v(0)=-1}, {y(x), z(x), u(x), v(x)}, type = numeric, value
= array([1,2]));
```

give this output:

```
                    x, y(x), z(x), u(x), v(x)
1   -3.183672016  3.291053188  .2557593258  -2.555374357
2    5.052211938 -29.02733440 -247.8533442  -1286.070148
```

of which the underlined values are the ones called for.

7.
```
with(DEtools):
dsolve({diff(Y(x),x) = U(x), diff(U(x),x)=V(x), diff(V(x),x) = x^2*Y(x), Y(0)=0,
U(0)=0, V(0)=1}, {Y(x),U(x),V(x)}, type=numeric, value = array([.5,1,2]));
```

(where Y, U, V are Y_3, U_3, V_3) gives x = 0.5, 1, 2

$Y_3(x)$ = .125018599, .502382754, 2.31220823

Next,
```
dsolve({diff(Y(x),x)=U(x), diff(U(x),x)=V(x), diff(V(x),x)=x^2*Y(x)-x^4,
Y(0)=0, U(0)=0, V(0)=0}, {Y(x),U(x),V(x)}, type=numeric, value = array([.5,1,2]));
```

(where this time Y, U, V are Y_p, U_p, V_p) gives x = 0.5 1 2

$Y_p(x)$ = −.00003720 −.00476551 −.62441644

Finally, (17) gives $y(0.5) = (2.0)(.125018599) - .00003720 = 0.25000000$

$y(1) = (2.0)(.502382754) - .00476551 = 1.00000000$

Indeed, the analytical solution was rigged to be simple, namely, $y(x) = x^2$, and the numerical results given above agree with this exact solution.

8. (a) $y'' - 2xy' + y = 3\sin x$; $y(0)=1$, $y(2)=3$. $y(1)=?$

$y(x) = C_1 Y_1(x) + C_2 Y_2(x) + Y_p(x)$, where

$$\begin{aligned} L[Y_1]=0:\ & Y_1' = u & ;\ & Y_1(0)=1 \\ & u' = 2xu - Y_1 & ;\ & Y_1'(0) = u(0) = 0 \\ L[Y_2]=0:\ & Y_2' = v & ;\ & Y_2(0)=0 \\ & v' = 2xv - Y_2 & ;\ & Y_2'(0) = v(0) = 1 \\ L[Y_p]=3\sin x:\ & Y_p' = w & ;\ & Y_p(0) = a \\ & w' = 2xw - Y_p + 3\sin x & ;\ & Y_p'(0) = w(0) = b. \end{aligned}$$

Then, $y(0)=1 = C_1 Y_1(0) + C_2 Y_2(0) + Y_p(0)$
$= C_1 + a$. Thus, let us choose $a=1$, because then $C_1=0$ and we do not need to compute $Y_1(x)$.

$y(2)=3 = \overset{0}{\cancel{C_1}} Y_1(2) + C_2 Y_2(2) + Y_p(2)$
$= C_2 Y_2(2) + Y_p(2)$ gives $C_2 = [3 - Y_p(2)]/Y_2(2)$. Can choose $b=0$, say.

so

$$y(x) = \frac{3 - Y_p(2)}{Y_2(2)} Y_2(x) + Y_p(x). \quad \bigstar$$

If we seek $y(x)$ at $x = 0.5, 1, 1.5$, say, (actually, only $x=1$ is asked for here) then use the Maple commands

```
with(DEtools):
dsolve({diff(Y(x),x)=v(x), diff(v(x),x)= 2*x*v(x)-Y(x), Y(0)=0, v(0)=1},
   {Y(x),v(x)}, type=numeric, value=array([0, 0.5, 1.0, 1.5, 2.0]));
```

(Y is Y_1 here)

which gives

$x =$ 0 , 0.5 , 1.0 , 1.5 , 2.0
$Y_2(x) =$ 0 , 0.522208452, 1.219161180, 2.623792372, 7.639140121

and then

```
dsolve({diff(Y(x),x)=w(x), diff(w(x),x)=2*x*w(x)-Y(x)+3*sin(x),
   Y(0)=1, w(0)=0}, {Y(x), w(x)}, type=numeric, value=array([0,0.5,1.0,1.5,2.0]));
```

(here Y is Y_p) which gives

$x =$ 0, 0.5 , 1.0 , 1.5 , 2.0
$Y_p(x) =$ 0, 0.9325135480, 0.9656913008, 1.733975626, 6.156037371.

Then $\bigstar$ becomes $y(x) = -0.413140396\, Y_2(x) + Y_p(x)$

so $x =$ 0 , 0.5 , 1.0 , 1.5 , 2.0
$y(x) =$ 1 (given), 0.716768141 , $\underline{0.462006567}$, 0.649981006, 3 (given)

Section 6.5

1. (b) No. The general solution of $y' = y - 2e^{-x}$ is $y(x) = Ae^{x} + e^{-x}$ and the initial condition $y(0) = 1 = A + 1$ gives $A = 0$, so $y(x) = e^{-x}$. In effect, numerical error will cause A to differ from 0, in which case the Ae^{x} term will cause the solution to differ drastically, as x increases, from the exact solution e^{-x}. Let us see:

The Maple commands

```
with(DEtools):
dsolve({diff(y(x),x)=y(x)-2*exp(-x), y(0)=1}, {y(x)}, type=numeric,
   value=array([0,2,5,20,40]));
```

which uses the RKF45 method, gives these results

$x =$ 0, 2, 5, 20, 40

RKF45 = 1, 0.13533528, 0.0067369, −3.288, -1.60×10^{9}

whereas the exact solution is $y(x) =$ 1, 0.13533524, 0.0067379, 2.06×10^{-9}, 4.25×10^{-18}

Of course, it takes a while for this error to develop, so the results are not so bad if x is sufficiently small.

(c) No, similar to (b). This time $y(x) = Ae^{x} - e^{-4x}$ and $y(0) = -1$ gives $y(x) = -e^{-4x}$

(d) No, similar to (b). This time $y(x) = Ae^{3x} + x$ and $y(0) = 0$ gives $y(x) = x$

4. Each of the terms $(Ah+\sqrt{\ })^n$ and $(Ah-\sqrt{\ })^n$ in (12) are solutions of (7). Let us check the first of them and leave the other for you: Putting $e_n = C_1(Ah+\sqrt{\ })^n$ into (7) gives

$$\cancel{C_1}(Ah+\sqrt{1+A^2h^2})^{n+1} = \cancel{C_1}(Ah+\sqrt{1+A^2h^2})^{n-1} + 2Ah\,\cancel{C_1}(Ah+\sqrt{1+A^2h^2})^{n}.$$

Dividing through by $(\ )^{n-1}$ gives

$$(Ah+\sqrt{1+A^2h^2})^2 = 1 + 2Ah(Ah+\sqrt{1+A^2h^2})$$

or

$$A^2h^2 + 2Ah\sqrt{\ } + 1 + A^2h^2 = 1 + 2A^2h^2 + 2Ah\sqrt{\ } \quad \checkmark$$

5. $(Ah+\sqrt{1+A^2h^2})^n$ does $\to 1$ if $h \to 0$ with n fixed. However, we wish to take the limit so as to stay at a fixed x point. Thus, n is not a constant; rather, $x_n = nh$ so $n = x_n/h$ is a function of n, not a constant. In that case

$$(Ah+\sqrt{1+A^2h^2})^n = (Ah+\sqrt{1+A^2h^2})^{x_n/h} \sim (1+Ah)^{x_n/h}$$

and it is NOT obvious (nor even true) that the latter $\to 1$ as $h \to 0$.

8. (a) Improved Euler:

$$\begin{aligned} y_{n+1} &= y_n + (h/2)\{f(x_n, y_n) + f[x_{n+1}, y_n + hf(x_n, y_n)]\} \\ &= y_n + (h/2)[Ay_n + A(y_n + hAy_n)] \quad \text{for } y' = Ay \\ &= (1 + Ah + \tfrac{A^2h^2}{2})y_n = (1+\alpha+\alpha^2/2)y_n, \text{ where } \alpha = Ah. \end{aligned}$$

Thus,

$$y_n = (1+\alpha+\alpha^2/2)^n y_0 = e^{n\ln(1+\alpha+\alpha^2/2)} y_0 \sim e^{n\ln(1+Ah)} y_0 \sim e^{Anh} y_0 = y_0 e^{Ax_n},$$

so we do have convergence (for any A). But we're asked to consider stability. To do so, perturb y_0 by a small amount ϵ (roundoff error) and generate the subsequent y_n^*'s according to

$$y_{n+1}^* = (1+\alpha+\alpha^2/2)y_n^* \;; \quad y_0^* = y_0 - \epsilon$$

Subtracting the latter from

$$y_{n+1} = (1+\alpha+\alpha^2/2)y_n \;; \quad y_0 = y_0$$

gives

$$e_{n+1} = (1+\alpha+\alpha^2/2)\,e_n \;; \quad e_0 = \epsilon$$

on the e_n's.

The latter gives $e_n = (1+\alpha+\alpha^2/2)^n e_0$

Consider first the case where $A<0$. Then the exact solution $y_n = e^{Ax_n}$ starts out at $y_0=1$ and diminishes, so evidently we have stability if and only if $|1+\alpha+\alpha^2/2| < 1$; i.e., $-1 < 1+\alpha+\alpha^2/2 < 1$

↳ $\alpha+\alpha^2/2 \overset{\ddagger}{<} 0$, $Ah(1+\frac{Ah}{2}) < 0$ or, since $A<0$, we need $1-|A|h/2 > 0$ or $h < \frac{1}{2|A|}$.

↳ $Ah + \frac{A^2h^2}{2} + 2 > 0$.

Let $-|A|h + \frac{|A|^2h^2}{2} + 2 \equiv g(\alpha)$ where $\alpha = |A|h$.

$g'(\alpha) = 0 = \alpha - 1$ gives $\alpha = 1$.

$g''(1) = 1 > 0$, so g has a min. @ $\alpha = 1$ and it $= 3/2$ there.

Thus $g(\alpha) > 0$ for all α; hence, this inequality imposes no restriction.

The upshot is that if $A<0$ then we have stability "iff" (if and only if) $h < 1/2|A|$. Next, consider $A>0$. In that case $e_n = (1+\alpha+\alpha^2/2)^n e_0$ grows with n because $\alpha > 0$. (Or, the ‡ inequality, above, cannot be satisfied because $\alpha>0$.) However, if $A>0$ we don't need e_n to diminish with n as n increases because the solution $y_n = e^{Ax_n}$ itself grows exponentially. In this case let us proceed differently and observe that

$$\left|\frac{e_n}{y_n}\right| = \frac{(1+Ah+A^2h^2/2)^n|e_0|}{e^{Ax_n}} = \frac{(1+Ah+A^2h^2/2)^n}{e^{Ahn}}|e_0| = \left(\frac{1+Ah+\frac{1}{2}A^2h^2}{e^{Ah}}\right)^n|e_0| < |e_0|$$

because $e^{Ah} = 1 + Ah + \frac{1}{2}A^2h^2 + \cdots > 1 + Ah + \frac{1}{2}A^2h^2$. Thus, if $A>0$ then we have stability with no restriction on h.

In any event, then, (i.e., $A>0$ or <0) $h < \frac{2}{|A|} \equiv h_{cr}$ ensures stability, as for the ordinary Euler method.

(b) Modified Euler method:

$$\begin{aligned} y_{n+1} &= y_n + hf[x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}f(x_n,y_n)] \\ &= y_n + hA[y_n + \tfrac{h}{2}Ay_n] = (1 + Ah + \tfrac{1}{2}A^2h^2)y_n, \end{aligned}$$

which is the same as in part (a), so the final result is the same: $h < 2/|A| \equiv h_{cr}$ for stability.

9. (a) $y_{n+1} = y_n + (55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})\frac{h}{24}$ where $f_n \equiv f(x_n, y_n)$

and if $y' = Ay$, then $y_{n+1} = (1+\frac{55}{24}Ah)y_n - \frac{59}{24}Ahy_{n-1} + \frac{37}{24}Ahy_{n-2} - \frac{9}{24}Ahy_{n-3}$.

Then, $y_n = \rho^n$ gives

$$\rho^4 - (1+55\alpha)\rho^3 + 59\alpha\rho^2 - 37\alpha\rho + 9\alpha = 0. \quad (\alpha \equiv Ah/24)$$

$\alpha \to 0$ gives $\rho^4 - \rho^3 = 0$ so $\rho = 0,0,0,1$

and

$$y_n = C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n + C_4\rho_4^n$$
$$\sim 0 + 0 + 0 + C_4\enclose{circle}{1^n}$$

↖This could go to ∞, a finite value, or 0, depending on how close ρ_4 is to 1, so seek $\rho = 1 + a\alpha + \cdots$ and find a. Plugging,

$$(1+4a\alpha+\cdots) - (1+55\alpha)(1+3a\alpha+\cdots) + 59\alpha(1+2a\alpha+\cdots) - 37\alpha(1+a\alpha+\cdots) + 9\alpha = 0$$

α^0: $1-1=0$ ✓

α^1: $4a-3a-55+59-37+9=0 \rightarrow a=24.$

Thus,

$$y_n \sim C_4(1+24\alpha)^n = C_4(1+Ah)^n = C_4 e^{n\ln(1+Ah)} \sim C_4 e^{nAh} = C_4 e^{Ax}$$

for $A>0$ or $A<0$; hence, strong stability.

11. (b) $y_n = \rho^n$ gives $\rho^{n+2} - \rho^n = (\rho^2-1)\rho^n = 0$ so $\rho^2 - 1 = 0$, $\rho = \pm 1$.

Thus, $y_n = C_1(1)^n + C_2(-1)^n = C_1 + C_2(-1)^n$.

$$\left.\begin{aligned} y_0 &= 1 = C_1 + C_2 \\ y_1 &= 3 = C_1 - C_2 \end{aligned}\right\} \rightarrow C_1 = 2,\ C_2 = -1,$$

so $y_n = 2 - (-1)^n$.

(c) $y_n = \rho^n$ gives $\rho^{n+2} + \rho^{n+1} - 6\rho^n = (\rho^2+\rho-6)\rho^n = 0$

$\rho^2+\rho-6 = 0 \rightarrow \rho = 2, -3$

$$y_n = C_1 2^n + C_2(-3)^n$$
$$\left.\begin{aligned} y_0 &= 9 = C_1 + C_2 \\ y_1 &= -2 = 2C_1 - 3C_2 \end{aligned}\right\} \rightarrow C_1 = 5,\ C_2 = 4$$

$y_n = 5(2)^n + 4(-3)^n$.

(f) $y_n = \rho^n$ gives $\rho^{n+3} - \rho^{n+2} - 4\rho^{n+1} + 4\rho^n = (\rho^3 - \rho^2 - 4\rho + 4)\rho^n = 0$

$\rho^3 - \rho^2 - 4\rho + 4 = 0 \rightarrow \rho = 1, 2, -2$

$$y_n = C_1 1^n + C_2 2^n + C_3(-2)^n$$
$$\left.\begin{aligned} y_0 &= 3 = C_1 + C_2 + C_3 \\ y_1 &= 5 = C_1 + 2C_2 - 2C_3 \\ y_2 &= 9 = C_1 + 4C_2 + 4C_3 \end{aligned}\right\} \rightarrow C_1 = 1,\ C_2 = 2,\ C_3 = 0$$

so $y_n = 1 + 2^{n+1}$

12. (b) rsolve({y(n+2) - y(n) = 0, y(0) = 1, y(1) = 3}, y(n));

gives $y_n = 2 - (-1)^n$

13. $y_n = \rho^n$ gives $\rho^{n+2} - 2b\rho^{n+1} + b^2\rho^n = (\rho^2 - 2b\rho + b^2)\rho^n = 0$

$\rho^2 - 2b\rho + b^2 = 0 \rightarrow \rho = b, b.$

By analogy with ODE theory, try $y_n = (C_1 + C_2 n)b^n$

Plugging it in, $[C_1 + C_2(n+2)]b^{n+2} - 2b[C_1 + C_2(n+1)]b^{n+1} + b^2(C_1 + C_2 n)b^n = 0,$

$C_1(b^2 - 2b^2 + b^2) + C_2[(n+2)b^2 - 2(n+1)b^2 + nb^2] = 0$ ✓

14. Plugging it in gives

$$a_0(n)[C_1 y_{n+2}^{(1)} + C_2 y_{n+2}^{(2)} + Y_{n+2}] + a_1(n)[C_1 y_{n+1}^{(1)} + C_2 y_{n+1}^{(2)} + Y_{n+1}] + a_2(n)[C_1 y_n^{(1)} + C_2 y_n^{(2)} + Y_n] = f_n$$

$$C_1[\cancelto{0}{a_0(n)y_{n+2}^{(1)} + a_1(n)y_{n+1}^{(1)} + a_2(n)y_n^{(1)}}] + C_2[\cancelto{0}{a_0(n)y_{n+2}^{(2)} + a_1(n)y_{n+1}^{(2)} + a_2(n)y_n^{(2)}}]$$

$$+ [a_0(n)Y_{n+2} + a_1(n)Y_{n+1} + a_2(n)Y_n] = f_n,$$

so $C_1[0] + C_2[0] + [f_n] = f_n$ ✓

15. (b) Homogeneous solution: $y_n=\rho^n$ gives $\rho^{n+1}-2\rho^n=(\rho-2)\rho^n=0 \to \rho=2$ so $y_n=C_1 2^n$.

Particular solution: Seek $y_n = A\sin n + B\cos n$. Then,

$$A\sin(n+1)+B\cos(n+1)-2A\sin n-2B\cos n=3\sin n$$

$$A(\sin n\cos 1+\sin 1\cos n)+B(\cos n\cos 1-\sin n\sin 1)-2A\sin n-2B\cos n=3\sin n$$

$$\left.\begin{array}{ll}\sin n: & A\cos 1-B\sin 1-2A=3\\ \cos n: & A\sin 1+B\cos 1-2B=0\end{array}\right\}\to \begin{array}{l}A=3(2-\cos 1)/(4\cos 1-5)\\ B=3\sin 1/(4\cos 1-5)\end{array}$$

so $$y_n=C_1 2^n+\frac{3(2-\cos 1)}{4\cos 1-5}\sin n+\frac{3\sin 1}{4\cos 1-5}\cos n$$

(e) Homogeneous solution: $y_n=\rho^n$ gives $\rho^{n+2}+\rho^{n+1}-2\rho^n=(\rho^2+\rho-2)\rho^n=0\to\rho=1,-2$

so $y_n=C_1 1^n+C_2(-2)^n=C_1+C_2(-2)^n$

Particular solution: $y_n=An^2+Bn+C$. Plugging gives

$$A(n+2)^2+B(n+2)+C+A(n+1)^2+B(n+1)+C-2An^2-2Bn-2C=n^2$$

n^2: $A+A-2A=1$ ← Not working. The problem is that the C term in the assumed particular solution is duplicating the C_1 term in the homogeneous solution, so seek the particular solution as

n: etc

1: etc

$$y_n=n(An^2+Bn+C)$$

instead (by analogy with the method of undetermined coefficients for ODE's).

Plug: $(n+2)[A(n+2)^2+B(n+2)+C]+(n+1)[A(n+1)^2+B(n+1)+C]-2n(An^2+Bn+C)=n^2$

$$\begin{array}{ll}n^3: & A+A-2A=0\ \checkmark\end{array}$$

$$\left.\begin{array}{ll}n^2: & 2A+4A+B+A+2A+B-2B=1\\ n: & 8A+2B+4A+2B+C+2A+B+A+B+C-2C=0\\ 1: & 8A+4B+2C+A+B+C=0\end{array}\right\}\to\begin{array}{l}A=1/9\\ B=-5/18\\ C=7/54\end{array}$$

so

$$y_n=C_1+C_2(-2)^n+\frac{n}{9}\left(n^2-\frac{5}{2}n+\frac{7}{6}\right).$$

16. (b) `rsolve(y(n+1)-2*y(n)=3*sin(n),y(n));`

gives $$y_n=y(0)\,2^n+3\,\frac{4\sin(n-1)\cos 1-\sin(n-1)-2\sin(n-2)}{-5+4\cos 1}-3\,\frac{\sin 1\ 2^n}{-5+4\cos 1}$$

To see that this agrees with our result, above, in 15(b), re-express it as

$$y_n=\underbrace{\left[y(0)-\frac{3\sin 1}{4\cos 1-5}\right]}_{C_1,\ \text{say}}2^n+\frac{3}{4\cos 1-5}\Big[4\sin n\,\overset{(1+\cos 2)/2}{\cancel{\cos^2 1}}-\overset{2\sin 2}{\boxed{4\sin 1\cos 1}}\cos n-\sin n\cos 1+\sin 1\cos n$$

$$-2\sin n\cos 2+2\sin 2\cos n\Big]$$

$$=C_1 2^n+\frac{3}{4\cos 1-5}\Big[(2+\cancel{2\cos 2}-\cos 1-\cancel{2\cos 2})\sin n+(\cancel{2\sin 2}+\sin 1-\cancel{2\sin 2})\cos n\Big]\ \checkmark$$

CHAPTER 7

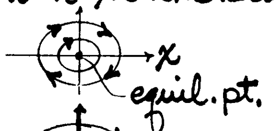
Section 7.2

1. $\frac{m}{k}y^2+x^2=A^2$, $y=\frac{dx}{dt}=\sqrt{\frac{k}{m}}\sqrt{A^2-x^2}$, $\int\frac{dx}{\sqrt{A^2-x^2}}=\sqrt{\frac{k}{m}}\int dt$, arcsin $\frac{x}{A}=\sqrt{\frac{k}{m}}t+B$, $x(t)=A\sin(\sqrt{\frac{k}{m}}t+B)$.

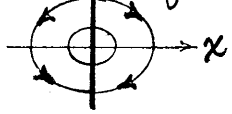
2. $m^2+x^2(x+1)^2\sim m^2+4$ as $x\to 1$.

3. The hard spring case is given by (6): $x''+x+x^3=0$. For small x, $x+x^3\sim x$ and the frequency of small motions is ~ 1, but for larger motions the x^3 term causes the effective stiffness (average slope of $x+x^3$) to increase.

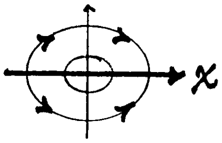
4. (a) $dy/dx=-x/y$, $x^2+y^2=C^2$



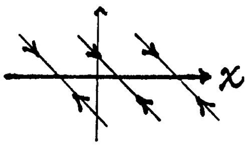
(b) $xy=0, -x^2=0$ give equil. pts all along $x=0$. $dy/dx=-x^2/xy$ $=-x/y$ gives $x^2+y^2=C^2$ again.



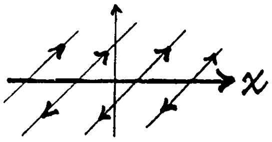
(c) $y^2=0, -xy=0$ give equil. pts all along $y=0$. $dy/dx=-xy/y^2$ $=-x/y$ gives $x^2+y^2=C^2$ again.



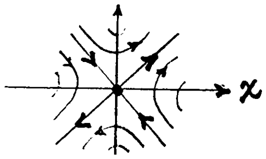
5. (a) $dy/dx=-y/y=-1$ gives $y=-x+C$, and $y=0, -y=0$ gives equil. pts all along $y=0$.



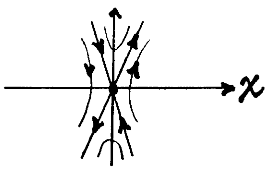
(b) $y=0, y=0$ give sing. (equil.) pts all along $y=0$. $dy/dx=y/y=1$ so $y=x+C$



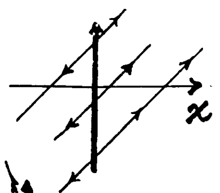
(c) Equil pt. at origin. $dy/dx=x/y$ $x^2-y^2=C^2$.



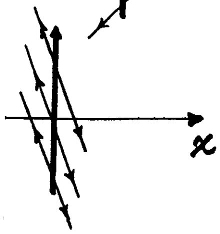
(d) Equil. pt at origin. $dy/dx=9x/y$ $x^2-(\frac{y}{3})^2=C^2$



(e) Equil. pts all along $x=0$.
$dy/dx=1$, $y=x+C$.

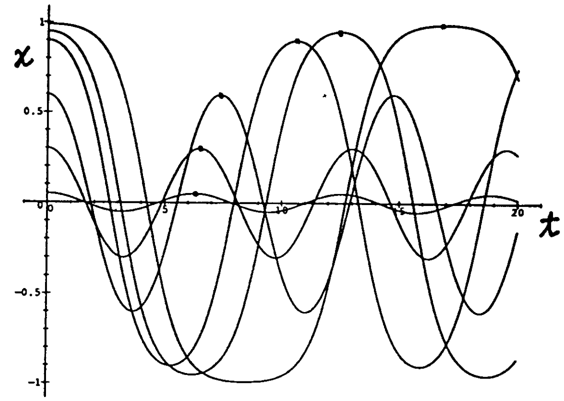


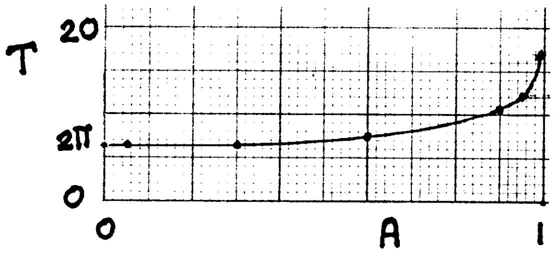
(f) Equil. pts all along $x=0$.
$dy/dx=-4$, $y=-4x+C$



6. I used $x=.05,.3,.6,.9,.95,.99$

```
phaseportrait([y,-x+x^3],
[t,x,y],t=0..20,{[0,.05,0],[0,.3,0],
[0,.6,0],[0,.9,0],[0,.95,0],[0,.99,0]},
stepsize=.05, scene=[t,x]);
```



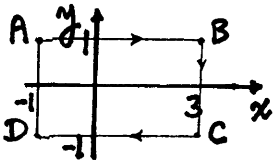


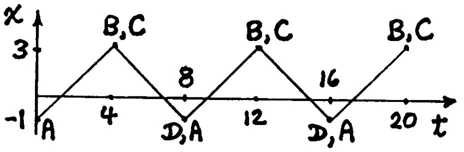
The period T does → 2π as A→0, and does give signs of tending to ∞ as A→1.

7. For a motion to be periodic it is necessary (but not sufficient) that its phase trajectory be closed and, outside the football, the trajectories are not closed.

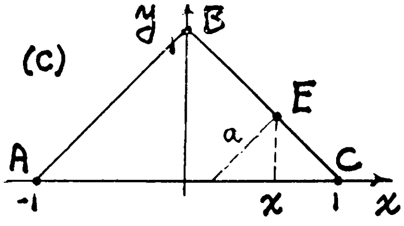
8. (a) By similar triangles, $a/y = s'/x'$, and since $x'=y$ it follows that $s'=a$.

(b)





(c)

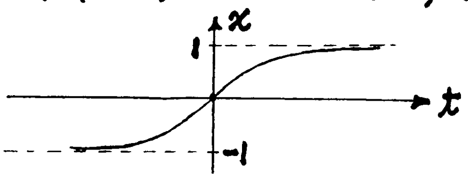


BC ($0<t<\infty$):

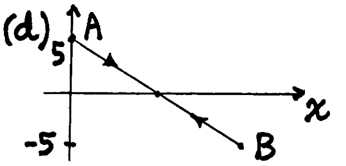
$s' = a = \sqrt{2}(1-x)$ and $x' = \frac{s'}{\sqrt{2}}$, so $x' = 1-x$, $x(t) = 1-e^{-t}$

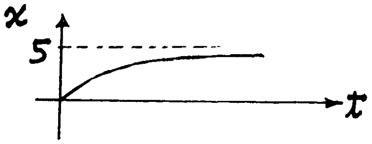
AB ($-\infty<t<0$):

$s' = a = \sqrt{2}(x+1)$ and $x' = s'/\sqrt{2}$, so $x' = x+1$, $x(t) = e^{t}-1$

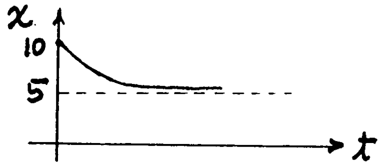


(d)





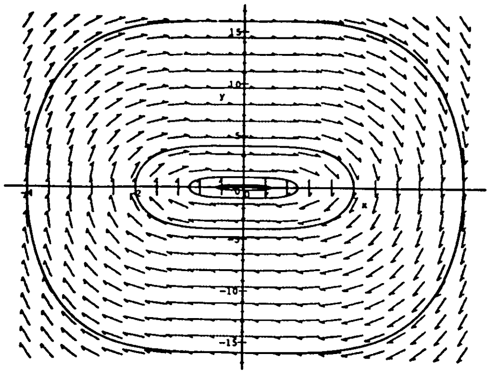
(e)



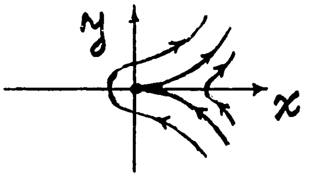
9. (a) $\left.\begin{aligned} x' &= y \\ y' &= -2x^3 \end{aligned}\right\} \frac{dy}{dx} = -\frac{2x^3}{y}$

so $y^2 + x^4 = C$. Over $-1<x<1$ the x^4 term becomes negligible as C is increased (i.e., for larger & larger motions), so $y \sim \sqrt{C}$ over $-1<x<1$ as $C \to \infty$.

(b)
```
phaseportrait([y,-2*x^3],[t,x,y],t=0..20,
 {[0,.5,0],[0,1,0],[0,2,0],[0,4,0]}, stepsize=.05,
 scene=[x,y],
 arrows=THIN);
```

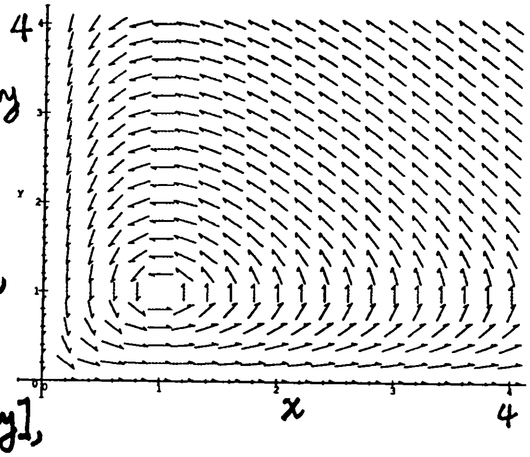


10. $x'=y$, $y'=-x^2$; $\frac{dy}{dx}=-\frac{x^2}{y}$,
$\frac{y^2}{2}+\frac{x^3}{3}=A$, $y=\pm\sqrt{\frac{2x^3}{3}+B}$.



Trajectories through the origin (which is an equil. pt) are $y=\pm\sqrt{2x^3/3}$.

11. $x'=(1-y)x$
$y'=-(1-x)y$
has equil. pts at (0,0) and (1,1).



```
phaseportrait([(1-y)*x,
-(1-x)*y],[t,x,y],
t=0..2, {[0,0,0]}, x=0
..4, y=0..4, scene=[x,y],
arrows=THIN);
```

No, one really can't distinguish periodic orbits from very weak spirals, by eye, from a lineal element field.

Section 7.3

1. (a) $0=0$ and $2x-y=0$ give the line of singular pts. $y=2x$. Hence, not isolated.
(b) $2x-2y=0$ and $x-y=0$ give the line of sing. pts $y=x$. Hence, not isolated.
(c) The intersection of the unit circle and the line $y=x$ gives two isolated sing. pts: $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$.
(d) Isolated sing. pts at $(-n\pi, n\pi)$, for $n=0,\pm1,\pm2,\ldots$
(e) Isolated sing. pt at $x=y=0$.
(f) Isolated sing. pts at $(1,0)$ and $(-1,0)$.
(g) Isolated sing. pts at $(2\sqrt{2},\sqrt{2})$ and $(-2\sqrt{2},-\sqrt{2})$.
(h) $x=y+n\pi/2$ $(n=0,\pm1,\pm2,\cdots)$ and $xy=1$ give $y^2+\frac{n\pi}{2}y-1=0$, so have isolated sing. pts. at

$\left(\frac{n\pi}{4}+\sqrt{\ },\ -\frac{n\pi}{4}+\sqrt{\ }\right)$ and $\left(\frac{n\pi}{4}-\sqrt{\ },\ -\frac{n\pi}{4}-\sqrt{\ }\right)$, where $\sqrt{\ }\equiv\sqrt{(n\pi/4)^2+1}$ and $n=0,\pm1,\pm2,\ldots$

2. Yes. For ex., $x'=x^2+y^2$ and $y'=x^2+y^2+1$ has none.

3. $$\begin{array}{l|l} c & (D-a)X-bY=0 \\ D-a & -cX+(D-d)Y=0 \end{array}\Bigg\}\to [(D-a)(D-d)-bc]Y=0$$ & similarly ["] $X=0$.

Thus, $Y''-(a+d)Y'+(ad-bc)Y=0$ and $Y=e^{\lambda t}$ gives $\lambda^2-(a+d)\lambda+(ad-bc)=0$, $\lambda=\frac{a+d\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$ $=[a+d\pm\sqrt{(a-d)^2+4bc}]/2$. ✓ To find C_3, C_4 in terms of C_1, C_2, put (10) into (9): Putting into (9a) gives

$$\lambda_1 C_1 e^{\lambda_1 t}+\lambda_2 C_2 e^{\lambda_2 t}=aC_1e^{\lambda_1 t}+aC_2e^{\lambda_2 t}+bC_3e^{\lambda_1 t}+bC_4e^{\lambda_2 t}$$

and since $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are LI, it follows that

$$\left.\begin{array}{l}\lambda_1 C_1=aC_1+bC_3\\ \lambda_2C_2=aC_2+bC_4\end{array}\right\}\to\begin{array}{l}C_3=\left(\frac{\lambda_1-a}{b}\right)C_1\\ C_4=\left(\frac{\lambda_2-a}{b}\right)C_2.\end{array}$$

Putting (10) into (9b) would give the same result.

4. No. For a given ϵ, the corresponding δ could be enormous.

5. (b) Let $\cos\alpha\equiv c$, $\sin\alpha\equiv s$. (5.1) into (16) gives

$$\bar{x}'c-\bar{y}'s=\frac{\sqrt{8}}{3}(\bar{x}c-\bar{y}s)+\frac{4}{3}(\bar{x}s+\bar{y}c)$$
$$\bar{x}'s+\bar{y}'c=-\frac{11}{3}(\bar{x}c-\bar{y}s)-\frac{\sqrt{8}}{3}(\bar{x}s+\bar{y}c).$$

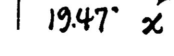
Elimination gives

$$\bar{x}'=\left(\frac{\sqrt{8}c^2+4cs-11cs-\sqrt{8}s^2}{3}\right)\bar{x}+\left(\frac{4c^2-\sqrt{8}cs+11s^2-\sqrt{8}cs}{3}\right)\bar{y}$$
$$\bar{y}'=\left(\frac{-\sqrt{8}sc-4s^2-11c^2-\sqrt{8}sc}{3}\right)\bar{x}+\left(\frac{-4cs+\sqrt{8}s^2+11cs-\sqrt{8}c^2}{3}\right)\bar{y}.$$

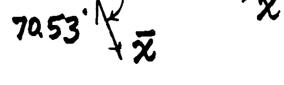
To have $\bar{x}'=\beta^2\bar{y}$ and $\bar{y}'=-\gamma^2\bar{x}$, set

$$\left.\begin{array}{l}\sqrt{8}c^2+4cs-11cs-\sqrt{8}s^2=0 \text{ or, } \sqrt{8}(c^2-s^2)-7cs=0\\ \text{\& } -4cs+\sqrt{8}s^2+11cs-\sqrt{8}c^2=0 \text{ or, } \sqrt{8}(s^2-c^2)+7cs=0\end{array}\right\rangle \text{same}$$

With $s/c=\tan\alpha\equiv t$, $\sqrt{8}(1-t^2)-7t=0$ so $t=1/\sqrt{8},-\sqrt{8}$ so $\alpha=\tan^{-1}(1/\sqrt{8}),\tan^{-1}(-\sqrt{8})\ =19.47^\circ,-70.53^\circ$, which are equivalent:



&

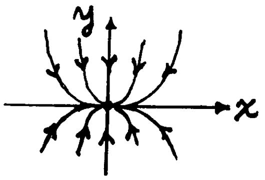


Finally, though we don't care about the values of β and γ we do need to verify that $\beta^2 = (4C^2 - \sqrt{8}CS + 11S^2 - \sqrt{8}CS)/3$ and $\gamma^2 = (4CS - \sqrt{8}S^2 - 11CS + \sqrt{8}C^2)/3$ are >0. Well, $t = \frac{1}{\sqrt{8}}$ gives $\beta^2 = \frac{4 - 2\sqrt{8}t + 11t^2}{3/C^2} = \frac{4 - 2\sqrt{8}(1/\sqrt{8}) + 11/8}{3/(8/9)} = 1$ ✓

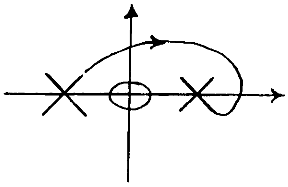
and $\gamma^2 = \frac{2\sqrt{8}t + 4t^2 + 11}{3/C^2} = 4$ ✓ Similarly, $t = -\sqrt{8}$ gives $\beta^2 > 0$ and $\gamma^2 > 0$.

6. $\frac{y}{x} = \frac{C_2 + \lambda C_1 + \lambda C_2 t}{C_1 + C_2 t} \sim \lambda$ as $t \to \infty$, $y \sim \lambda x$.

7. From below (19a), $\lambda_1 \to 0$ and $\lambda_2 \to -\infty$ as $p \to \infty$, so phase portrait is as sketched:



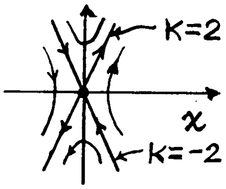
8. No, I can't. Suppose we try this, → for instance. Then we find that we cannot complete the sketch without introducing at least one additional singular point.



9. Use $\lambda = [a + d \pm \sqrt{(a-d)^2 + 4bc}\,]/2$.

(a) $\lambda = (2 \pm \sqrt{16})/2 = 3, -1 \rightarrow$ saddle. Seeking $y = Kx$,

$$\left.\begin{aligned} x' &= x + Kx \\ y' &= Kx' = 4x + Kx \end{aligned}\right\} \rightarrow \begin{aligned} 1 + K &= \frac{4+K}{K} \\ K^2 + K &= 4 + K \\ K &= \pm 2. \end{aligned}$$

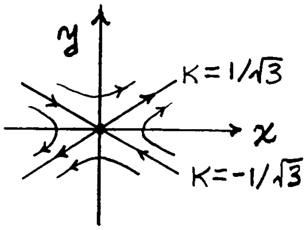


On $K = +2$, $x' = 3x \rightarrow$ unstable manifold, and on $K = -2$, $x' = -x \rightarrow$ stable manifold.

(d) $\lambda = (-2 \pm \sqrt{0 + 12})/2 = -1 \pm \sqrt{3}$ $\rightarrow$ saddle. Seeking $y = Kx$,

$$\left.\begin{aligned} x' &= -x + 3Kx \\ Kx' &= x - Kx \end{aligned}\right\} \rightarrow \begin{aligned} -1 + 3K &= \frac{1-K}{K} \\ 3K^2 &= 1 \end{aligned}$$

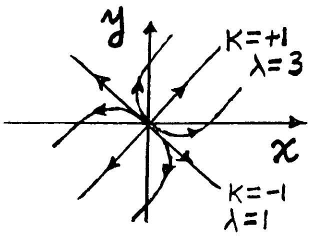
$K = +1/\sqrt{3}$ (unstable), $-1/\sqrt{3}$ (stable).



(g) $\lambda = (4 \pm \sqrt{0+4})/2 = 3, 1$
$\rightarrow$ unstable node. Seeking
$y = Kx$ gives

$$\left.\begin{array}{l} x' = 2x + Kx \\ Kx' = x + 2Kx \end{array}\right\} \begin{array}{c} 2 + K = \frac{1+2K}{K} \\ K = \pm 1 \end{array}$$

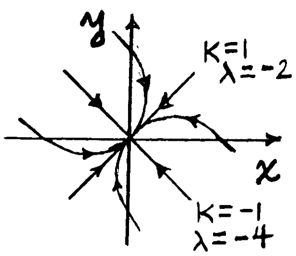
$K = +1$ gives $\lambda = 3$, $K = -1$ gives $\lambda = 1$



(j) $\lambda = (-6 \pm \sqrt{4})/2 = -2, -4$
$\rightarrow$ stable node. $y = Kx$ gives

$$\left.\begin{array}{l} x' = -3x + Kx \\ Kx' = x - 3Kx \end{array}\right\} \rightarrow \begin{array}{c} -3 + K = \frac{1-3K}{K} \\ K = \pm 1 \end{array}$$

$K = +1$ gives $\lambda = -2$, $K = -1$ gives $\lambda = -4$



NOTE: How did we know that the "pitchfork" runs NE by SW (northeast by southwest) in (g), and NW by SE in (j)? In (g), $x = Ae^{3t} + Be^{t}$ and $y = x' - 2x = Ae^{3t} - Be^{t}$, so $y \sim x$ as $t \rightarrow +\infty$, whereas in (j) $x = Ae^{-2t} + Be^{-4t}$ and $y = x' + 3x = Ae^{-2t} - Be^{-4t}$, so $y \sim -x$ as $t \rightarrow -\infty$ (and $y \sim x$ as $t \rightarrow +\infty$).

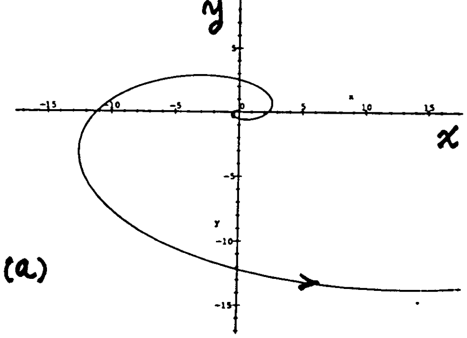
10. $y = Kx$ gives $x' = ax + Kbx$, $y' = Kx' = cx + Kdx$, so $a + bK = (c + dK)/K$ gives quadratic eqn. on K, which can have 0, 1, or 2 real roots.

11. Use $\lambda = [a + d \pm \sqrt{(a-d)^2 + 4bc}\,]/2$.
   (a) $\lambda = (2 \pm \sqrt{0 - 16})/2 = 1 \pm 2i \rightarrow$ unstable focus.
   (d) $\lambda = (0 \pm \sqrt{4 - 12})/2 = \pm \sqrt{2}\, i \rightarrow$ center
   (g) $\lambda = (5 \pm \sqrt{1+4})/2 = (5 \pm \sqrt{5})/2 \rightarrow$ unstable node.

12. Let us do (a), (d), and (g). (a) and (d) are easy (in terms of selecting initial points that will reveal all the key features, but for (g) we need to locate the straight line trajectories: $y = Kx$ gives

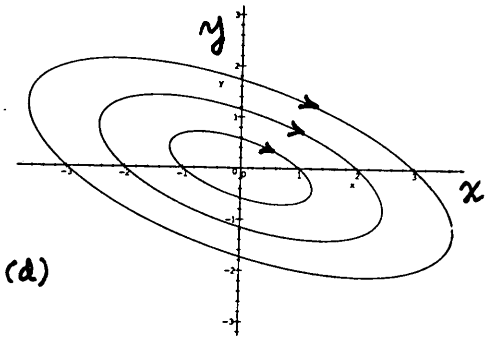
$$\left.\begin{array}{l} x' = 2x - Kx \\ Kx' = -x + 3Kx \end{array}\right\} \rightarrow 2 - K = \frac{-1+3K}{K}, \quad K = .618034 \text{ and } -1.618034$$

For (a), use `phaseportrait([x-4*y,x+y],[t,x,y], t=0..5, {[0,.1,0]}, stepsize=.05, x=-16..16, y= -16..8, scene=[x,y]);`
For (d), use `phaseportrait([x+3*y,-x-y],[t,x,y], t=0..5, {[0,1,0],[0,2,0],[0,3,0]}, stepsize=.05, x= -3.5..3.5, y=-3..3, scene=[x,y]);`
For (g), use `phaseportrait([2*x-y,-x+3*y],[t,x,y], t=-5..5, {[0,1,.618034],[0,-1,-.618034], [0,1,-1.618034],[0,-1,1.68034],[0,1,0],[0,-1,0], [0,0,1],[0,0,-1]}, stepsize=.05, x=-3..3, y=-3..3, scene=[x,y]);`

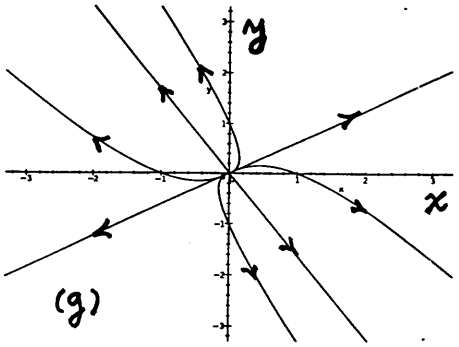


(a)

How do we know the direction of the arrows? From $x'=x+3y$, say, we see that for $x=0$ and $y>0$, we have $x'>0$, so x is increasing.

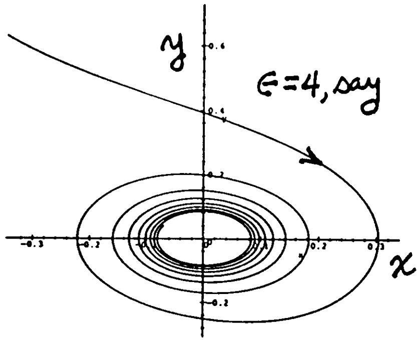


(d)



(g)

Section 7.4



1. (a) phaseportrait([y,-x-4*y^3],[t,x,y],t=0..50,{[0,-.4,.6]}, stepsize=.05, scene=[x,y]);

(b) When $|x'|=|y|>1$, the cubic damping is much greater than linear, but once the motion is reduced the damping grows weaker and weaker since $x'^3 \to 0$ as $x'=y\to 0$. This weakening is clearly seen in the phaseportrait output in (a).

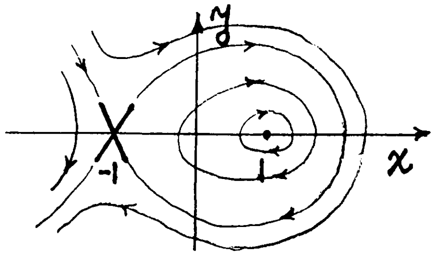
(c) again, the linearized version implies a center at (0,0), but that borderline case (Fig. 1) could be a center <u>or</u> a focus. Just as the $\epsilon>0$ case was a stable focus, a phaseportrait plot for $\epsilon<0$ will no doubt reveal an <u>unstable</u> focus.

2. (a) $y=0$ & $x^4-1=0$ gives sing. pts @ $(-1,0)$ and $(1,0)$.

$(-1,0)$: $$\left.\begin{array}{l} x'=0(x+1)+y \\ y'=4(x+1)+0y \end{array}\right\} \begin{array}{l} a=0, b=1, c=4, d=0, \text{ so} \\ \lambda=(0\pm\sqrt{0+16})/2=\pm 2 \to \text{saddle.} \end{array}$$

$(1,0)$: $$\left.\begin{array}{l} x'=0(x-1)+y \\ y'=-4(x-1)+0y \end{array}\right\} \begin{array}{l} a=0, b=1, c=-4, d=0, \text{ so} \\ \lambda=(0\pm\sqrt{0-16})/2=\pm 2i \to \text{center} \end{array}$$

or focus, but since system is equivalent to $x''+x^4=1$, which is conservative, we know it must be a center.



(b) $(1,1)$: $$\left.\begin{aligned} x' &= 0(x-1)-2(y-1) \\ y' &= -1(x-1)+0(y-1) \end{aligned}\right\} \lambda = (0 \pm \sqrt{0+8})/2 \rightarrow \text{saddle}$$

$(1,-1)$: $$\left.\begin{aligned} x' &= 0(x-1)+2(y+1) \\ y' &= -1(x-1)+0(y+1) \end{aligned}\right\} \lambda = (0 \pm \sqrt{0-8})/2 \rightarrow \text{center or focus}$$

(c) $(-1,0)$: $$\left.\begin{aligned} x' &= 0(x+1)+1(y) \\ y' &= 1(x+1)+0(y) \end{aligned}\right\} \lambda = (0 \pm \sqrt{0+4})/2 \rightarrow \text{saddle}$$

$(1,0)$: $$\left.\begin{aligned} x' &= 0(x-1)+1(y) \\ y' &= -1(x-1)+0(y) \end{aligned}\right\} \lambda = (0 \pm \sqrt{0-4})/2 \rightarrow \text{center or focus}$$

actually, it will be a center because the system is equivalent to $x'' = (1-x^2)/(1+x^2)$, which is conservative.

(d) Sing. pts at $x = y = n\pi/2$ $(n = 0, \pm 1, \pm 2, \dots)$

$$x' = (x - n\pi/2) - (y - n\pi/2)$$
$$y' = \cos n\pi\,(x - n\pi/2) + \cos n\pi\,(y - n\pi/2)$$

so $a = 1$, $b = -1$, $c = d = \cos n\pi = (-1)^n$, so

$$\lambda = (1 + (-1)^n \pm \sqrt{(1-(-1)^n)^2 - 4(-1)^n})/2 .$$

n even $\Rightarrow \lambda = 1 \pm i \rightarrow$ unstable focus

n odd $\Rightarrow \lambda = \pm\sqrt{2} \rightarrow$ saddle

(e) $(0,0)$: $$\left.\begin{aligned} x' &= 0x + 1y \\ y' &= -x - 2y \end{aligned}\right\} \lambda = (-2 \pm \sqrt{4-4})/2 = -1, -1$$

The repeated root means that $p^2 - 4q = 0$: borderline case. Specifically, $p = a + d = -2$ and $q = ad - bc = 1$, so have stable focus or stable proper node or stable improper node.

$(-1, 1/2)$: $$\left.\begin{aligned} x' &= 1(x+1) + 0(y - 1/2) \\ y' &= -(x+1) - 2(y - 1/2) \end{aligned}\right\} \lambda = (-2 \pm \sqrt{9})2 \rightarrow \text{saddle}$$

$(1, -1/2)$: $$\left.\begin{aligned} x' &= +(x-1) + 0(y + 1/2) \\ y' &= -(x-1) - 2(y + 1/2) \end{aligned}\right\} \lambda = (-1 \pm \sqrt{9})/2 \rightarrow \text{saddle}$$

(f) $(0,0)$: $$\left.\begin{aligned} x' &= 0x + 1y \\ y' &= -x + 2y \end{aligned}\right\} \lambda = (2 \pm \sqrt{4-4})/2 = 1, 1$$

Another borderline case: $p=a+d=2$, $q=ad-bc=1$, $p^2=4q$, and we have unst. focus or unst. proper node or unst. improper node

$$(1,1/2):\ \left.\begin{aligned} x' &= -(x-1)+0(y-1/2) \\ y' &= -(x-1)+2(y-1/2) \end{aligned}\right\} \begin{aligned} &\lambda=(1\pm\sqrt{9})/2 \\ &\rightarrow \text{saddle} \end{aligned}$$

$$(-1,-1/2):\ \left.\begin{aligned} x' &= -(x+1)+0(y+1/2) \\ y' &= -(x+1)+2(y+1/2) \end{aligned}\right\} \begin{aligned} &\lambda=(1\pm\sqrt{9})/2 \\ &\rightarrow \text{saddle} \end{aligned}$$

(g) $$(0,0):\ \left.\begin{aligned} x' &= -2x-y \\ y' &= x+0y \end{aligned}\right\} \lambda=(-2\pm\sqrt{4-4})/2=-1,-1$$

Borderline: $p=a+d=-2$, $q=ad-bc=1$, so stable focus or stable proper node or stable improper node *

(j) $$(0,0):\ \left.\begin{aligned} x' &= 0x-2y \\ y' &= 2x-y \end{aligned}\right\} \begin{aligned} &\lambda=(-1\pm\sqrt{1-16})/2 \\ &\rightarrow \text{stable focus} \end{aligned}$$

$$(4,8):\ \left.\begin{aligned} x' &= 8(x-4)-2(y-8) \\ y' &= 2(x-4)-(y-8) \end{aligned}\right\} \begin{aligned} &\lambda=(7\pm\sqrt{81-16})/2 \\ &\rightarrow \text{saddle} \end{aligned}$$

(m) $x+2y=0 \rightarrow x=-2y$. Then, $-x-\sin y$ is $2y-\sin y = 0$ at y's such that $2y=\sin y$. Graphs of $2y$ and $\sin y$ show that the only root is $y=0$. Thus,

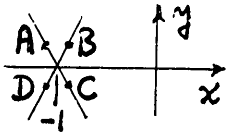
$$(0,0):\ \left.\begin{aligned} x' &= x+2y \\ y' &= -x-y \end{aligned}\right\} \begin{aligned} &\lambda=(0\pm\sqrt{4-8})/2 \\ &\rightarrow \text{center or focus} \end{aligned}$$

3. (a) From our solution to Exercise 2(a) we know we have a saddle at $(-1,0)$ and a center at $(1,0)$ so to generate the phase portrait all we need is the stable and unstable manifolds (of the linearized version) at $(-1,0)$. Putting $Y=y=K(x+1)=KX$ in $x'=0(x+1)+y$ and $y'=4(x+1)+0y$ gives $X'=0X+KX$ and $KX'=4X+0X$, so $0+K=(4+0)/K$, $K=\pm 2$. Thus, as initial points on the stable and unst. manifolds through the saddle, use $(x,y)=$ as

---

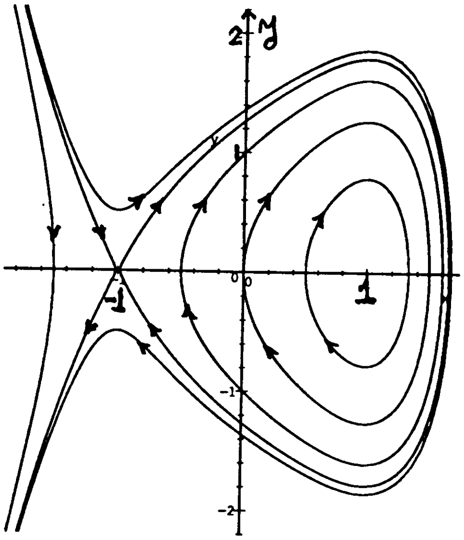
* a phaseportrait plot shows it to be a stable improper node.

shown:

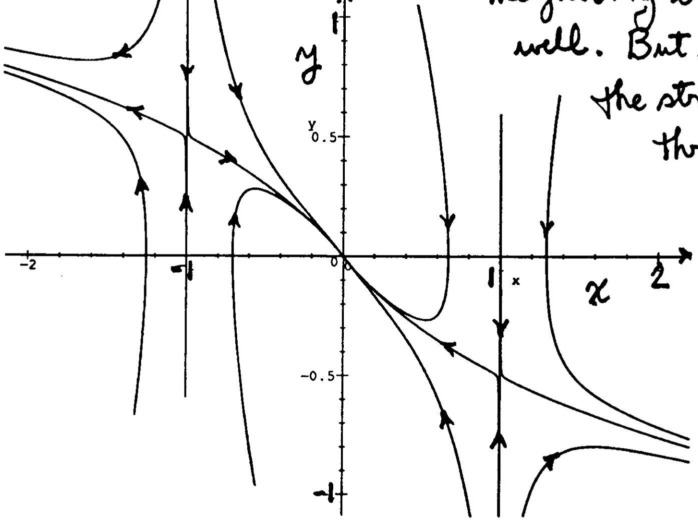


$A=(-1.02,.02)$, $B=(-.98,.02)$, $C=(-.98,-.02)$, $D=(-1.02,-.02)$

```
phaseportrait([y,1-x^4],
[t,x,y], t=-2..10, {[0,-1.02,
.02],[0,-.98,.02],[0,-.98,-.02],
[0,-1.02,-.02],[0,-1.5,0],[0,0,0],
[0,.5,0],[0,-1,.5],[0,-1,-.5],
[0,-.5,0]}, stepsize=.05, x=
-2.3, y=-2..2, scene=[x,y]);
```



(e) From our solution to Exercise 2(e) we know we have a stable focus or node @(0,0) and saddles @ (-1,1/2) & (1,-1/2). Instead of finding the straight line trajectories thru those pts, as above in (a), I first used phaseportrait to get lineal element field, which revealed vertical trajectories thru the saddles, and *seemed* to show horizontal ones as well. Then I ran phasportrait with these initial pts: [x,y] = [-1.02,.5],[-.98,.5],[-1,.48],[-1,.52],[1.02,-.5],[.98,-.5],[1,-.48],[1,-.52], and clicked on 8 other pts so as to give some neighboring trajectories as well. But let us also find the straight line trajs. thru [-1,1/2] analytically. Recall, from 2(e), the linearized equations at that singular point:



$$\left.\begin{array}{l} x' \approx (x+1) + 0(y-1/2) \\ y' \approx -(x+1) - 2(y-1/2) \end{array}\right\} \text{or, } \begin{array}{l} X' = X + 0Y \\ Y' = -X - 2Y. \end{array}$$

If we seek $Y = KX$ line solutions, we get $X' = X$, $KX' = -X - 2KX$ so $1 = (-1-2K)/K$ or, $K = -1/3$. But there should be a second one. The second one is simply the line $X = 0$ $(x = -1/2)$. That one has an infinite slope, so we can't find it by seeking $Y = KX$ — though we could by seeking $X = KY$: Then

$$KY' = KY$$
$$Y' = -KY - 2Y,$$

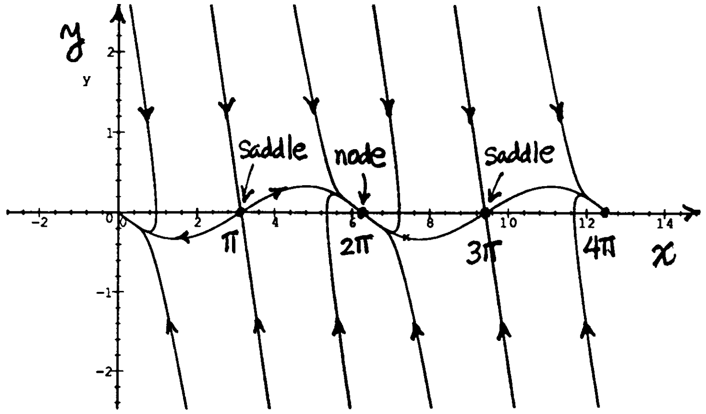
which gives $\frac{K}{K} = -K - 2$ or $K = -3$ (as above), but also the solution $K = 0$, $Y = Ce^{-2t}$.

4. (a) no; not of the form $x'' = f(x)$ due to the x' term.
(b) no (c) yes (d) no

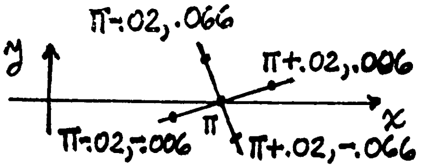
5. (a) & (b): See Figs. 5 & 6. Let's focus on:
(c) $r_{crit} = 2$, so let's choose $r = 3$, say. The flow will be similar to that in Fig. 6, but without the oscillations about $(0,0)$, $(2\pi, 0)$, $(4\pi, 0)$, ... Mostly, what we need is the straight line solns. thru the saddles — say at $(\pi, 0)$ and $(3\pi, 0)$. It'll suffice to study $(\pi, 0)$. The linearized eqns. there are

$$x' = y \approx 0(x-\pi) + 1y$$
$$y' = -3y - \sin x \approx 1(x-\pi) - 3y$$

or $X' = 0X + 1Y$, $Y' = 1X - 3Y$.
$Y = KX$ gives $X' = KX$, $KX' = X - 3KX$, so $K = (1-3K)/K$, $K \approx .3028$



and −3.3028. So choose initial pts as shown. →



Similarly for the saddle @ 3π.
Thus, I ran phaseportrait([y,-3*y-sin(x)],[t,x,y], t=-35..35, {[0,3.16159,.006],[0,3.16159,-.066], [0,3.12159,-.006],[0,3.12159,.066],[0,9.44478,.006], [0,9.44478,-.066],[0,9.40478,-.006],[0,9.40478,.066]}, stepsize=.05, x=-2..14, y=-3..3, scene=[x,y]);
Then I clicked on 8 more initial points to "fill in" the gaps. Result is shown above.

6.(a) If $r = .3$, then (32) gives $S_+ = (x_+, y_+) = (3, .9)$ and $S_- = (x_-, y_-) = (1/3, .1)$. Expand and linearize (31) about $S_\pm$:

$$x' = -r(x - x_\pm) + (y - y_\pm)$$
$$y' = \frac{2x_\pm}{(1+x_\pm^2)^2}(x - x_\pm) - (y - y_\pm)$$

At S_+ this becomes $\left.\begin{array}{l} X' = -.3X + Y \\ Y' = .06X - Y \end{array}\right\}$ $\lambda = (-1.3 \pm \sqrt{.49 + .24})/2 \approx -.22, -1.08$, implies a stable improper node.

Seek straight-line trajectories thru S_+: $Y = kX$,
$\left.\begin{array}{l} X' = -.3X + kX \\ kX' = +.06X - kX \end{array}\right\}$ so $-.3 + k = (.06 - k)/k$, gives $k = -.7772$ & $.0772$

At S_- we have $\left.\begin{array}{l} X' = -.3X + Y \\ Y' = .54X - Y \end{array}\right\}$ $\lambda = (-1.3 \pm \sqrt{.49 + 2.16})/2 \approx .16, -1.46$, so saddle

Seek straight line trajectories thru S_-: $Y = kX$,
$\left.\begin{array}{l} X' = -.3X + kX \\ kX' = .54X - kX \end{array}\right\}$ so $-.3 + k = (.54 - k)/k$, so $k = -1.16394$ & $.46394$

↳ gives $X' = -.3X + kX = .16X$, $X = Ce^{+.16t}$, so $k = .46$ gives the unst. manifold

and $K = -1.16$ gives $X' = -.3X + KX$
$= -1.46X$ gives $X = Ce^{-1.46t}$,
corresponds to the stable
manifold.

Now consider the sing. at the origin. Linearizing there, gives

$$\left.\begin{aligned} x' &= -.3x + y \\ y' &= 0x - y \end{aligned}\right\} \lambda = (-1.3 \pm \sqrt{.49})/2 = -1, -.3, \text{ so } (0,0) \text{ is a stable node.}$$

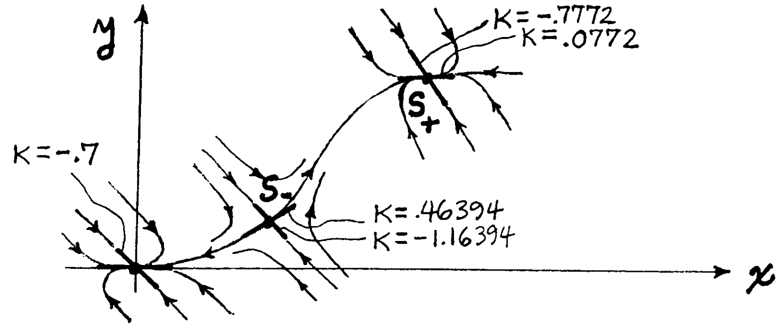
If we seek straight line trajs. thru (0,0) by $y = Kx$, we obtain

$$x' = -.3x + Kx$$
$$Kx' = \quad -Kx,$$

comparison of which gives $-.3 + K = -K/K$, so $K = -.7$ but there should be a second straight line traj. as well. Can always fall back on simply looking at the solution. Solution of $x' = -.3x + y$, $y' = -y$ is $y(t) = Ae^{-t}$ and $x(t) = -(A/.7)e^{-t} + Be^{-.3t}$ or,

$$\left.\begin{aligned} y(t) &= Ae^{-t} + B(0)e^{-.3t} \\ x(t) &= -(A/.7)e^{-t} + Be^{-.3t}. \end{aligned}\right\} *$$

If initial conditions give $B = 0$, then $*$ gives $y = -.7x$, as found above. If they give $A = 0$, then $*$ gives $y = 0x = 0$; i.e, the x axis. Thus (not to scale):



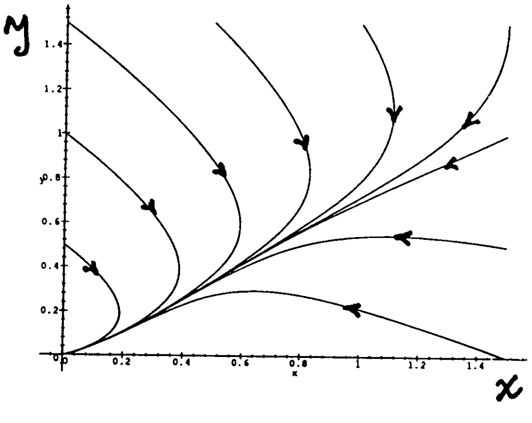
(b) If $r = 1/2$, (32) gives $S_{\pm} = (1, 1/2)$. Linearizing (31) about that point gives $X' = -\frac{1}{2}X + Y$, $Y' = \frac{1}{2}X - Y$. Since $ad - bc = \frac{1}{2} - \frac{1}{2} = 0$, the singularity is not elementary.

(c) If $r = 1$, then (31) gives only one sing., at (0,0). Linearizing (31) at (0,0) gives

$$\begin{aligned} x' &= -x + y \\ y' &= 0x - y \end{aligned}$$

so $a = d = -1$, $b = 1$, $c = 0$, and $\lambda = (-2 \pm \sqrt{0})/2$ is repeated. Looks like borderline case so let's be careful and use Fig. 1: $p = -2$, $q = 1$ is on the lower branch of the parabola $p^2 = 4q$, so could be a stable focus or a stable proper node or a stable improper node. We'll have to use our phaseportrait to see which one applies.

```
phaseportrait([-x+y, x^2/(1+x^2)-y], [t,x,y], t=0.10,
{[0,0,.5],[0,0,1],[0,0,1.5],[0,.5,1.5],[0,1,1.5],[0,1.5,1.5],
[0,1.5,1],[0,1.5,.5],[0,1.5,0]}, stepsize=0.05, x=0..1.5,
y=0..1.5, scene = [x,y]);
```



From the phase-portrait we can see that the singularity at (0,0) is a stable improper node.

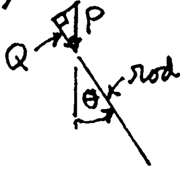
7. (a)

$$\begin{aligned} x' &= y & &= 0x + 1y \\ y' &= -\frac{k}{m}x + \frac{2P}{ml}x\left(1 - \frac{x^2}{l^2}\right)^{-1/2} & &\approx \left(\frac{2P}{ml} - \frac{k}{m}\right)x + 0y \end{aligned}$$

so $\lambda = \left(0 \pm \sqrt{0 + 4\left(\frac{2P}{ml} - \frac{k}{m}\right)}\right)/2 = \pm\sqrt{P - kl/2}\,\sqrt{2/ml}$, so the origin is a center (or focus, but we can rule out the focus since (7.1) is conservative) if $P < kl/2$, and a saddle if $P > kl/2$. Since the saddle is unstable, $P_{cr} = kl/2$.

(b) $P = kl/2$ is a bifurcation pt. on a P axis.

(c)



Since the rod $\simeq$ massless it must be in equilibrium (Σ forces $=0$, Σ moments $=0$). Thus it must simply be in a state of compression by $Q = P\cos\theta$. Since each of the two rods push on m with a force whose x component is $P\cos\theta\sin\theta$, Newton's 2nd law gives $mx'' = 2P\cos\theta\sin\theta - kx$. Putting $\cos\theta = \sqrt{l^2 - x^2}/l$ and $\sin\theta = x/l$ gives (7.1).

8. (a) $\left.\begin{aligned} x' &= y \\ y' &= -x + \frac{r}{1-x} \end{aligned}\right\}$ gives sing. pts. at $x_{\pm} = (1 \pm \sqrt{1-4r})/2$ & $y = 0$.

If $r < 1/4$ there are two sing. pts, "$S_{\pm}$" $= (x_{\pm}, 0)$, where $0 < x_- < x_+ < 1$. To linearize about those points, expand

$$-x + \frac{r}{1-x} = 0 + \left[-1 + \frac{r}{(1-x_{\pm})^2}\right](x - x_{\pm}) + \cdots$$

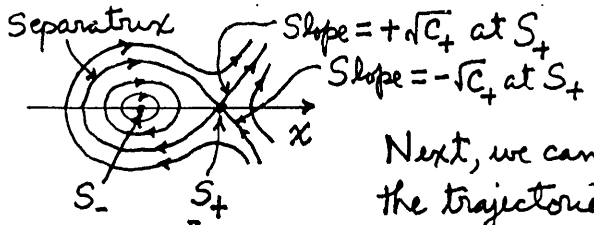
$$= \text{etc} = \boxed{\frac{1-4r \pm \sqrt{1-4r}}{2r}}(x - x_{\pm}) + \cdots$$

call this $c_{\pm}$

Thus, the linearized eqns. are (with $X = x - x_{\pm}$, $Y = y - 0$)

$\left.\begin{aligned} X' &= 0X + Y \\ Y' &= c_{\pm}X + 0Y \end{aligned}\right\}$ thus, $\lambda = (0 \pm \sqrt{0 + 4c_{\pm}})/2$

Since $c_+ > 0$ and $c_- < 0$, these λ's imply that S_+ is a saddle and S_- is a center or a focus — but we can rule out a focus because the system is conservative. To get the straight line trajs. through the saddle, seek $Y = \kappa X$. Then the linearized equations are $X' = \kappa X$ and $\kappa X' = c_+ X$, so $\kappa = c_+/\kappa$, so $\kappa^2 = c_+$, $\kappa = \pm\sqrt{c_+}$. Thus, as a first cut the phase portrait looks like this:



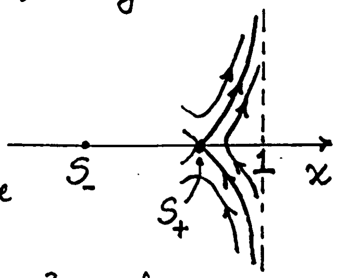
Next, we can get the equations of the trajectories and separatrix:

$$\frac{dy}{dx} = \frac{-x+\frac{r}{1-x}}{y}, \quad y\,dy + \left(x+\frac{r}{x-1}\right)dx = 0$$

$$\tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + r\ln|x-1| = \text{const}.$$

or, since $x<1$, $\quad y^2+x^2+2r\ln(1-x) = C$.

Observe that as $x \to 1^-$, $\ln(1-x) \to -\infty$, so $y^2 \to 0$. Thus, a more accurate phase portrait would be as shown at the right.



For the particular trajectory that is the separatrix, find C so that the traj. passes through S_+. Thus, separatrix is $y^2+x^2+2r\ln(1-x) = x_+^2+2r\ln(1-x_+)$.

(b)

```
phaseportrait([y,-x+.1/(1-x)],[t,x,y], t=-10..10,
  {[0,.8973,.0262164],[0,.8973,-.0262164],[0,.8773,
  .0262164]}, stepsize=.01, x=-1..1, y=-1..1, scene=[x,y]);
```

should generate the separatrix (and you could add a few more initial points to fill out the display). There is difficulty in the numerical integration, when the trajectory gets very close to the line $x=1$. I reduced that problem by reducing the stepsize to .02, but even then had some trouble – and it was slow in running. Reducing further to stepsize $=.01$ might suffice. NOTE: For $r=.1$, $x_+=.8873$, $x_-=.1127$, and $K = \pm\sqrt{C_+} = \pm\sqrt{(.6+\sqrt{.6})/2} = \pm 2.62164$.

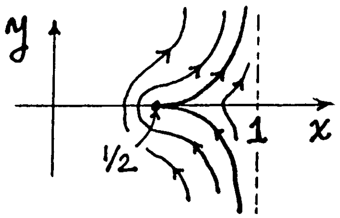
(c) As $r \to 1/4$, $x_\pm$ merge at $1/2$. Then,

$$-x+\frac{r}{1-x} = 0 + 0(x-1/2) + \frac{1}{2!}\frac{2r}{(1-x)^3}\bigg|_{\substack{r=1/4\\ x=.5}}(x-1/2)^2 + \cdots \sim 2\left(x-\tfrac{1}{2}\right)^2$$

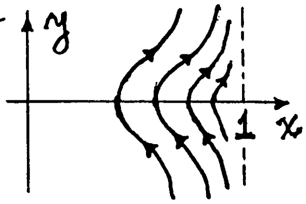
so we have the nonelementary singularity (second order)

$$\left.\begin{aligned} X' &= Y \\ Y' &= 2X^2 \end{aligned}\right\} \text{ so } \frac{dY}{dX} = 2\frac{X^2}{Y}, \quad \tfrac{1}{2}Y^2 = \tfrac{2}{3}X^3 + \text{const.},$$

$Y = \pm\sqrt{\frac{4}{3}X^3 + A}$. The trajectory through the sing. pt. is $Y \sim \pm\frac{2}{\sqrt{3}}X^{3/2}$, so it is a cusp.



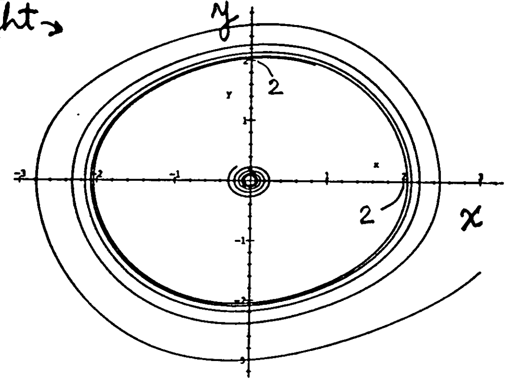
(e) If $r > 1/4$ then the roots $x_{\pm} = (1 \pm \sqrt{1-4r})/2$ are no longer real, so there are no singular pts., and the flow is as sketched at the right.



(g) Of course, $r = 1/4$ is a bifurcation value: for $r < 1/4$ there is a center and a saddle, at $r = 1/4$ they coalesce into a second order (i.e., nonelementary) singularity, and for $r > 1/4$ there is no singularity at all. Physically, if $r > 1/4$ then the force of attraction becomes so great as to rule out the possibility of oscillations.

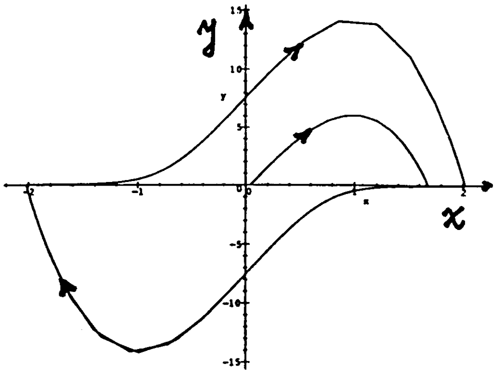
Section 7.5

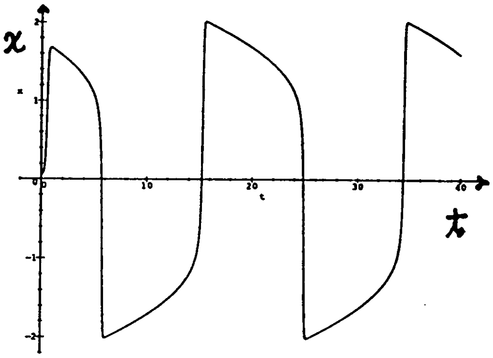
1.(a) `phaseportrait([y,-x+.1*(1-x^2)*y],[t,x,y],t=0..30,{[0,.05,.05],[0,3,-1.5]},stepsize=.05,scene=[x,y]);` and then again with `scene=[t,x]`. In the x,y plot we see very slow approach to limit cycle, shown at right→

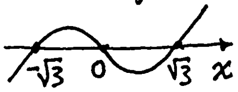


(b) Now we have a relaxation oscillation, with narrow t intervals of rapid change, so reduce stepsize to .01. Run `phaseportrait( [y,-x+10*(1-x^2)*y],`

[t,x,y], t=0..60,
{[0,.05,.05]},
stepsize = .03,
scene = [x,y]);
and again, for
scene = [t,x].





2. (a) $f(x) = x^2 - 1$, $g(x) = x^3$. f is even, g is odd ✓ $g(x) > 0$ for all $x > 0$ ✓ $g'(x) = 3x^2$ continuous for all x. ✓
$F(x) = \int_0^x f(\xi)\,d\xi = \frac{x^3}{3} - x =$  so, with $x_0 = \sqrt{3}$,

$F(x) < 0$ for $0 < x < x_0$ ✓
$F(x) > 0$ for $x > x_0$ ✓ and $F(x)$ is monotone increasing for $x > x_0$ ✓, with $F(x) \to \infty$ as $x \to \infty$. ✓ Thus, thm 7.5.1 says the equation does have a single limit cycle, enclosing the origin.

(b) $x_0 = 1$ in this case ⋯

3. (a) $$\left.\begin{aligned} x' &= y \\ y' &= -(x^2+y^2-1)y - x \end{aligned}\right\}$$ Sing. pt only at (0,0). There,
$$x' = y$$
$$y' = -x + y$$
so $\lambda = (+1 \pm \sqrt{1-4})/2 = +\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, hence an unst. focus at the origin. Further, the "damping coefficient" x^2+y^2-1 is < 0 inside the unit circle and > 0 outside it, so it seems clear that the unit circle is a stable limit cycle, as can

be verified by a phaseportrait plot. At the least, one can easily see that the unit circle is a trajectory because $x(t) = 1\cos(t+\phi)$, $y(t) = x' = -\sin(t+\phi)$ satisfy the differential equation.

4. (a) $\sin\theta \equiv S$, $\cos\theta \equiv C$, for brevity. Then

$$x = rC \rightarrow x' = r'C - rS\theta' = \epsilon rC + rS - rCr^2 \quad ①$$

$$y = rS \rightarrow y' = r'S + rC\theta' = -rC + \epsilon rS - rSr^2 \quad ②$$

C times ① + S times ② gives $r' = \epsilon r - r^3$, and C times ② − S times ① gives $r\theta' = -r$ or $\theta' = -1$ ✓.

(b) $r' = r(\epsilon - r^2)$ shows that for $r < \sqrt{\epsilon}$ the flow is outward, and that for $r > \sqrt{\epsilon}$ it is inward. ($\theta' = -1$ merely gives the uniform clockwise motion $\theta = -t + \text{const}$)

From (4.1), $\left.\begin{array}{l} x' = \epsilon x + y \\ y' = -x + \epsilon y \end{array}\right\}$ $\lambda = (2\epsilon \pm \sqrt{0-4})/2 = \epsilon \pm i$, so $\epsilon < 0$ $\Rightarrow$ st. focus, $\epsilon > 0 \Rightarrow$ unst. focus

(c) $r(t) = \sqrt{\epsilon}$, $\theta = -t + C$ satisfies (4.2) and is therefore a trajectory. Further, $r' = r(\epsilon - r^2)$ shows an approach from both outside and inside.

5. (a) (5.1) follows directly by taking d/dt of $L\,di/dt + f(i) + \frac{1}{C}\int i\,dt = 0$.

(b) $t = \alpha\tau$, $i = \beta I$ give $L\frac{d}{\alpha d\tau}\frac{d}{\alpha d\tau}\beta I + (3a\beta^2 I^2 - b)\frac{d}{\alpha d\tau}\beta I + \frac{1}{C}\beta I = 0$ or,

$$I'' + \frac{\alpha^2 b}{\beta L}\left(\frac{3a\beta^2}{b}I^2 - 1\right)\frac{\beta}{\alpha}I' + \frac{\beta}{C}\frac{\alpha^2}{\beta L}I = 0$$

Set $3\alpha\beta^2/b = 1$ and $\cancel{\beta}\alpha^2/C\cancel{\beta}L = 1$. These give $\alpha = \sqrt{CL}$, $\beta = \sqrt{b/3\alpha} = \sqrt{b/3\sqrt{CL}}$. Further, if we set $\epsilon = \frac{\alpha^2 b}{\beta L}\frac{\beta}{\alpha} = \frac{\alpha b}{L} = b\sqrt{\frac{C}{L}}$, then (5.1) gives (5.2).

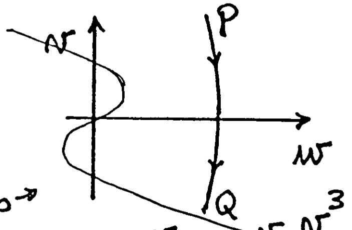
6. (a) $z''' - \epsilon(1 - z'^2)z'' + z' = 0$. $\int z'^2 z''\,dt = \int z'^2\,dz' = z'^3/3$, so we obtain $z'' - \epsilon(z' - z'^3/3) + z = C$. Then, $z = u + C$ gives $u'' - \epsilon(u' - u'^3/3) + u = 0$.

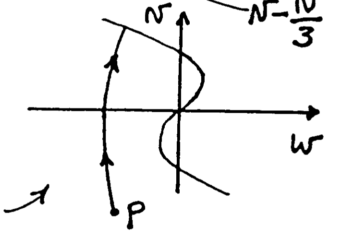
(b) $u' = v$, $v' = -u + \epsilon(1 - \frac{v^2}{3})v$ } has sing. pt. only at (0,0).

There, $u' = 0u + v$, $v' = -u + \epsilon v$ } $\lambda = (\epsilon \pm \sqrt{\epsilon^2 - 4})/2$ which is an unstable focus if $\epsilon < 2$ and an unstable node if $\epsilon > 2$.

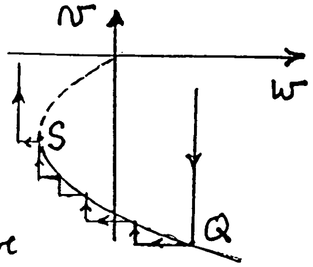
(d) $\epsilon w' = v$, $v' = -\epsilon w + \epsilon(1 - \frac{v^2}{3})v$ } $\frac{dv}{dw} = \epsilon^2 \frac{(v - \frac{v^3}{3}) - w}{v}$ ♀

As $\epsilon \to \infty$, it follows that $dv/dw \to \infty$ everywhere in the w, v phaseplane _except_ on the curve $w = v - v^3/3$. To determine the direction of the arrow on PQ (and other such vertical trajectories) note that to the _right_ of the curve $w = v - v^3/3$ and for $v > 0$ ♀ gives dv/dw = a very large negative number and for $v < 0$ it gives a very large _positive_ number, so for large (but not infinite) ϵ we see that PQ must really be like this → Similarly, if P is to the _left_ of $w = v - v^3/3$ then for $v < 0$ ♀ says dv/dw is large and negative, and for $v > 0$ ♀ says its large and positive. ↗

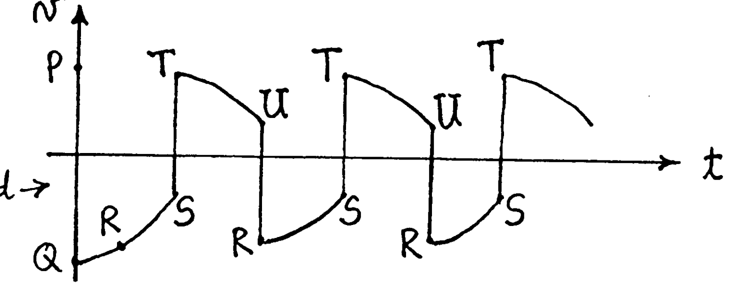




In the limit, then, as $\epsilon \to \infty$, the arrows are downward on straight vertical trajectories to the right of the ≀ curve, and upward on such trajectories to the left of it. Beginning at P, we therefore drop straight to Q (see figure in text). But Q is on the curve $w = v - v^3/3$, so we then move infinitesimally—horizontally—to the left. But that puts us below & to the left of the curve, so we go straight up, til we hit the curve again. And so on, til we reach S. There, when we step to the left & go straight up we keep going til T. Of course, the



little zig zags are infinitesimal, so from Q to S we simply move along the curve, then jump up to T, move smoothly to U, then jump to R, and so on. Observe that PQ, ST, and UR take zero time because the phase speed "$s'=a$" $=\infty$ on those trajectories. Plot $v(t)$ using the formula $s'=a$ to help us & we get the result sketched → But $v=\epsilon w' = u' = z' = x$, so the graph is actually $x(t)$ vs. t. Observe how it's easier to work with the Rayleigh equation because then ♀ immediately gives the actual equation of the limit cycle, namely, $w = v - v^3/3$ together with the straight segments UR and ST.



7.(a) $\int x''dx - \epsilon\int(1-x^2)x'dx + \int x\,dx = 0$

$\int x''dx = \int \frac{dx'}{dt}dx = \int x'dx' = x'^2/2$ & $\int x\,dx = x^2/2 \Rightarrow (7.1)$.

But x' initial $= x'$ final, and x initial $= x$ final, over a closed cycle, so (7.1) gives (7.2).

(b) $\int_0^{2\pi}(1-a^2c^2)(-as)(-as\,dt) = a^2\int_0^{2\pi}(s^2-a^2s^2c^2)dt$

$= a^2\left[\pi - \frac{a^2}{8}(2\pi)\right]$ (from a table, or maple)

$= 0$ gives $a^2=4$, $a=2$.

Section 7.6

1.(b) $x''_{n+1} = -\alpha x_n + F_0\cos\Omega t$. Let $x_0 = A\cos\Omega t$. Then

$x_1'' = (-\alpha A + F_0)\cos\Omega t$, $x_1' = (-\alpha A + F_0)\frac{\sin\Omega t}{\Omega} + B$,

$x_1 = \frac{\alpha A - F_0}{\Omega^2}\cos\Omega t + \cancelto{0}{B}t + \cancelto{0}{C}$

Since $A \to \frac{\alpha A-F_0}{\Omega^2}$, then $x_2 = \frac{\alpha\left(\frac{\alpha A-F_0}{\Omega^2}\right)-F_0}{\Omega^2}\cos\Omega t$
$=\left\{\left(\frac{\alpha}{\Omega^2}\right)^2 A - \frac{F_0}{\Omega^2}\left[1+\frac{\alpha}{\Omega^2}\right]\right\}\cos\Omega t$, and so on.

(c) If $|\alpha/\Omega^2|<1$, (1.5) $\to$ $x(t) = \left(0-\frac{F_0}{\Omega^2}\frac{1}{1-\frac{\alpha}{\Omega^2}}\right)\cos\Omega t$
$= \frac{F_0}{\alpha-\Omega^2}\cos\Omega t$ as $n\to\infty$.

(d) $A = (\alpha A-F_0)/\Omega^2$, $\left(1-\frac{\alpha}{\Omega^2}\right)A = -F_0/\Omega^2$, $A = -F_0/(\Omega^2-\alpha)$.

(e) $x_1'' = -\alpha x_0 - \beta x_0^3 + F_0\cos\Omega t$

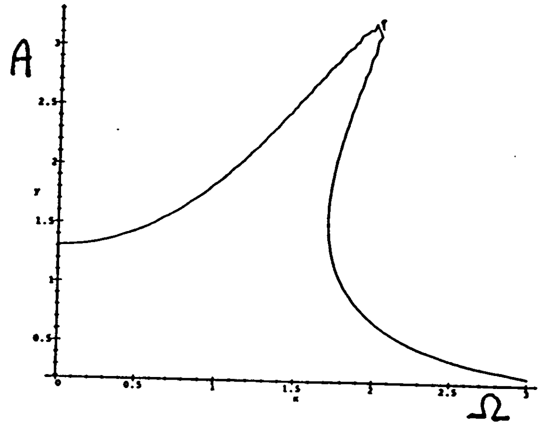
$$= -\alpha A\cos\Omega t - \frac{\beta}{4}A^3(3\cos\Omega t+\cos 3\Omega t) + F_0\cos\Omega t$$

$$x_1' = \left(-\alpha A - \frac{\beta A^3}{4}3 + F_0\right)\frac{\sin\Omega t}{\Omega} + \cancel{B} - \frac{\beta A^3}{4}\frac{\sin 3\Omega t}{3\Omega}$$

$$x_1 = \left(-\alpha A - \frac{3\beta A^3}{4} + F_0\right)\frac{\cos\Omega t}{-\Omega^2} - (\text{etc})\cos 3\Omega t$$

so set $(\alpha A + \frac{3}{4}\beta A^3 - F_0)/\Omega^2 = A$, so $\Omega^2 = \alpha + \frac{3}{4}\beta A^2 - \frac{F_0}{A}$.

2. (a) $4 = [(1-\Omega^2)A + \frac{3}{4}(.4)A^3]^2 + (.3\Omega A)^2$,
so use maple command `implicitplot(4=((1-x^2)*y+.3*y^3)^2+(.3*x*y)^2, x=0..3, y=0..4, numpoints=1000);` where x is Ω and y is A. The 1000 pts was not nearly enough so I re-ran for 5000 pts and even for 5000 the peak was a bit messed up. (10,000 should suffice)

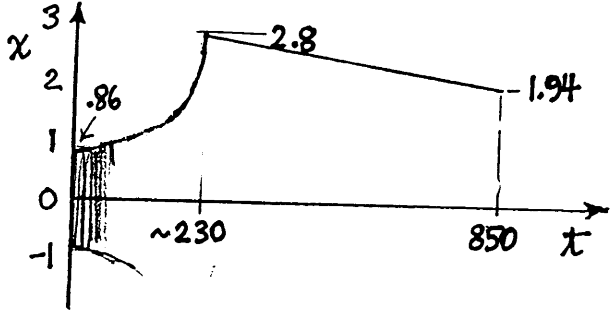


(b) `fsolve(4=(.3*A^3)^2+(.3*A)^2);`
gives $A = 1.856909894$. To generate $x(t)$, run
`phaseportrait([y,-x-.4*x^3-.3*y+2*cos(t)],[t,x,y], t=0..50, {[0,0,0]}, stepsize=.05, scene=[t,x]);`
The resulting graph covers around 8 oscillations, by which time the amplitude has settled down to 1.91, compared with Duffing's approximate value 1.8569...

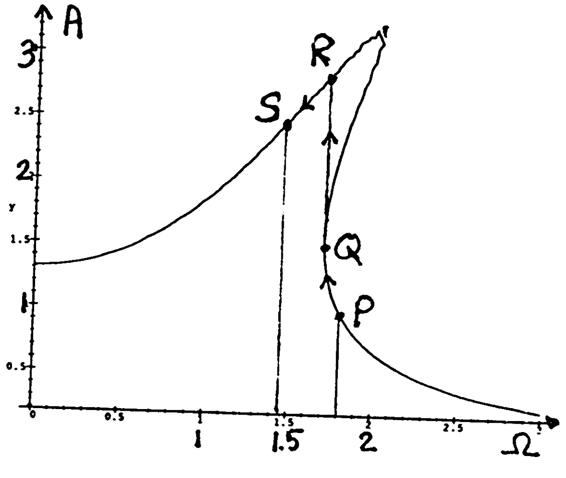
(c) `fsolve(4=(-2.24*A+.3*A^3)^2+(.54*A)^2);` gives

$A = .990, 2.27, 2.97$. Using same phaseportrait command as above, but with cos(t) changed to cos(1.8*t), we get the graph of $x(t)$ which, after around a dozen oscillations, settles down to an amplitude $A = 0.98$, which corresponds to the lower branch. Re-running, each time increasing $x(0)$, we find that we continue to get the lower-branch oscillation until we reach $x(0) \simeq 0.95$, whereupon we get the upper branch oscillation with amplitude $\simeq 3$.

(g) `phaseportrait([y,-x-.4*x^3-.3*y+2*cos((1.9-.0005*t)*t)],[t,x,y],t=0..800,{[0,0,0]},stepsize=.05,scene=[t,x]);` the output is as sketched at the right, where we show only the envelope, for there are a great many oscillations within.

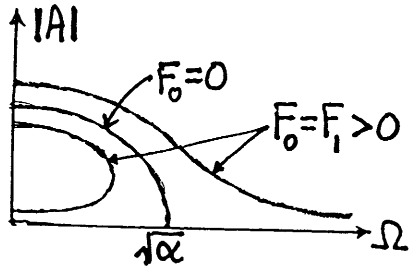


That result is in excellent agreement with the Duffing A, Ω plot which we reproduce here from Exercise 2a, above, with the correspondence of the points P, Q, R, S shown in the two figures.



3.(a) As $A \to \infty$ in (4), (4) gives the hyperbola
$$\Omega^2 \sim \alpha + \frac{3}{4}\beta A^2.$$

(b) The key is to first seek the "spine", $F_0 = 0$: $\Omega^2 = \alpha - \frac{3}{4}|\beta| A^2$, which is the ellipse $\frac{3}{4}|\beta| A^2 + \Omega^2 = \alpha$, as sketched:



4. $\left.\begin{aligned} x' &= y = 0 \\ y' &= -ry + x - x^3 = 0 \end{aligned}\right\} \Rightarrow y = 0$ and $x = 0, \pm 1$. Linearize about these points:

(0,0): $\left.\begin{aligned} x' &= 0x + 1y \\ y' &= 1x - ry \end{aligned}\right\}$ $\lambda = (-r \pm \sqrt{r^2+4})/2 \Rightarrow$ saddle

(1,0): $\left.\begin{aligned} x' &= 0(x-1) + 1y \\ y' &= -2(x-1) - ry \end{aligned}\right\}$ $\lambda = (-r \pm \sqrt{r^2-8}) \Rightarrow$ stable focus if $r < \sqrt{8}$ and stable node if $r > \sqrt{8}$.

(−1,0): $\left.\begin{aligned} x' &= 0(x+1) + 1y \\ y' &= -2(x+1) - ry \end{aligned}\right\}$ likewise, stable focus if $r < \sqrt{8}$ and stable node if $r > \sqrt{8}$.

5. (a) phaseportrait([y, −.3*y + x − x^3 + $\overset{F_0}{\boxed{.2}}$*cos(1.2*t)], [t,x,y], $\boxed{t=0..44}$, {[0,1,.2]}, stepsize=.04, scene=$\boxed{[x,y]}$); and again, with the circled items changed to .2, t = 0..44, [t,x]. In parts (b)–(f) use the F_0 given in Fig. 5, use both [x,y] and [t,x], and use these t intervals:

(b) t=150..200 (c) t=400..450 (d) t=400..450

(e) t=300..400 (f) t=150..200

6. For F_0=.292 I obtain a period-8 oscillation, using t=600..700.

7. Jordan & Smith obtain a period-1 solution, but going out to t=600..700 my maple results still are not settling into a periodic solution. You may wish to carry this calculation further.

8. Yes.

CHAPTER 8

Section 8.2

1. (a) False. $x^2+x+3=0$ is algebraic but not linear.
   (b) False. $3x+2=0$ is algebraic and linear.
   (c) False. $2\sin x+x=0$ is transcendental and nonlinear.
   (d) True.
   (e) True.
   (f) False. $xe^x+4\sin x=5$ is nonlinear but not algebraic
   (g) False. $5x-7=0$ is linear but not transcendental.
   (h) False. $2x^3-5x^2+x+1=0$ is nonlinear but not transcendental; it is algebraic.

2. $a_{11}x_1+a_{12}x_2=c_1$ (7a)
   $a_{21}x_1+a_{22}x_2=c_2.$ (7b)

Suppose $a_{11}\neq 0$. Then $x_1=(c_1-a_{12}x_2)/a_{11}$ from (7a). Putting that in (7b) gives $a_{21}(c_1-a_{12}x_2)/a_{11}+a_{12}x_2=c_1$ or $(a_{11}a_{22}-a_{21}a_{12})x_2=a_{11}c_2-a_{21}c_1$. The latter gives a unique solution for x_2 if and only if $a_{11}a_{22}-a_{21}a_{12}\neq 0$. (If $a_{11}a_{22}-a_{21}a_{12}$ does $=0$ then the latter gives <u>no</u> solution for x_2 if $a_{11}c_2-a_{21}c_1\neq 0$, and it gives x_2 as arbitrary if $a_{11}c_2-a_{21}c_1=0$.) And we already found x_1 uniquely. Thus, if $a_{11}\neq 0$ then (7) admits a unique solution iff (if and only if) "Δ"$=a_{11}a_{22}-a_{21}a_{12}\neq 0$

Finally, suppose $a_{11}=0$: Then, in (7a) we need $a_{12}\neq 0$, for if $a_{12}=0$ then there is <u>no</u> solution if $c_1\neq 0$ and there is a <u>nonunique</u> solution if $c_1=0$, namely, all points on the straight line defined by (7b). With $a_{12}\neq 0$, (7a) gives $x_2=c_1/a_{12}$, and putting that in (7b) gives $a_{21}x_1=c_2-c_1a_{22}/a_{12}$, which gives a unique solution for x_2 iff $a_{21}\neq 0$. Thus, if $a_{11}=0$ then there is a unique solution iff $a_{12}\neq 0$ and $a_{21}\neq 0$, i.e., iff $\Delta=\underset{0}{\cancel{a_{11}}}a_{22}-a_{12}a_{21}\neq 0$.

3. (a) If $c_1\neq 0$ then (9a) is $0x_1+0x_2+0x_3=c_1$ cannot be satisfied by <u>any</u> choice of x_j's so there will be NO SOLUTION to (9). If $c_1=0$ then $0x_1+0x_2+0x_3=0$ is the identity $0=0$ and can be discarded. In that case there will be NO SOLUTION to (9) if the planes (9b) and (9c) are parallel and noncoincident and there will be a NONUNIQUE solution if those planes are coincident, namely, all points in that plane. A UNIQUE solution is not possible.
   (b) If c_1 and/or c_2 is $\neq 0$ then (9) has NO SOLUTION. If $c_1=c_2=0$ then (9a) and (9b) can be discarded, leaving the plane (9c). Hence, there will be a NONUNIQUE solution, namely, all points in that plane.
   (c) If c_1 and/or c_2 and/or c_3 is $\neq 0$ then (9) has NO SOLUTION. If $c_1=c_2=c_3=0$ then (9) has a NONUNIQUE solution, namely, <u>all</u> points in x_1,x_2,x_3 space.

1. (b) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \end{pmatrix} \xrightarrow{\text{eq2} \to \text{eq2} + (-3/2)\text{eq1}} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -\frac{7}{2} & 0 \end{pmatrix}$ gives unique solution $x_2 = 0,\ x_1 = 0$

(e) $2x_1 - x_2 - x_3 - 5x_4 = 6$ is already in Gauss-eliminated form. Nonunique solution: $x_4 = \alpha_1,\ x_3 = \alpha_2,\ x_2 = \alpha_3,\ x_1 = 6 + 5\alpha_1 + \alpha_2 + \alpha_3$ (α's arbitrary)

(f) $\begin{pmatrix} 2 & -1 & -1 & -3 & 0 \\ 1 & -1 & 4 & 0 & 2 \end{pmatrix} \xrightarrow{\text{eq2} \to \text{eq2} + (-1/2)\text{eq1}} \begin{pmatrix} 2 & -1 & -1 & -3 & 0 \\ 0 & -1/2 & 9/2 & 3/2 & 2 \end{pmatrix} \xrightarrow[\text{eq2} \to -2\,\text{eq2}]{\text{eq1} \to 1/2\,\text{eq1}} \begin{pmatrix} 1 & -1/2 & -1/2 & -3/2 & 0 \\ 0 & 1 & -9 & -3 & -4 \end{pmatrix}$

Nonunique soln.: $x_4 = \alpha_1,\ x_3 = \alpha_2,\ x_2 = -4 + 6\alpha_1 + 9\alpha_2,\ x_1 = \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}(-4 + 3\alpha_1 + 9\alpha_2$
$= -2 + 3\,\alpha_1 + 5\alpha_2$

(h) $\begin{pmatrix} 1 & 1 & -2 & 3 \\ 1 & -1 & -3 & 1 \\ 1 & -3 & -4 & -1 \end{pmatrix} \xrightarrow[\text{eq3} \to \text{eq3} - \text{eq1}]{\text{eq2} \to \text{eq2} - \text{eq1}} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -2 & -1 & -2 \\ 0 & -4 & -2 & -4 \end{pmatrix} \xrightarrow{\text{eq3} \to \text{eq3} - (2)\text{eq2}} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{eq2} \to (-\frac{1}{2})\text{eq2}} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Nonunique solution: $x_3 = \alpha,\ x_2 = 1 - \frac{1}{2}\alpha,\ x_1 = 3 + 2\alpha - (1 - \frac{1}{2}\alpha) = 2 + \frac{5}{2}\alpha$

(i) $\begin{pmatrix} 2 & -1 & 6 \\ 3 & 2 & 4 \\ 1 & 10 & -12 \\ 6 & 11 & -2 \end{pmatrix} \xrightarrow{\substack{\text{eq2} \to \text{eq2} + (-3/2)\text{eq1} \\ \text{eq3} \to \text{eq3} + (-1/2)\text{eq1} \\ \text{eq4} \to \text{eq4} + (-3)\text{eq1}}} \begin{pmatrix} 2 & -1 & 6 \\ 0 & \frac{7}{2} & -5 \\ 0 & \frac{21}{2} & -15 \\ 0 & 14 & -20 \end{pmatrix} \xrightarrow{\substack{\text{Scale eqns} \\ \text{2,3,4 by} \\ \frac{2}{7},\ \frac{2}{21},\ \frac{1}{14}}} \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -\frac{10}{7} \\ 0 & 1 & -\frac{10}{7} \\ 0 & 1 & -\frac{10}{7} \end{pmatrix} \xrightarrow{\substack{\text{eq3} \to \text{eq3} + (-1)\text{eq2} \\ \text{eq4} \to \text{eq4} + (-1)\text{eq2}}} \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -\frac{10}{7} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

so unique soln: $x_2 = -10/7,\ x_1 = 16/7$

(k) no solution

(l) Nonunique solution $x_4 = \alpha,\ x_3 = 2 - \alpha,\ x_2 = (1 - \alpha)/2,\ x_1 = (1 + \alpha)/2$

(m) $c = 10$: no solution; $c = 11$: unique solution $x_3 = 1,\ x_2 = 0,\ x_1 = 2$

(n) Unique solution $x_3 = 15/4,\ x_2 = -3/4,\ x_1 = 7/2$

(o) Unique solution $x_4 = 2/5,\ x_3 = 1/5,\ x_2 = 1/5,\ x_1 = 2/5$

(p) Unique solution $x_3 = -5/2,\ x_2 = 1,\ x_1 = -1/2$

(q) Nonunique solution $x_5 = \alpha,\ x_4 = 0,\ x_3 = -\alpha,\ x_2 = 0,\ x_1 = -\alpha$

2. We already did the Gauss elim. in problem 1, so let's merely finish the steps.

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & -\frac{7}{2} & 0 \end{pmatrix} \xrightarrow[\text{eq2} \to (-\frac{2}{7})\text{eq2}]{\text{eq1} \to (\frac{1}{2})\text{eq1}} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{eq1} \to \text{eq1} + (-\frac{1}{2})\text{eq2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ gives $x_2 = 0$,
$x_1 = 0$.

(f) $\begin{pmatrix} 1 & -1/2 & -1/2 & -3/2 & 0 \\ 0 & 1 & -9 & -6 & -4 \end{pmatrix} \xrightarrow{\text{eq1} \to \text{eq1} + (\frac{1}{2})\text{eq2}} \begin{pmatrix} 1 & 0 & -5 & -9/2 & -2 \\ 0 & 1 & -9 & -6 & -4 \end{pmatrix}$ gives $x_4 = \alpha_1,\ x_3 = \alpha_2$,
$x_2 = -4 + 6\alpha_1 + 9\alpha_2$
$x_1 = -2 + \frac{9}{2}\alpha_1 + 5\alpha_2$, as above.

(q) Let's show all the steps for this one:

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -3 & 2 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1/2 & -1 & -1/2 & -1 & 0 \\ 0 & 1/2 & 1 & -7/2 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1/2 & -1 & -1/2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1/2 & -1 & -1/2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 0 \\ 0 & 0 & 0 & 3/2 & 0 & 0 \end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1&\frac{1}{2}&0&\frac{1}{2}&1&0\\0&1&-2&-1&-2&0\\0&0&1&-\frac{3}{2}&1&0\\0&0&0&1&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&\frac{1}{2}&0&0&1&0\\0&1&-2&0&-2&0\\0&0&1&0&1&0\\0&0&0&1&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&\frac{1}{2}&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&1&0\\0&0&0&1&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&1&0\\0&0&0&1&0&0\end{pmatrix}$$

so $x_5=\alpha$, $x_4=0$, $x_3=-\alpha$, $x_2=0$, $x_1=-\alpha$

3. (l) The commands

```
with(linalg):
A:= array([[0,0,1,1],[0,4,-1,1],[1,-1,2,1]]);
b:= array([2,0,4]);
linsolve(A,b);
```

gives the solution $[-\underline{t}_1+1 \quad \underline{t}_1 \quad 2\underline{t}_1+1 \quad -2\underline{t}_1+1]$ where $\underline{t}_1$ is an arbitrary constant; we use α instead. Thus, $x_4=1-2\alpha$, $x_3=1+2\alpha$, $x_2=\alpha$, $x_1=1-\alpha$. This looks different from our result in 1(l) but is equivalent to it. To see the equivalence, set $x_4=1-2\alpha\equiv\beta$, say, so that $\alpha=(1-\beta)/2$. Then the Maple result becomes $x_4=\beta$, $x_3=2-\beta$, $x_2=(1-\beta)/2$, $x_1=(1+\beta)/2$, which is equivalent to our result in 1(l).

5. Easy: $x_1=x_2=x_3=0$ is seen to be a solution of the system, so the line must pass through the origin.

6. (a) These are not linear in x_1,x_2,x_3, but they are linear in x_1^2,x_2^2,x_3^2. Thus we can apply Gauss elimination:

$$\begin{pmatrix}1&2&-1&29\\1&1&1&19\\3&4&0&67\end{pmatrix}\rightarrow\begin{pmatrix}1&2&-1&29\\0&-1&2&-10\\0&-2&3&-20\end{pmatrix}\rightarrow\begin{pmatrix}1&2&-1&29\\0&-1&2&-10\\0&0&-1&0\end{pmatrix}$$

gives $x_3^2=0$, $x_2^2=10$, $x_1^2=9$
so $x_3=0$, $x_2=\pm\sqrt{10}$, $x_1=\pm3$
so there are four solutions.

(b) Not possible. (I.e., I dont see how it could be done.)

(c) Not linear in x,y,z, but they are linear in $\sin x$, $\sin y$, $\cos z$, so we can use Gauss elimination:

$$\begin{pmatrix}1&1&0&1\\1&-1&4&1.2\\1&1&2&1.6\end{pmatrix}\rightarrow\begin{pmatrix}1&1&0&1\\0&-2&4&0.2\\0&0&2&0.6\end{pmatrix}\rightarrow\begin{pmatrix}1&1&0&1\\0&1&-2&-0.1\\0&0&1&0.3\end{pmatrix}$$

so $\cos z=0.3$, $\sin y=0.5$, $\sin x=0.5$.
Thus, $z=1.27$ rad (72.5°)
or 5.02 rad (287.5°),
$y=0.524$ rad (30°),
$x=0.524$ rad (30°).

Thus, there are two solutions within the specified intervals.

7. (b) $(2-\lambda)x-y=0$
$-x+(2-\lambda)y=0$, $\begin{pmatrix}2-\lambda&-1&0\\-1&2-\lambda&0\end{pmatrix}$

If $\lambda=2$ we obtain only the trivial solution $x=y=0$, so we can assume that $\lambda\neq2$. Then $\begin{pmatrix}2-\lambda&-1&0\\-1&2-\lambda&0\end{pmatrix}\rightarrow\begin{pmatrix}1&\frac{1}{\lambda-2}&0\\1&\lambda-2&0\end{pmatrix}\rightarrow\begin{pmatrix}1&\frac{1}{\lambda-2}&0\\0&\lambda-2-\frac{1}{\lambda-2}&0\end{pmatrix}$

gives nontrivial solution only if $\lambda-2-\frac{1}{\lambda-2}=0$, i.e., if $\lambda=1,3$.

If $\lambda=1$, then we have $\begin{pmatrix}1&-1&0\\0&0&0\end{pmatrix}$, with nontrivial solutions $y=\alpha$, $x=\alpha$.

If $\lambda=3$, then we have $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with nontrivial solutions $y=\beta,\ x=-\beta$

(e) Proceeding as in (b), we find that nontrivial solutions exist only for the values $\lambda=1,2$. For $\lambda=1$ we have $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, with nontrivial solutions $z=0, y=0, x=\alpha$

and for $\lambda=2$ we have $\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, with nontrivial solutions $z=\beta, y=\beta, x=2\beta$

8. (b) Passing from the original system to $x_1+x_2-4x_3=2x_1-x_2+x_3$ is equivalent to these steps: $\begin{pmatrix} 1 & 1 & -4 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{eq2 \to eq2+(-1)eq1} \begin{pmatrix} 1 & 1 & -4 & 0 \\ 1 & -2 & 5 & 0 \end{pmatrix} \xrightarrow{eq1 \to (0)eq1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 5 & 0 \end{pmatrix}$

However, the last step is not an elementary row operation; multiplying eq1 by 0 simply amounts to throwing that equation away.

11. (a) Vertical: $T_1\sin\theta_1+T_2\sin\theta_2=F$

Horizontal: $T_1\cos\theta_1-T_2\cos\theta_2=0$.

For $\theta_1=\theta_2=\pi/2$ the system becomes $\begin{matrix} T_1+T_2=F \\ 0=0 \end{matrix}$ and there is a nonunique soln.

For $\theta_1=\theta_2=0$ the system becomes $\begin{matrix} 0=F \\ T_1-T_2=0 \end{matrix}$ and there is no solution.

(b) Vertical: $T_1\sin 45° + T_2\sin 60° + T_3\sin 30° = F$ or, $0.71T_1+0.87T_2+0.5T_3=F$

Horizontal: $T_1\cos 45° - T_2\cos 60° - T_3\cos 30° = 0$ $\quad 0.71T_1-0.5T_2-0.87T_3=0$.

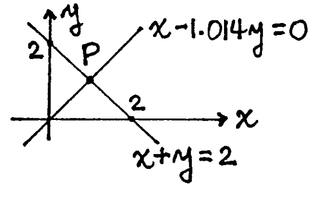
(c) Putting (11.4) into the latter equations gives

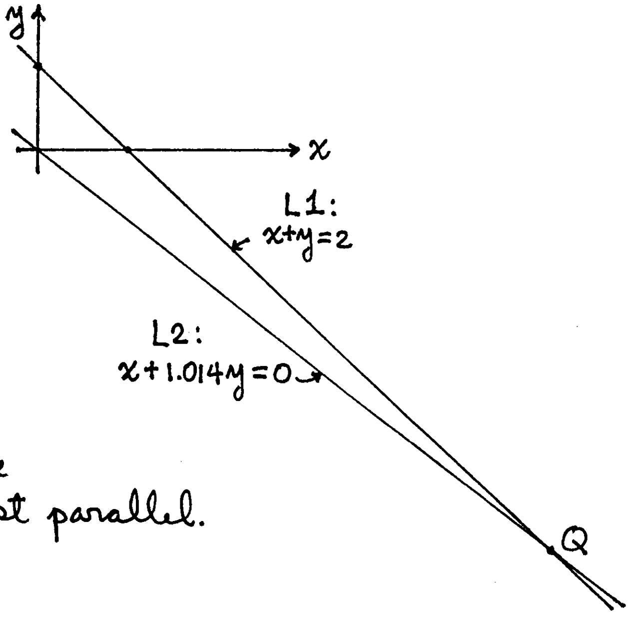
$$0.71\frac{k_1}{\sqrt2}(x-y)+0.87\left(-\frac{k_2}{2}\right)(x+\sqrt3 y)+0.5\left(-\frac{k_3}{2}\right)(\sqrt3 x+y)=F$$

$$0.71\frac{k_1}{\sqrt2}(x-y)-0.5\left(-\frac{k_2}{2}\right)(x+\sqrt3 y)-0.87\left(-\frac{k_3}{2}\right)(\sqrt3 x+y)=0.$$

We could solve these by the linsolve command and put the results into (11.4) to obtain T_1, T_2, T_3.

13.



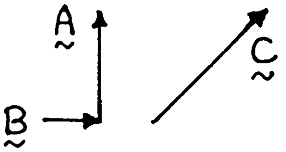


Changing the 1.014 to 1.01 causes the intersection Q (and hence the solution x, y) to move far more than it causes P to move, because the lines L1 and L2 are almost parallel.

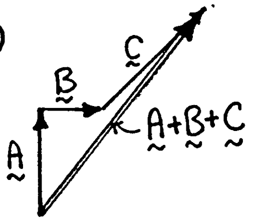
CHAPTER 9

Section 9.2

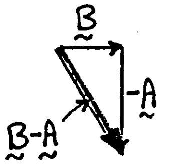
1.



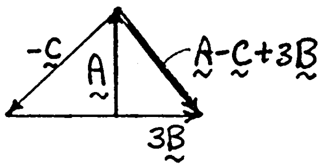
(a)



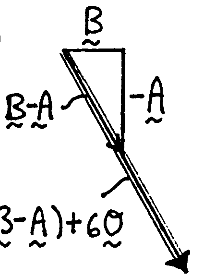
(b)



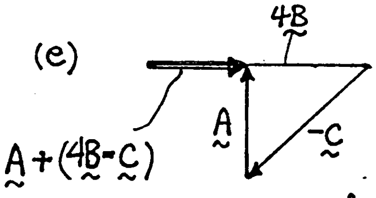
(c)



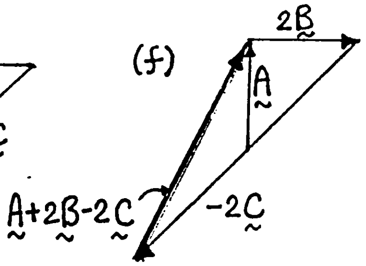
(d)



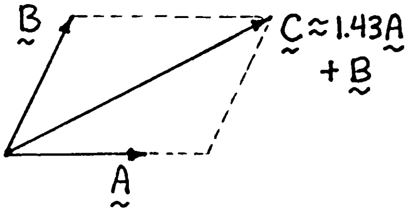
(e)



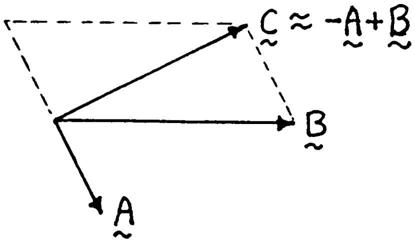
(f)



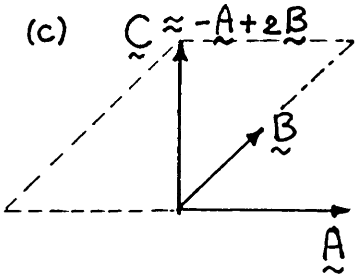
2. (a)



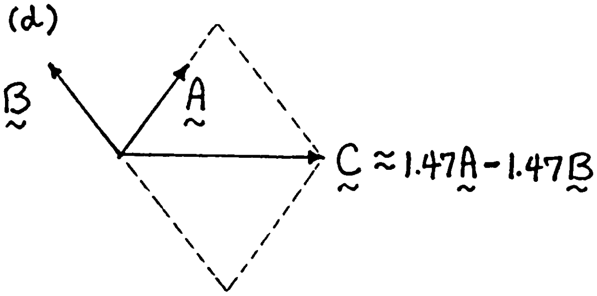
(b)



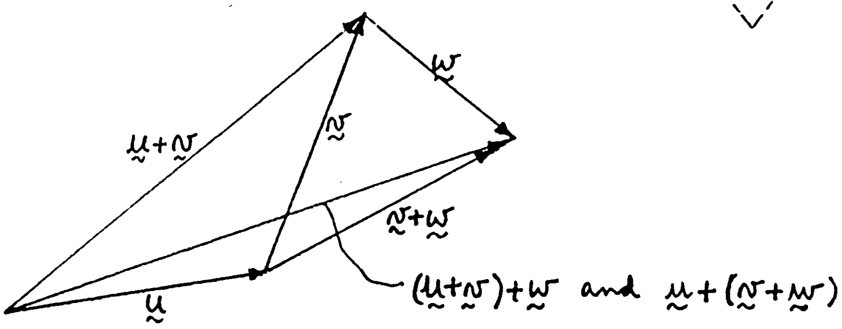
(c)



(d)



3.



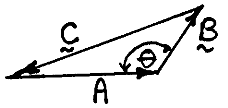
4. (a) Suppose α and/or β and/or $\underset{\sim}{u}$ are zero. Then, according to the definition of scalar multiplication it follows immediately that $\alpha(\beta\underset{\sim}{u}) = \underset{\sim}{0}$ and that $(\alpha\beta)\underset{\sim}{u} = \underset{\sim}{0}$ so that $\alpha(\beta\underset{\sim}{u}) = (\alpha\beta)\underset{\sim}{u}$. Next, suppose $\alpha, \beta, \underset{\sim}{u}$ are all nonzero. Then $\beta\underset{\sim}{u}$ is $|\beta|$ times as long as $\underset{\sim}{u}$ and $\alpha(\beta\underset{\sim}{u})$ is $|\alpha|$ times as long as $\beta\underset{\sim}{u}$ and $|\alpha||\beta|$ times as long as $\underset{\sim}{u}$. Likewise, $(\alpha\beta)\underset{\sim}{u}$ is $|\alpha\beta|$ or $|\alpha||\beta|$ times as long as $\underset{\sim}{u}$. Thus, $\alpha(\beta\underset{\sim}{u})$ and $(\alpha\beta)\underset{\sim}{u}$ have the same length. Next, consider their directions. If α and β have the same sign then each of the two scalar multiplications in $\alpha(\beta\underset{\sim}{u})$ give either no change in direction (if $\alpha>0$, $\beta>0$) or a direction reversal (if $\alpha<0, \beta<0$); in the latter case the two reversals give no net change. Thus, $\alpha(\beta\underset{\sim}{u})$ is in the same direction as $\underset{\sim}{u}$. Likewise, $\alpha\beta>0$ (since α, β are the same sign) so that $(\alpha\beta)\underset{\sim}{u}$ is in the same direction as $\underset{\sim}{u}$. Thus, $\alpha(\beta\underset{\sim}{u})$ and $(\alpha\beta)\underset{\sim}{u}$ have the same direction. Finally, suppose α and β are of opposite sign. Then both $\alpha(\beta\underset{\sim}{u})$ and $(\alpha\beta)\underset{\sim}{u}$ have directions opposite to that of $\underset{\sim}{u}$ and hence the same as each other. We conclude that (4a) holds for all $\alpha, \beta, \underset{\sim}{u}$.

(b) If $\underset{\sim}{u} = \underset{\sim}{0}$ then $(\alpha+\beta)\underset{\sim}{u} = (\alpha+\beta)\underset{\sim}{0} = \underset{\sim}{0}$ and $\alpha\underset{\sim}{u} + \beta\underset{\sim}{u} = \alpha\underset{\sim}{0} + \beta\underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$. ✓ Next, suppose $\underset{\sim}{u} \neq \underset{\sim}{0}$. We can treat the cases where α, β are both positive, both negative, and of opposite sign separately. These stories are quite similar so we'll do only the case where $\alpha, \beta > 0$. Then $(\alpha+\beta)\underset{\sim}{u}$ is $\alpha+\beta$ times as long as $\underset{\sim}{u}$ and in same direction as $\underset{\sim}{u}$; $\alpha\underset{\sim}{u}$ and $\beta\underset{\sim}{u}$ are α and β times as long as $\underset{\sim}{u}$, respectively, and in the same direction as $\underset{\sim}{u}$, so $\alpha\underset{\sim}{u} + \beta\underset{\sim}{u}$ is $\alpha+\beta$ times as long as $\underset{\sim}{u}$ and in same direction as $\underset{\sim}{u}$. Finally, if $\alpha = 0, \beta \neq 0$ then $(\alpha+\beta)\underset{\sim}{u} = (\alpha+0)\underset{\sim}{u} = \alpha\underset{\sim}{u}$, and $\alpha\underset{\sim}{u} + \beta\underset{\sim}{u} = \alpha\underset{\sim}{u} + 0\underset{\sim}{u} = \alpha\underset{\sim}{u} + \underset{\sim}{0} = \alpha\underset{\sim}{u}$; similarly if $\alpha \neq 0, \beta = 0$; if $\alpha = \beta = 0$ then $(\alpha+\beta)\underset{\sim}{u} = 0\underset{\sim}{u} = \underset{\sim}{0}$ and $\alpha\underset{\sim}{u} + \beta\underset{\sim}{u} = 0\underset{\sim}{u} + 0\underset{\sim}{u} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$.

(C) It is evident from Fig. 3a that the $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{u}+\underset{\sim}{v}$ and $\alpha\underset{\sim}{u}, \alpha\underset{\sim}{v}, \alpha\underset{\sim}{u}+\alpha\underset{\sim}{v}$ triangles are similar. Thus, $\alpha\underset{\sim}{u}+\alpha\underset{\sim}{v}$ is α times as long as $\underset{\sim}{u}+\underset{\sim}{v}$ and in the same direction as $\underset{\sim}{u}+\underset{\sim}{v}$, and in the same direction as $\underset{\sim}{u}+\underset{\sim}{v}$. The same is true of $\alpha(\underset{\sim}{u}+\underset{\sim}{v})$, so (4C) follows.

(d) If $\underset{\sim}{u} = \underset{\sim}{0}$ then $1\underset{\sim}{u} = 1\underset{\sim}{0} = \underset{\sim}{0}$ so (4d) holds. If $\underset{\sim}{u} \neq \underset{\sim}{0}$ then $1\underset{\sim}{u}$ is $|1|$ times as long as $\underset{\sim}{u}$ and in the same direction as $\underset{\sim}{u}$, so it is equal to $\underset{\sim}{u}$.

5. (a) Suppose $\underset{\sim}{A}+\underset{\sim}{B}+\underset{\sim}{C}$ does $= \underset{\sim}{0}$. Then  must close, to form a triangle:



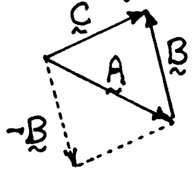
Applying the law of cosines, $\|\underset{\sim}{C}\|^2 = \|\underset{\sim}{A}\|^2 + \|\underset{\sim}{B}\|^2 - 2\|\underset{\sim}{A}\|\|\underset{\sim}{B}\|\cos\theta$,

$$25 = 1 + 4 - 2(1)(2)\cos\theta,$$

$\cos\theta = 5$, which is impossible. Thus, the $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$ figure cannot close.

(b) This time $25 = 16 + 4 - 2(4)(2)\cos\theta$, so $|\cos\theta| = 5/16 < 1$ so the $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$ figure can close (by adjusting the directions of $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$).

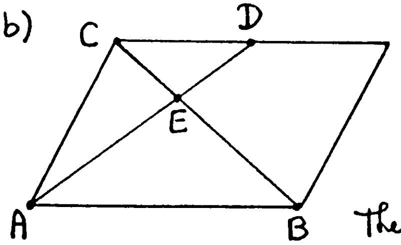
6.



Given that $\underset{\sim}{A}+\underset{\sim}{B} = \underset{\sim}{C}$, observe that $\underset{\sim}{C}-\underset{\sim}{B}$, i.e. $\underset{\sim}{C}+(-\underset{\sim}{B})$, is $= \underset{\sim}{A}$.

7. (a) If $\underset{\sim}{A}+\underset{\sim}{B}=\underset{\sim}{0}$ then $\underset{\sim}{A}+\underset{\sim}{B}+(-\underset{\sim}{B})=\underset{\sim}{0}+(-\underset{\sim}{B})$, $\underset{\sim}{A}+\underset{\sim}{0}=\underset{\sim}{0}+(-\underset{\sim}{B})$, $\underset{\sim}{A}=-\underset{\sim}{B}$, so either A and B are of equal length and oppositely directed, or they are both $\underset{\sim}{0}$. The former does not hold, by assumption, so $\underset{\sim}{A}=\underset{\sim}{B}=\underset{\sim}{0}$.

(b)



$$\left.\begin{aligned} AB+BC&=AC \leftarrow \text{solve for AB and use to eliminate AB}\\ BE&=\alpha BC\\ AB+BE&=AE\\ \beta AE-AC&=\tfrac{1}{2}AB\end{aligned}\right\}\Rightarrow \begin{aligned}BE&=\alpha BC\\ AC-BC+BE&=AE\\ \beta AE-AC&=\tfrac{1}{2}(AC-BC)\end{aligned}$$

Then eliminate BE,

$$\left.\begin{aligned}AC-BC+\alpha BC&=AE\\ \beta AE+\tfrac{1}{2}BC&=\tfrac{3}{2}AC\end{aligned}\right\}\Rightarrow$$

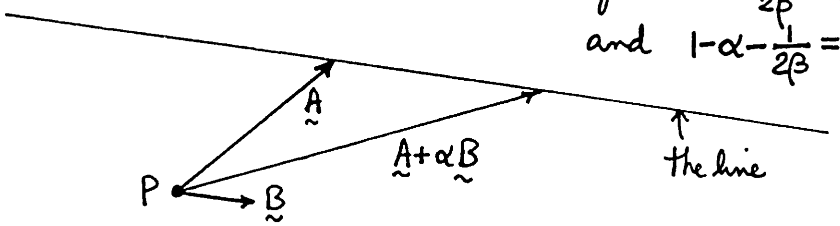
Eliminate AE,

$$AC-(1-\alpha)BC=\frac{3}{2\beta}AC-\frac{1}{2\beta}BC$$

$$\left(1-\frac{3}{2\beta}\right)AC=\left(1-\alpha-\frac{1}{2\beta}\right)BC$$

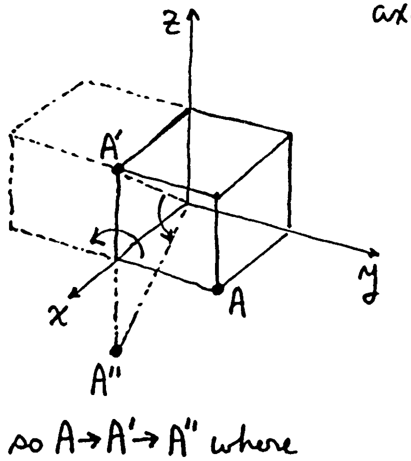
$$\left.\begin{aligned}&\text{AC, BC not aligned}\Rightarrow 1-\frac{3}{2\beta}=0\\ &\text{and}\quad 1-\alpha-\frac{1}{2\beta}=0\end{aligned}\right\}\text{ so }\alpha=\frac{2}{3}$$

8.



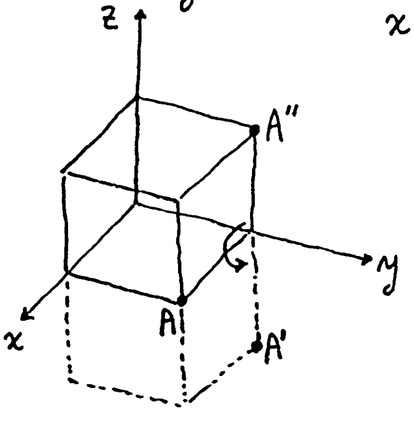
9.
$$\begin{aligned}\underset{\sim}{OB}&=\underset{\sim}{OA}+\underset{\sim}{AB}\\ &=\underset{\sim}{OA}+\alpha\underset{\sim}{AC}\\ &=\underset{\sim}{OA}+\alpha(\underset{\sim}{OC}-\underset{\sim}{OA})\\ &=(1-\alpha)\underset{\sim}{OA}+\alpha\underset{\sim}{OC}.\end{aligned}$$

10. $\overbrace{\pi/2\text{ about }x\text{ axis}}^{\underset{\sim}{\theta}_1\text{, say}}$ then $\overbrace{\pi/2\text{ about }y\text{ axis}}^{\underset{\sim}{\theta}_2\text{, say}}$:



so A→A'→A" where A" is at (1,0,−1).

π/2 about y axis then π/2 about x axis:



so A→A'→A" where A" is at (0,1,1). Thus, $\underset{\sim}{\theta}_1+\underset{\sim}{\theta}_2\neq\underset{\sim}{\theta}_2+\underset{\sim}{\theta}_1$ whereas $\underset{\sim}{\theta}_1+\underset{\sim}{\theta}_2=\underset{\sim}{\theta}_2+\underset{\sim}{\theta}_1$ would have to hold if $\underset{\sim}{\theta}$ were a true vector.

Section 9.3

1. (a) $\underset{\sim}{u}\cdot\underset{\sim}{v} = \|\underset{\sim}{u}\|\|\underset{\sim}{v}\|\cos\theta = (5)(5/\cos 30^\circ)\cos(90^\circ+60^\circ) = -25$

(b) $\underset{\sim}{u}\cdot\underset{\sim}{v} = (3)(6)\cos 0 = 18$

(c) $\underset{\sim}{u}\cdot\underset{\sim}{v} = (6)(6)\cos 180^\circ = -36$

(d) $\underset{\sim}{u}\cdot\underset{\sim}{v} = (4/\cos 30^\circ)(4/\cos 60^\circ)\cos 30^\circ = 32$

2. (a) $\underset{\sim}{v}\cdot\underset{\sim}{u} = \begin{cases} \|\underset{\sim}{v}\|\|\underset{\sim}{u}\|\cos\theta, & \text{if } \underset{\sim}{u},\underset{\sim}{v} \neq \underset{\sim}{0} \\ 0, & \text{if } \underset{\sim}{v} \text{ or } \underset{\sim}{u} = \underset{\sim}{0} \end{cases} = \underset{\sim}{u}\cdot\underset{\sim}{v}$ defined by (2).

(b) $\underset{\sim}{u}\cdot\underset{\sim}{u} = \begin{cases} \|\underset{\sim}{u}\|^2\cos 0 & \text{if } \underset{\sim}{u}\neq\underset{\sim}{0} \\ 0 & \text{if } \underset{\sim}{u}=\underset{\sim}{0} \end{cases} \quad \begin{matrix} >0 \text{ if } \underset{\sim}{u}\neq\underset{\sim}{0} \\ =0 \text{ if } \underset{\sim}{u}=\underset{\sim}{0} \end{matrix}$

(c) $(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})\cdot\underset{\sim}{w} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) + \beta(\underset{\sim}{v}\cdot\underset{\sim}{w})$ ①

For $\alpha=\beta=1$: $(\underset{\sim}{u}+\underset{\sim}{v})\cdot\underset{\sim}{w} = \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{w}$ ②

For $\beta=0$: $(\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w})$ ③

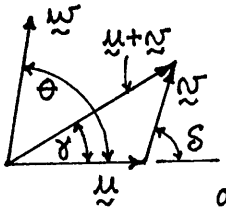
Thus, ① implies ② and ③. Conversely, ② and ③ imply ① because

$(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})\cdot\underset{\sim}{w} = (\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} + (\beta\underset{\sim}{v})\cdot\underset{\sim}{w}$ per ②

$= \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) + \beta(\underset{\sim}{v}\cdot\underset{\sim}{w})$ per ③.

Thus, instead of proving ① it will suffice to prove ② and ③.

First ②: $(\underset{\sim}{u}+\underset{\sim}{v})\cdot\underset{\sim}{w} = \|\underset{\sim}{u}+\underset{\sim}{v}\|\|\underset{\sim}{w}\|\cos(\theta-\gamma)$



$= \|\underset{\sim}{u}+\underset{\sim}{v}\|\|\underset{\sim}{w}\|(\cos\theta\cos\gamma + \sin\theta\sin\gamma)$

$= \cancel{\|\underset{\sim}{u}+\underset{\sim}{v}\|}\|\underset{\sim}{w}\|\left(\cos\theta\,\dfrac{\|\underset{\sim}{u}\|+\|\underset{\sim}{v}\|\cos\delta}{\cancel{\|\underset{\sim}{u}+\underset{\sim}{v}\|}} + \sin\theta\,\dfrac{\|\underset{\sim}{v}\|\sin\delta}{\cancel{\|\underset{\sim}{u}+\underset{\sim}{v}\|}}\right)$

$= \|\underset{\sim}{w}\|\left[\|\underset{\sim}{u}\|\cos\theta + \|\underset{\sim}{v}\|\cos(\theta-\delta)\right]$

and $\underset{\sim}{u}\cdot\underset{\sim}{w}+\underset{\sim}{v}\cdot\underset{\sim}{w} = \|\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta - \|\underset{\sim}{v}\|\|\underset{\sim}{w}\|\cos(\theta-\delta)$ $>\checkmark$

Next, ③: If $\alpha\geq 0$, $(\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} = \|\alpha\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta = \alpha\|\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta$

$\alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) = \alpha\|\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta$ $>\checkmark$

If $\alpha<0$, $(\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} = \|\alpha\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos(\pi-\theta)$

$= |\alpha|\,\|\underset{\sim}{u}\|\|\underset{\sim}{w}\|(-\cos\theta) = \alpha\|\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta$

$\alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) = \alpha\|\underset{\sim}{u}\|\|\underset{\sim}{w}\|\cos\theta$ $>\checkmark$

and we are done.

3. $(\underset{\sim}{u}+\underset{\sim}{v})\cdot(\underset{\sim}{w}+\underset{\sim}{x}) = \underset{\sim}{u}\cdot(\underset{\sim}{w}+\underset{\sim}{x}) + \underset{\sim}{v}\cdot(\underset{\sim}{w}+\underset{\sim}{x})$ per ② above

$= (\underset{\sim}{w}+\underset{\sim}{x})\cdot\underset{\sim}{u} + (\underset{\sim}{w}+\underset{\sim}{x})\cdot\underset{\sim}{v}$ per commutativity (Exercise 2a)

$= \underset{\sim}{w}\cdot\underset{\sim}{u} + \underset{\sim}{x}\cdot\underset{\sim}{u} + \underset{\sim}{w}\cdot\underset{\sim}{v} + \underset{\sim}{x}\cdot\underset{\sim}{v}$ per ② above

$= \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{u}\cdot\underset{\sim}{x} + \underset{\sim}{v}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{x}$ per commutativity

4. (b) $\underset{\sim}{BA}\cdot\underset{\sim}{OP} = BA\cdot(OA+AD+\frac{1}{2}AB+\frac{1}{2}AO)$ since $DC=AB$

$= 0+0-\frac{1}{2}(1)^2+0 = -1/2$

(c) $AC\cdot OP = (AD+DC)\cdot(OA+AD+\frac{1}{2}DC-\frac{1}{2}OA)$

$= AD\cdot(OA+AD+\frac{1}{2}DC-\frac{1}{2}OA) + DC\cdot(OA+AD+\frac{1}{2}DC-\frac{1}{2}OA)$

$= 0+1+0+0+0+\frac{1}{2} = 3/2$

(e) $OC\cdot OP = OC\cdot(OC+CP) = 3+OC\cdot CP = 3-1$ (since $OC\cdot CP=-1$ from part (d))

$=2$

(f) $BC\cdot OP = BC\cdot(OA+AD+DP) = 0+1+0=1$

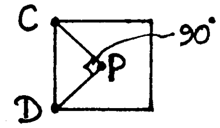
(g) $AO\cdot OP = AO\cdot(\tfrac{1}{2}OA+\tfrac{1}{2}AB+AD) = -\tfrac{1}{2}+0+0=-\tfrac{1}{2}$

(h) $CP\cdot DP = (\tfrac{1}{2}AO+\tfrac{1}{2}BA)\cdot(DC+\tfrac{1}{2}AO+\tfrac{1}{2}BA)$

$= \tfrac{1}{2}AO\cdot(DC+\tfrac{1}{2}AO+\tfrac{1}{2}BA)+\tfrac{1}{2}BA\cdot(DC+\tfrac{1}{2}AO+\tfrac{1}{2}BA)$

$= 0+\tfrac{1}{4}(0)+\tfrac{1}{2}(-1)+0+\tfrac{1}{4}=0,$

as could have been seen more readily from a sketch (shown at the right) of the $y=1$ face.



(i) $BP\cdot DB = (BC+CP)\cdot(DC+CB)$

$= BC\cdot(DC+CB)+CP\cdot(DC+CB) = 0-1-\tfrac{1}{2}+0=-\tfrac{3}{2}$

(j) $PB\cdot CO = (PC+CB)\cdot(CD+DA+AO)$

$= PC\cdot(CD+DA+AO)+CB\cdot(CD+DA+AO) = -\tfrac{1}{2}+0-\tfrac{1}{2}-0+1+0=0$

(k) $AP\cdot PB = (AD+\tfrac{1}{2}DC+\tfrac{1}{2}AO)\cdot(PC+CB)$

$= PC\cdot(AD+\tfrac{1}{2}DC+\tfrac{1}{2}AO)+CB\cdot(AD+\tfrac{1}{2}DC+\tfrac{1}{2}AO)=0+\tfrac{1}{4}-\tfrac{1}{2}(\tfrac{1}{2})-1+0+0=-1$

(l) $AO\cdot PA = AO\cdot(\tfrac{1}{2}BA+\tfrac{1}{2}OA+DA) = 0-\tfrac{1}{2}+0=-\tfrac{1}{2}$.

5. (a) $$\sphericalangle APO = \cos^{-1}\frac{PA\cdot PO}{\|PA\|\,\|PO\|} = \cos^{-1}\frac{(\tfrac{1}{2}OA+\tfrac{1}{2}BA+DA)\cdot(\tfrac{1}{2}AO+\tfrac{1}{2}BA+DA)}{\|\quad''\quad\|\;\|\quad''\quad\|}$$

$$= \cos^{-1}\frac{-\tfrac{1}{4}+\tfrac{1}{4}+1}{\sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1}\,\sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1}} = \cos^{-1}\frac{2}{3} = 48.19^\circ$$

(b) $$\sphericalangle APB = \cos^{-1}\frac{PA\cdot PB}{\|PA\|\,\|PB\|} = \cos^{-1}\frac{1}{\sqrt{\frac{6}{4}}\sqrt{\frac{6}{4}}} = \cos^{-1}\frac{2}{3} = 48.19^\circ$$

(c) $$\sphericalangle APC = \cos^{-1}\frac{PA\cdot PC}{\|PA\|\,\|PC\|} = \cos^{-1}0 = 90^\circ$$

(d) $$\sphericalangle APD = \cos^{-1}\frac{PA\cdot PD}{\|PA\|\,\|PD\|} = \cos^{-1}\frac{\frac{1}{2}}{\sqrt{\frac{6}{4}}\sqrt{\frac{1}{2}}} = \cos^{-1}\frac{1}{\sqrt{3}} = 54.74^\circ$$

(e) $$\sphericalangle ABP = \cos^{-1}\frac{BA\cdot BP}{\|BA\|\,\|BP\|} = \cos^{-1}\frac{\frac{1}{2}}{(1)\sqrt{\frac{6}{4}}} = \cos^{-1}\frac{1}{\sqrt{6}} = 65.91^\circ$$

(f) $$\sphericalangle ACP = \cos^{-1}\frac{CA\cdot CP}{\|CA\|\,\|CP\|} = \cos^{-1}\frac{\frac{1}{2}}{\sqrt{2}\sqrt{\frac{1}{2}}} = \cos^{-1}\frac{1}{2} = 60^\circ$$

(g) $$\sphericalangle BPO = \cos^{-1}\frac{PB\cdot PO}{\|PB\|\,\|PO\|} = \cos^{-1}\frac{\frac{1}{2}}{\sqrt{\frac{6}{4}}\sqrt{\frac{6}{4}}} = \cos^{-1}\frac{1}{3} = 70.53^\circ$$

(h) $$\sphericalangle BPC = \cos^{-1}\frac{PB\cdot PC}{\|PB\|\,\|PC\|} = \cos^{-1}\frac{\frac{1}{2}}{\sqrt{\frac{6}{4}}\sqrt{\frac{1}{2}}} = \cos^{-1}\frac{1}{\sqrt{3}} = 54.74^\circ$$

(i) $$\sphericalangle BPD = \cos^{-1}\frac{PB\cdot PD}{\|PB\|\,\|PD\|} = \cos^{-1}0 = 90^\circ$$

(k) $\measuredangle CPO = \cos^{-1}\frac{PC\cdot PO}{\|PC\|\|PO\|} = \cos^{-1}\frac{-\frac{1}{2}}{\sqrt{\frac{1}{2}}\sqrt{\frac{6}{4}}} = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) = 125.3°$

6. We want to show that $\cos^{-1}\frac{\underset{\sim}{w}\cdot\underset{\sim}{u}}{\|\underset{\sim}{w}\|\|\underset{\sim}{u}\|} = \cos^{-1}\frac{\underset{\sim}{w}\cdot\underset{\sim}{v}}{\|\underset{\sim}{w}\|\|\underset{\sim}{v}\|}$ or, equivalently, that $\frac{\underset{\sim}{w}\cdot\underset{\sim}{u}}{\|\underset{\sim}{u}\|} = \frac{\underset{\sim}{w}\cdot\underset{\sim}{v}}{\|\underset{\sim}{v}\|}$. Well, $\frac{\underset{\sim}{w}\cdot\underset{\sim}{u}}{\|\underset{\sim}{u}\|} = \frac{\|\underset{\sim}{v}\|\cancelto{\|\underset{\sim}{u}\|^2}{\underset{\sim}{u}\cdot\underset{\sim}{u}} + \|\underset{\sim}{u}\|\underset{\sim}{v}\cdot\underset{\sim}{u}}{\|\underset{\sim}{u}\|} = \|\underset{\sim}{v}\|\|\underset{\sim}{u}\| + \underset{\sim}{v}\cdot\underset{\sim}{u}$
and $\frac{\underset{\sim}{w}\cdot\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{\|\underset{\sim}{v}\|\,\underset{\sim}{u}\cdot\underset{\sim}{v} + \|\underset{\sim}{u}\|\cancelto{\|\underset{\sim}{v}\|^2}{\underset{\sim}{v}\cdot\underset{\sim}{v}}}{\|\underset{\sim}{v}\|} = \underset{\sim}{u}\cdot\underset{\sim}{v} + \|\underset{\sim}{u}\|\|\underset{\sim}{v}\|.$ ✓

Section 9.4

1. (b) $3\underset{\sim}{t} - 5\underset{\sim}{u} = (5,5,-12,-14)$ per (9b), then (9d), then (9a)
   (c) $4[\underset{\sim}{u} + 5(\underset{\sim}{w} - 2\underset{\sim}{u})] = (-92,-4,-8,-24)$
   (e) $-\underset{\sim}{w} + \underset{\sim}{t} = (6,2,-4,-4)$
   (f) division of vectors not defined
   (g) $\underset{\sim}{t} + 2\underset{\sim}{u} + 3\underset{\sim}{w} = \underset{\sim}{t} + (2\underset{\sim}{u} + 3\underset{\sim}{w})$ per (10b)
   $= \underset{\sim}{t} + [(4,-2,6,8) + (-3,-6,15,18)]$ per (9b)
   $= \underset{\sim}{t} + (1,-8,21,26)$ per (9a)
   $= (6,-8,22,28)$ per (9a)
   (h) $\underset{\sim}{t} - 2\underset{\sim}{u} - 4\underset{\sim}{v}$ = not defined since $\underset{\sim}{t}, \underset{\sim}{u}$ are 4-tuples and $\underset{\sim}{v}$ is a 3-tuple
   (i through m) also not defined

2. $\underset{\sim}{x} + \underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} = \underset{\sim}{0}$ so $\underset{\sim}{x} = -(\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}) = -(7,6,-3,-3) = (-7,-6,3,3)$

3. (a) $3\underset{\sim}{x} + 2(\underset{\sim}{u} - 5\underset{\sim}{v}) = \underset{\sim}{w}$
   $3\underset{\sim}{x} + 2\{\underset{\sim}{u} + [-(5\underset{\sim}{v})]\} = \underset{\sim}{w}$ per (9e)
   $3\underset{\sim}{x} + 2\{\underset{\sim}{u} + [-(10,0,-25,0)]\} = \underset{\sim}{w}$ per (9e)
   $3\underset{\sim}{x} + 2\{\underset{\sim}{u} + (-10,0,25,0)\} = \underset{\sim}{w}$ per (9d) then (9b)
   $3\underset{\sim}{x} + 2(-9,3,25,-2) = \underset{\sim}{w}$ per (9a)
   $3\underset{\sim}{x} + (-18,6,50,-4) = \underset{\sim}{w}$ per (9b)
   $[3\underset{\sim}{x} + (-18,6,50,-4)] + (18,-6,-50,4) = \underset{\sim}{w} + (18,-6,-50,4)$ adding equals to equals
   $3\underset{\sim}{x} + [(-18,6,50,-4) + (18,-6,-50,4)] = \underset{\sim}{w} + (18,-6,-50,4)$ per (10b)
   $3\underset{\sim}{x} + \underset{\sim}{0} = (22,-3,-48,3)$ per (9a)
   $3\underset{\sim}{x} = (22,-3,-48,3)$ per (10c)
   $(\frac{1}{3})(3\underset{\sim}{x}) = (\frac{1}{3})(22,-3,-48,3)$ multiplying equals by equals
   $1\underset{\sim}{x} = (\frac{1}{3})($ " $)$ per (10e)
   $\underset{\sim}{x} = (\frac{1}{3})($ " $)$ per (10h)
   $\underset{\sim}{x} = (\frac{22}{3},-1,-16,1)$ per (9b)

Naturally, in applications we do not go through all this detail; our purpose here was to show that we _could_ justify our detailed steps if we chose to.

(b) $3\underset{\sim}{x} = 4\underset{\sim}{0} + (1,0,0,0)$

$= \underset{\sim}{0} + (1,0,0,0)$ per (10i)

$= (1,0,0,0)$ per (10a) then (10C)

$(\frac{1}{3})(3\underset{\sim}{x}) = \frac{1}{3}(1,0,0,0)$ multiplying equals by equals

$1\underset{\sim}{x} =$ " per (10e)

$\underset{\sim}{x} =$ " per (10h)

$= (\frac{1}{3}, 0, 0, 0)$ per (9b)

(c) $\underset{\sim}{u} - 4\underset{\sim}{x} = \underset{\sim}{0}$

$(\underset{\sim}{u} - 4\underset{\sim}{x}) + 4\underset{\sim}{x} = \underset{\sim}{0} + 4\underset{\sim}{x}$ adding equals to equals

$(\underset{\sim}{u} + (-4\underset{\sim}{x})) + 4\underset{\sim}{x} = 4\underset{\sim}{x} + \underset{\sim}{0}$ per (9e) on LHS and (10a) on RHS

$\underset{\sim}{u} + [4\underset{\sim}{x} + (-4\underset{\sim}{x})] = 4\underset{\sim}{x}$ per (10b) then (10a) on LHS and (10c) on RHS

$\underset{\sim}{u} + \underset{\sim}{0} = 4\underset{\sim}{x}$ per (10d)

$\underset{\sim}{u} = 4\underset{\sim}{x}$ per (10c)

$\frac{1}{4}\underset{\sim}{u} = \frac{1}{4}(4\underset{\sim}{x})$ multiplying equals by equals

$= (\frac{1}{4}4)\underset{\sim}{x}$ per (10e)

$= 1\underset{\sim}{x}$

$= \underset{\sim}{x}$ per (10h), so $\underset{\sim}{x} = (\frac{1}{4}, \frac{3}{4}, 0, -\frac{1}{2})$ per (9b)

4. (a) $\alpha_1(2,1,3) + \alpha_2(1,2,-4) + \alpha_3(0,1,1) = (0,0,0)$

$(2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3, 3\alpha_1 - 4\alpha_2 + \alpha_3) = (0,0,0)$

$$\begin{aligned} 2\alpha_1 + \alpha_2 \quad &= 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 &= 0 \\ 3\alpha_1 - 4\alpha_2 + \alpha_3 &= 0 \end{aligned}$$

and Gauss elimination gives the unique solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

(b) As in (a), we get

$$\begin{aligned} 2\alpha_1 \qquad - 2\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 3\alpha_1 + \alpha_2 - \alpha_3 &= 0 \end{aligned}$$

and Gauss elim. gives the 1-parameter family of solutions $\alpha_1 = C, \alpha_2 = -2C, \alpha_3 = C$ where C is arbitrary

(c) As in (a), we get

$$\begin{aligned} 2\alpha_1 + \alpha_2 - 2\alpha_3 &= 1 \\ \alpha_1 + 2\alpha_2 + \alpha_3 &= 3 \\ 3\alpha_1 - 4\alpha_2 - \alpha_3 &= 2 \end{aligned}$$

and Gauss elim gives unique soln. $\alpha_3 = 29/28$, $\alpha_2 = 2/7$, $\alpha_1 = 39/28$.

(d) As in (a), we get

$$\begin{aligned} 2\alpha_1 \qquad - 2\alpha_3 &= 2 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 3\alpha_1 + \alpha_2 - \alpha_3 &= -1 \end{aligned}$$

and Gauss elim. shows there is NO solution for the α's.

(e) As in (a), we get

$$\begin{aligned} \alpha_1 \qquad &= 0 \\ 2\alpha_1 + \alpha_2 &= 0 \\ -4\alpha_1 + \alpha_2 &= 0 \end{aligned}$$

gives unique solution $\alpha_1 = \alpha_2 = 0$

(f) As in (a), we get

$$\begin{aligned} \alpha_1 \qquad + 2\alpha_3 &= -2 \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ -4\alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

and Gauss elim. gives unique solution $\alpha_3 = -6/7$, $\alpha_2 = -2/7, \alpha_1 = -2/7$.

5. (a) No. E.g., if $\underset{\sim}{u} = (1,0,0,0)$ and $\underset{\sim}{v} = (0,1,0,0)$ then $\alpha_1 \underset{\sim}{u} + \alpha_2 \underset{\sim}{v} = \underset{\sim}{0}$ gives $(\alpha_1, \alpha_2, 0, 0) = (0,0,0,0)$, which holds iff $\alpha_1 = \alpha_2 = 0$

(b) No. E.g., if $\underset{\sim}{u} = (1,0,0)$ and $\underset{\sim}{v} = (0,1,0)$ then $\alpha_1 \underset{\sim}{u} + \alpha_2 \underset{\sim}{v} = (\alpha_1, \alpha_2, 0) = (0,0,0)$ iff $\alpha_1 = \alpha_2 = 0$.

Section 9.5

1. (a) $\|\underset{\sim}{u}\| = \sqrt{4^2+3^2} = 5$, $\|\underset{\sim}{v}\| = \sqrt{2^2+(-1)^2} = \sqrt{5}$, $\theta = \cos^{-1} \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\| \|\underset{\sim}{v}\|} = \cos^{-1} \frac{8-3}{5\sqrt{5}} = 63.4°$, 1.11 rad

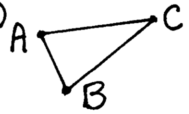
(c) $\|\underset{\sim}{u}\| = \sqrt{10}$, $\|\underset{\sim}{v}\| = \sqrt{49} = 7$, $\theta = \cos^{-1} 0 = 90°$; $\pi/2$ rad ; orthogonal

(e) $\|\underset{\sim}{u}\| = \sqrt{29}$, $\|\underset{\sim}{v}\| = \sqrt{116} = 2\sqrt{29}$, $\theta = \cos^{-1} 0 = 90°$, $\pi/2$ rad; orthogonal

(g) $\|\underset{\sim}{u}\| = \sqrt{15}$, $\|\underset{\sim}{v}\| = \sqrt{45} = 3\sqrt{5}$, $\cos^{-1} \frac{-13}{\sqrt{15}\, 3\sqrt{5}} = \cos^{-1}(-0.5004) = 120°$, 2.09 rad

2. (a), (c), (d), (e), (h), (i) are defined; the others are not.

3. (a)



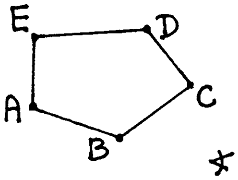
$\measuredangle CAB = \cos^{-1} \frac{AB \cdot AC}{\|AB\| \|AC\|} = \cos^{-1} \frac{(1,-1)\cdot(3,0)}{(\sqrt{2})(3)} = \cos^{-1} \frac{1}{\sqrt{2}} = 45°$

$\measuredangle ABC = \cos^{-1} \frac{BA \cdot BC}{\|BA\| \|BC\|} = \cos^{-1} \frac{(-1,1)\cdot(2,1)}{\sqrt{2}\sqrt{5}} = \cos^{-1}\left(-\frac{1}{\sqrt{10}}\right) = 108.4°$

$\measuredangle BCA = \cos^{-1} \frac{CB \cdot CA}{\|CB\| \|CA\|} = \cos^{-1} \frac{(-2,-1)\cdot(-3,0)}{(\sqrt{5})(3)} = \cos^{-1} \frac{2}{\sqrt{5}} = 26.6°$

Sum = 45 + 108.4 + 26.6 = 180.0° ✓ (Correct for any triangle.)

(c)



$\measuredangle EAB = \cos^{-1} \frac{AE \cdot AB}{\|AE\| \|AB\|} = \cos^{-1} \frac{(0,2)\cdot(1,-1)}{2\sqrt{2}} = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135°$

$\measuredangle ABC = 108.4°$ per part (a)

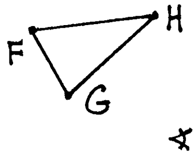
$\measuredangle BCD = \cos^{-1} \frac{CB \cdot CD}{\|CB\| \|CD\|} = \cos^{-1} \frac{(-2,-1)\cdot(-1,2)}{\sqrt{5}\sqrt{5}} = \cos^{-1}(0) = 90°$

$\measuredangle CDE = \cos^{-1} \frac{DC \cdot DE}{\|DC\| \|DE\|} = \cos^{-1} \frac{(1,-2)\cdot(-2,0)}{(\sqrt{5})(2)} = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) = 116.6°$

$\measuredangle DEA = \cos^{-1} \frac{ED \cdot EA}{\|ED\| \|EA\|} = \cos^{-1} \frac{(2,0)\cdot(0,-2)}{(2)(2)} = \cos^{-1} 0 = 90°$

Sum = 135 + 108.4 + 90 + 116.6 + 90 = 540° (Correct for any pentagon)

(f)

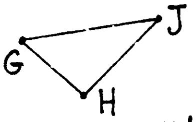


$\measuredangle HFG = \cos^{-1} \frac{FH \cdot FG}{\|FH\| \|FG\|} = \cos^{-1} \frac{(4,1,5)\cdot(1,-3,6)}{\sqrt{42}\sqrt{46}} = \cos^{-1} \frac{31}{\sqrt{1104}} = 45.1°$

$\measuredangle FGH = \cos^{-1} \frac{GF \cdot GH}{\|GF\| \|GH\|} = \cos^{-1} \frac{(-1,3,-6)\cdot(3,4,-1)}{\sqrt{46}\sqrt{26}} = \cos^{-1} \frac{15}{\sqrt{1196}} = 64.3°$

$\measuredangle GHF = \cos^{-1} \frac{HG \cdot HF}{\|HG\| \|HF\|} = \cos^{-1} \frac{(-3,-4,1)\cdot(-4,-1,-5)}{\sqrt{26}\sqrt{42}} = \cos^{-1} \frac{11}{\sqrt{1092}} = 70.6°$

Sum = 45.1 + 64.3 + 70.6 = 180° ✓

(j) 

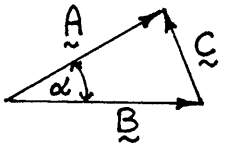
$$\measuredangle GHJ = \cos^{-1}\frac{HG\cdot HJ}{\|HG\|\|HJ\|} = \cos^{-1}\frac{(-3,-4,1)\cdot(-5,-4,-3)}{\sqrt{26}\sqrt{50}} = \cos^{-1}\frac{28}{\sqrt{26}\sqrt{50}} = 39.1^\circ$$

$$\measuredangle HJG = \cos^{-1}\frac{JH\cdot JG}{\|JH\|\|JG\|} = \cos^{-1}\frac{(5,4,3)\cdot(2,0,4)}{\sqrt{50}\sqrt{20}} = \cos^{-1}\frac{22}{\sqrt{1000}} = 45.9^\circ$$

$$\measuredangle JHG = \cos^{-1}\frac{GJ\cdot GH}{\|GJ\|\|GH\|} = \cos^{-1}\frac{(-2,0,-4)\cdot(3,4,-1)}{\sqrt{20}\sqrt{26}} = \cos^{-1}\frac{-2}{\sqrt{520}} = 95.0^\circ$$

$\text{Sum} = 39.1 + 45.9 + 95.0 = 180^\circ$ ✓

4. (d) $\hat{\underset{\sim}{u}} = \frac{1}{2\sqrt{3}}(2,2,2) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{77}}(-4,-5,-6) = (-4/\sqrt{77}, -5/\sqrt{77}, -6/\sqrt{77})$

5. 

$\underset{\sim}{C}\cdot\underset{\sim}{C} = (\underset{\sim}{A}-\underset{\sim}{B})\cdot(\underset{\sim}{A}-\underset{\sim}{B})$

$C^2 = A\cdot A - A\cdot B - B\cdot A + B\cdot B = A^2 + B^2 - 2AB\cos\alpha$

6. (d) $\underset{\sim}{u} = (1,2,0,1)$, $\underset{\sim}{v} = (1,0,1,1)$, $\underset{\sim}{w} = (2,-1,1,1)$.

We'll need these: $\underset{\sim}{u}\cdot\underset{\sim}{u} = 6$, $\underset{\sim}{v}\cdot\underset{\sim}{v} = 3$, $\underset{\sim}{w}\cdot\underset{\sim}{w} = 7$

$\underset{\sim}{u}\cdot\underset{\sim}{v} = 2$, $\underset{\sim}{u}\cdot\underset{\sim}{w} = 1$, $\underset{\sim}{v}\cdot\underset{\sim}{w} = 4$

$\underset{\sim}{u}_2\cdot\underset{\sim}{u}_1 = (\underset{\sim}{u}+\alpha\underset{\sim}{v})\cdot\underset{\sim}{u} = 6 + 2\alpha = 0$, so $\alpha = -3$.

$\underset{\sim}{u}_3\cdot\underset{\sim}{u}_1 = (\underset{\sim}{u}+\beta\underset{\sim}{v}+\gamma\underset{\sim}{w})\cdot\underset{\sim}{u} = 6 + 2\beta + \gamma = 0$ ①

$\underset{\sim}{u}_3\cdot\underset{\sim}{u}_2 = (\underset{\sim}{u}+\beta\underset{\sim}{v}+\gamma\underset{\sim}{w})\cdot(\underset{\sim}{u}-3\underset{\sim}{v})$

$= (6-6) + \beta(2-9) + \gamma(1-12) = -7\beta - 11\gamma = 0$ ②

① and ② give $\beta = -154/35$, $\gamma = 14/5$.

Also, $\alpha = -3$, $\underset{\sim}{u}_1 = \underset{\sim}{u} = (1,2,0,1)$,

$\underset{\sim}{u}_2 = \underset{\sim}{u} - 3\underset{\sim}{v} = (-2,2,-3,-2)$

$\underset{\sim}{u}_3 = \underset{\sim}{u} - \frac{154}{35}\underset{\sim}{v} + \frac{14}{5}\underset{\sim}{w} = (\frac{11}{5}, -\frac{4}{5}, -\frac{8}{5}, -\frac{3}{5})$

(g) $\underset{\sim}{u} = (1,2)$, $\underset{\sim}{v} = (0,2)$, $\underset{\sim}{w} = (1,-1)$

We'll need $\underset{\sim}{u}\cdot\underset{\sim}{u} = 5$, $\underset{\sim}{v}\cdot\underset{\sim}{v} = 4$, $\underset{\sim}{w}\cdot\underset{\sim}{w} = 2$

$\underset{\sim}{u}\cdot\underset{\sim}{v} = 4$, $\underset{\sim}{u}\cdot\underset{\sim}{w} = -1$, $\underset{\sim}{v}\cdot\underset{\sim}{w} = -2$

$\underset{\sim}{u}_2\cdot\underset{\sim}{u}_1 = (\underset{\sim}{u}+\alpha\underset{\sim}{v})\cdot\underset{\sim}{u} = 5 + 4\alpha = 0$, $\alpha = -5/4$

$\underset{\sim}{u}_3\cdot\underset{\sim}{u}_1 = (\underset{\sim}{u}+\beta\underset{\sim}{v}+\gamma\underset{\sim}{w})\cdot\underset{\sim}{u} = 5 + 4\beta - \gamma = 0$ ①

$\underset{\sim}{u}_3\cdot\underset{\sim}{u}_2 = (\underset{\sim}{u}+\beta\underset{\sim}{v}+\gamma\underset{\sim}{w})\cdot(\underset{\sim}{u}-\frac{5}{4}\underset{\sim}{v})$

$= (5-5) + \beta(4-5) + \gamma(-1+\frac{5}{2})$

$= -\beta + \frac{3}{2}\gamma = 0$ ②

① and ② give $\beta = -3/2$, $\gamma = -1$.

Also, $\alpha = -5/4$, $\underset{\sim}{u}_1 = \underset{\sim}{u} = (1,2)$

$\underset{\sim}{u}_2 = \underset{\sim}{u} - \frac{5}{4}\underset{\sim}{v} = (1, -1/2)$

$\underset{\sim}{u}_3 = \underset{\sim}{u} - \frac{3}{2}\underset{\sim}{v} - \underset{\sim}{w} = (0,0)$ is the zero vector, so we cannot find such an orthogonal set.

7. (a) $\underset{\sim}{u}-\underset{\sim}{v} = (-1,3,-4,-1)$, $\|\underset{\sim}{u}-\underset{\sim}{v}\| = \sqrt{1+9+16+1} = \sqrt{27} = 3\sqrt{3}$

(c) $\left\| \frac{1}{\|\underset{\sim}{u}\|}\underset{\sim}{u} \right\| = 1$ for every $\underset{\sim}{u} \neq \underset{\sim}{0}$.

8. (a) $\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 + \|\underset{\sim}{u}-\underset{\sim}{v}\|^2 = (\underset{\sim}{u}+\underset{\sim}{v})\cdot(\underset{\sim}{u}+\underset{\sim}{v}) + (\underset{\sim}{u}-\underset{\sim}{v})\cdot(\underset{\sim}{u}-\underset{\sim}{v})$

$= \underset{\sim}{u}\cdot\underset{\sim}{u} + 2\underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{v} + \underset{\sim}{u}\cdot\underset{\sim}{u} - 2\underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{v} = 2\|\underset{\sim}{u}\|^2 + 2\|\underset{\sim}{v}\|^2$

9. (b) $$\left.\begin{aligned} \underset{\sim}{u}\cdot(2,1,1) &= 2u_1+u_2+u_3 = 0 \\ \underset{\sim}{u}\cdot(1,2,3) &= u_1+2u_2+3u_3 = 0 \end{aligned}\right\} \begin{array}{l} \text{Gauss elim. gives } u_3 = 3\alpha,\ u_2 = -5\alpha,\ u_1 = \alpha \\ \text{so } \underset{\sim}{u} = \alpha(1,-5,3) \text{ for arbitrary } \alpha\text{; i.e., a line.} \end{array}$$

(g) $$\left.\begin{aligned} \underset{\sim}{u}\cdot(1,-2,0) &= u_1-2u_2 = 0 \\ \underset{\sim}{u}\cdot(2,3,1) &= 2u_1+3u_2+u_3 = 0 \\ \underset{\sim}{u}\cdot(7,0,2) &= 7u_1 + 2u_3 = 0 \end{aligned}\right\} \begin{array}{l} \text{Gauss elim. gives } u_3 = 7\alpha,\ u_2 = -\alpha,\ u_1 = -2\alpha \\ \text{so } \underset{\sim}{u} = \alpha(-2,-1,7) \text{ for arbitrary } \alpha\text{; i.e., a line.} \end{array}$$

(h) $$\left.\begin{aligned} \underset{\sim}{u}\cdot(2,1,-1) &= 2u_1+u_2-u_3 = 0 \\ \underset{\sim}{u}\cdot(1,1,1) &= u_1+u_2+u_3 = 0 \\ \underset{\sim}{u}\cdot(3,2,1) &= 3u_1+2u_2+u_3 = 0 \end{aligned}\right\} \begin{array}{l} \text{Gauss elim. gives only the ``trivial'' soln. } \alpha_1 = \alpha_2 = \alpha_3 = 0 \\ \quad \text{so } \underset{\sim}{u} = \underset{\sim}{0}. \end{array}$$

10. (a) $\underset{\sim}{u} = \underset{\sim}{u}_1 + \underset{\sim}{u}_2 = \alpha\underset{\sim}{v} + \underset{\sim}{u}_2$

$\underset{\sim}{v}\cdot\underset{\sim}{u} = \alpha\|\underset{\sim}{v}\|^2 + \cancelto{0}{\underset{\sim}{v}\cdot\underset{\sim}{u}_2}$ gives $\alpha = \underset{\sim}{v}\cdot\underset{\sim}{u}/\|\underset{\sim}{v}\|^2 = \hat{\underset{\sim}{v}}\cdot\underset{\sim}{u}/\|\underset{\sim}{v}\|$

so $\underset{\sim}{u}_1 = \alpha\underset{\sim}{v} = \frac{\hat{\underset{\sim}{v}}\cdot\underset{\sim}{u}}{\|\underset{\sim}{v}\|}\underset{\sim}{v} = (\hat{\underset{\sim}{v}}\cdot\underset{\sim}{u})\hat{\underset{\sim}{v}}$ and $\underset{\sim}{u}_2 = \underset{\sim}{u} - \underset{\sim}{u}_1$.

These formulas hold not only for 2- and 3-space, but for n-space $(n \geq 2)$.

(f) $\underset{\sim}{u} = (2,1,0,0,3)$, $\underset{\sim}{v} = (0,0,1,-2,1)$

$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{6}}(0,0,1,-2,1)$, $\hat{\underset{\sim}{v}}\cdot\underset{\sim}{u} = 3/\sqrt{6}$ so $\underset{\sim}{u}_1 = (\hat{\underset{\sim}{v}}\cdot\underset{\sim}{u})\hat{\underset{\sim}{v}} = \frac{3}{\sqrt{6}}\frac{1}{\sqrt{6}}(0,0,1,-2,1) = (0,0,\frac{1}{2},-1,\frac{1}{2})$

and $\underset{\sim}{u}_2 = \underset{\sim}{u} - \underset{\sim}{u}_1 = (2,1,-\frac{1}{2},1,\frac{5}{2})$.

11. (a) $$\left.\begin{aligned} (\alpha\underset{\sim}{u})\cdot\underset{\sim}{v} &= (\alpha u_1,\dots,u_n)\cdot(v_1,\dots,v_n) = \alpha u_1 v_1 + \cdots + \alpha u_n v_n \\ \alpha(\underset{\sim}{u}\cdot\underset{\sim}{v}) &= \alpha(u_1 v_1 + \cdots + u_n v_n) = \alpha u_1 v_1 + \cdots + \alpha u_n v_n \end{aligned}\right\rangle \text{the same} \checkmark$$

(b) $(\underset{\sim}{u}+\underset{\sim}{v})\cdot\underset{\sim}{w} = (u_1+v_1,\dots,u_n+v_n)\cdot(w_1,\dots,w_n)$

$= (u_1+v_1)w_1 + \cdots + (u_n+v_n)w_n = u_1w_1 + \cdots + u_nw_n$

$+ v_1w_1 + \cdots + v_nw_n = \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{w}$

(c) $(\alpha\underset{\sim}{u} + \beta\underset{\sim}{v})\cdot\underset{\sim}{w} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) + \beta(\underset{\sim}{v}\cdot\underset{\sim}{w})$

$\alpha = \beta = 1$ gives the distributive property $(\underset{\sim}{u}+\underset{\sim}{v})\cdot\underset{\sim}{w} = \underset{\sim}{u}\cdot\underset{\sim}{w} + \underset{\sim}{v}\cdot\underset{\sim}{w}$

$\beta = 0$ gives the associative property $(\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w})$

Now show the reverse:

$(\alpha\underset{\sim}{u} + \beta\underset{\sim}{v})\cdot\underset{\sim}{w} = (\alpha\underset{\sim}{u})\cdot\underset{\sim}{w} + (\beta\underset{\sim}{v})\cdot\underset{\sim}{w}$ per distributive property

$= \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w}) + \beta(\underset{\sim}{v}\cdot\underset{\sim}{w})$ per associative property. ✓

12. (a) $l_1 = \cos\alpha = u_1/\|\underset{\sim}{u}\| = u_1/\sqrt{u_1^2+u_2^2+u_3^2}$

$l_2 = \cos\beta = u_2/\|\underset{\sim}{u}\| = u_2/\sqrt{\ }$ "

$l_3 = \cos\gamma = u_3/\|\underset{\sim}{u}\| = u_3/\sqrt{\ }$ "

(d) If $\underset{\sim}{u} = (4,0,-3)$ then $l_1 = 4/5$, $l_2 = 0$, $l_3 = -3/5$

(e) $l_1^2 + l_2^2 + l_3^2 = (u_1/\sqrt{u_1^2+u_2^2+u_3^2})^2 + (u_2/\sqrt{u_1^2+u_2^2+u_3^2})^2 + (u_3/\sqrt{u_1^2+u_2^2+u_3^2})^2$

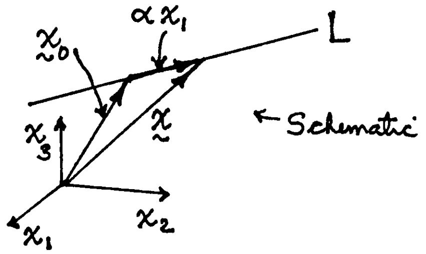
$= (u_1^2+u_2^2+u_3^2)/(u_1^2+u_2^2+u_3^2) = 1$

13. No. For ex., if $\underset{\sim}{u} = \underset{\sim}{w} = (1,0)$ and $\underset{\sim}{v} = (0,1)$, then $\underset{\sim}{u}\cdot\underset{\sim}{v} = 0$, $\underset{\sim}{v}\cdot\underset{\sim}{w} = 0$, but $\underset{\sim}{u}\cdot\underset{\sim}{w} = 1 \neq 0$.

14. (b) No (c) Yes (d) Yes

15. (b) $\left.\begin{array}{l} x_1 + x_2 \qquad = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{array}\right\}$ Gauss elim. give $x_3 = \alpha$, $x_2 = \alpha$, $x_1 = -\alpha$ so $\underset{\sim}{x} = \alpha(-1,1,1)$ A unit vector along that line of intersection is $\pm\frac{1}{\sqrt{3}}(-1,1,1)$.

(c) $\left.\begin{array}{l} x_1 - x_2 - 5x_3 = 0 \\ x_2 + 4x_3 = 6 \end{array}\right\}$ Gauss elim. gives $x_3 = \alpha$, $x_2 = -4\alpha + 6$, $x_1 = 6 + \alpha$ so $\underset{\sim}{x} = (6+\alpha, 6-4\alpha, \alpha)$. This vector is not the desired vector because it is not along the line of intersection, it is from the origin to the line of intersection, as is seen easily if we split it as



$$\underset{\sim}{x} = (6+\alpha, 6-4\alpha, \alpha) = (6,6,0) + \alpha(1,-4,1)$$
$$\equiv \underset{\sim}{x}_0 + \alpha\,\underset{\sim}{x}_1$$

The vector along L is $\alpha\underset{\sim}{x}_1$, not $\underset{\sim}{x}$. Scaling that vector to unit length gives the answer $\pm\frac{1}{\sqrt{18}}(1,-4,1)$. Alternatively, we can get a vector along L by differencing two $\underset{\sim}{x}$ vectors corresponding to different α's. For ex.,

$\underset{\sim}{x}\big|_{\alpha=2} - \underset{\sim}{x}\big|_{\alpha=1} = (8,-2,2) - (7,2,1) = (1,-4,1)$. Then, scaling to unit length gives $\pm\frac{1}{\sqrt{18}}(1,-4,1)$, as before.

NOTE: I believe this simple problem is a good one to assign and discuss. The common error is to simply take the resulting **x** vector and scale it to unit length, whereas the **x** vector is not along the line L (unless the two equations happen to be homogeneous). The key to understanding is a simple sketch, like the one above, and it is important for the student to understand that it does not even need to be to scale, but only schematic. Generally, students do not use sketching and visual aids enough, and this example offers strong encouragement for them to do so.

Section 9.6

1. (b) Let us see. Let $\underset{\sim}{u} = (u, 2u, 3u, \dots, nu)$ and $\underset{\sim}{v} = (v, 2v, 3v, \dots, nv)$ be any two vectors in the space. Then $\alpha\underset{\sim}{u} = (\alpha u, 2\alpha u, \dots, n\alpha u)$ is in the space and $\underset{\sim}{u} + \underset{\sim}{v} = (u+v, 2u+2v, \dots, nu+nv) = ((u+v), 2(u+v), \dots, n(u+v))$ is in the space, so the space is closed under scalar multiplication and vector addition. Further, $\underset{\sim}{u} = (u, 2u, \dots, nu)\big|_{u=0} = (0,0,\dots,0) = \underset{\sim}{0}$ is in the space, as is the negative inverse $-\underset{\sim}{u} = (-u, -2u, \dots, -nu)$ of $\underset{\sim}{u}$. Also, $\underset{\sim}{u} + \underset{\sim}{v} = (u+v, 2u+2v, \dots, nu+nv)$ $= (v+u, 2v+2u, \dots, nv+nu) = \underset{\sim}{v} + \underset{\sim}{u}$, and so on. Yes, it is a vector space.

NOTE: If we modified the space by having $0 < u < \infty$, say, instead of $-\infty < u < \infty$, then it would not be a vector space because it would not be closed under scalar multiplication [e.g., $(-2)(u, 2u, \dots, nu) = (-2u, -4u, \dots, -2nu)$ $= ((-2u), 2(-2u), \dots, n(-2u))$ is not in the space], nor – for any given $\underset{\sim}{u}$ vector – does the space contain a negative inverse $-\underset{\sim}{u}$.

(c) Yes

(d) No, there are several reasons why it is not a vector space (and it only takes one!):

(1) fails since $\underset{\sim}{u}+\underset{\sim}{v}=(u_1-v_1,\dots,u_n-v_n)$
whereas $\underset{\sim}{u}-\underset{\sim}{v}=(v_1-u_1,\dots,v_n-u_n)$ > not the same, in general

(2) fails since $(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w}=(u_1-v_1,\dots,u_n-v_n)+(w_1,\dots,w_n)$
$=(u_1-v_1-w_1,\dots,u_n-v_n-w_n)$
whereas $\underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})=\underset{\sim}{u}+(v_1-w_1,\dots,v_n-w_n)$ > not the same, in general
$=(u_1-v_1+w_1,\dots,u_n-v_n+w_n)$

Also, the negative inverse does not satisfy (4), in general, since
$\underset{\sim}{u}+(-\underset{\sim}{u})=(u_1-(-u_1),\dots,u_n-(-u_n))=(2u_1,\dots,2u_n)$

Also, (6) fails since $(\alpha+\beta)\underset{\sim}{u}=((\alpha+\beta)u_1,\dots,(\alpha+\beta)u_n)$
whereas $\alpha\underset{\sim}{u}+\beta\underset{\sim}{u}=(\alpha u_1-\beta u_1,\dots,\alpha u_n-\beta u_n)$.

(e) No: (3) fails since $\underset{\sim}{u}+\underset{\sim}{0}=\underset{\sim}{0}$ instead of $\underset{\sim}{u}$.
Also, there does not exist a unique vector $-\underset{\sim}{u}$ corresponding to each $\underset{\sim}{u}$ such that $\underset{\sim}{u}+(-\underset{\sim}{u})=\underset{\sim}{0}$, since $\underset{\sim}{u}+\underset{\sim}{v}=\underset{\sim}{0}$ for every vector $\underset{\sim}{v}$.
Also, (6) fails since $(\alpha+\beta)\underset{\sim}{u}=((\alpha+\beta)u_1,\dots,(\alpha+\beta)u_n)$
whereas $\alpha\underset{\sim}{u}+\beta\underset{\sim}{u}=\underset{\sim}{0}$.

2. Yes, it violates (2) since $(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w}=(u_1+2v_1,\dots,u_n+2v_n)+(w_1,\dots,w_n)$
$=(u_1+2v_1+2w_1,\dots,u_n+2v_n+2w_n)$
whereas $\underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})=\underset{\sim}{u}+(v_1+2w_1,\dots,v_n+2w_n)=(u_1+2v_1+4w_1,\dots,u_n+2v_n+4w_n)$
Also, $-\underset{\sim}{u}=(-u_1,\dots,-u_n)$ does not satisfy $\underset{\sim}{u}+(-\underset{\sim}{u})=\underset{\sim}{0}$ since
$\underset{\sim}{u}+(-\underset{\sim}{u})=(u_1+2(-u_1),\dots,u_n+2(-u_n))=(-u_1,\dots,-u_n)=-\underset{\sim}{u}$, not $\underset{\sim}{0}$.
Also, (6) fails since $(\alpha+\beta)\underset{\sim}{u}=((\alpha+\beta)u_1,\dots,(\alpha+\beta)u_n)$
whereas $\alpha\underset{\sim}{u}+\beta\underset{\sim}{u}=(\alpha u_1+2\beta u_1,\dots,\alpha u_n+2\beta u_n)$

3. $\underset{\sim}{u}+(-1)\underset{\sim}{u}=1\underset{\sim}{u}+(-1)\underset{\sim}{u}$ per (8)
$=(1+(-1))\underset{\sim}{u}$ per (6)
$=0\underset{\sim}{u}=\underset{\sim}{0}$ per (15a). Then $(-1)\underset{\sim}{u}$ must $=-\underset{\sim}{u}$ per (iii) in Definition 9.6.1.

4. $\alpha\underset{\sim}{u}+\alpha\underset{\sim}{0}=\alpha(\underset{\sim}{u}+\underset{\sim}{0})$ per (7)
$=\alpha\underset{\sim}{u}$ per (3).
Then, per (ii) in Definition 9.6.1, $\alpha\underset{\sim}{0}=\underset{\sim}{0}$.

5. $\alpha\underset{\sim}{u}=\underset{\sim}{0}$. Suppose $\alpha\neq0$. Then $\frac{1}{\alpha}(\alpha\underset{\sim}{u})=\frac{1}{\alpha}\underset{\sim}{0}$
$(\frac{1}{\alpha}\alpha)\underset{\sim}{u}=\underset{\sim}{0}$ per (5) and (13),
and $\underset{\sim}{u}=\underset{\sim}{0}$ per (8).
If, however, $\underset{\sim}{u}\neq\underset{\sim}{0}$, then the supposition $\alpha\neq0$ must be false, so that $\alpha=0$.

6. Want to show that $\underset{\sim}{u}\cdot\underset{\sim}{v}\equiv\sum_1^n w_j u_j v_j$ does satisfy (16a,b,c), where the w_j's are >0.
(16a): $\underset{\sim}{v}\cdot\underset{\sim}{u}=\sum_1^n w_j v_j u_j=\sum_1^n w_j u_j v_j=\underset{\sim}{u}\cdot\underset{\sim}{v}$ ✓
(16b): $\underset{\sim}{u}\cdot\underset{\sim}{u}=\sum_1^n w_j u_j^2>0$, clearly, for all $\underset{\sim}{u}\neq\underset{\sim}{0}$, and $=0$ for $\underset{\sim}{u}=\underset{\sim}{0}$.
(16c): $(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})\cdot\underset{\sim}{w}=\sum_1^n W_j(\alpha u_j+\beta v_j)w_j=\alpha\sum_1^n W_j u_j w_j+\beta\sum_1^n W_j v_j w_j=\alpha\underset{\sim}{u}\cdot\underset{\sim}{w}+\beta\underset{\sim}{v}\cdot\underset{\sim}{w}$.
Note: To avoid confusion between the weights and the components of $\underset{\sim}{w}$, we have used different

notations for them: W_j for the weights and w_j for the $\underset{\sim}{w}$ components.

7. Given a dot product such that
$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \underset{\sim}{v}\cdot\underset{\sim}{u} \quad (16a)$$
$$\left.\begin{aligned}\underset{\sim}{u}\cdot\underset{\sim}{u} &> 0 \text{ for all } \underset{\sim}{u}\neq\underset{\sim}{0}\\ &= 0 \text{ for } \underset{\sim}{u}=\underset{\sim}{0}\end{aligned}\right\} \quad (16b)$$
$$(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})\cdot\underset{\sim}{w} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{w})+\beta(\underset{\sim}{v}\cdot\underset{\sim}{w}) \quad (16c)$$
and a "natural norm" $\|\underset{\sim}{u}\| = \sqrt{\underset{\sim}{u}\cdot\underset{\sim}{u}}$, then
$$\begin{aligned}\|\alpha\underset{\sim}{u}\| = \sqrt{(\alpha\underset{\sim}{u})\cdot(\alpha\underset{\sim}{u})} &= \sqrt{\alpha\,[\underset{\sim}{u}\cdot(\alpha\underset{\sim}{u})]} && \text{per (16c)}\\ &= \sqrt{\alpha\,[(\alpha\underset{\sim}{u})\cdot\underset{\sim}{u}]} && \text{per (16a)}\\ &= \sqrt{\alpha^2\,(\underset{\sim}{u}\cdot\underset{\sim}{u})} && \text{per (16c)}\\ &= |\alpha|\sqrt{\underset{\sim}{u}\cdot\underset{\sim}{u}} = |\alpha|\,\|\underset{\sim}{u}\| \ \checkmark\end{aligned}$$
Next, $\|\underset{\sim}{u}\| = \sqrt{\underset{\sim}{u}\cdot\underset{\sim}{u}}\ \begin{aligned}&> 0 \text{ if } \underset{\sim}{u}\neq\underset{\sim}{0} && \text{per (16b)}\\ &= 0 \text{ if } \underset{\sim}{u}=\underset{\sim}{0} && \text{" \quad "}\end{aligned}\ \checkmark$

Finally,
$$\begin{aligned}\|\underset{\sim}{u}+\underset{\sim}{v}\|^2 = (\underset{\sim}{u}+\underset{\sim}{v})\cdot(\underset{\sim}{u}+\underset{\sim}{v}) &= \underset{\sim}{u}\cdot\underset{\sim}{u}+\underset{\sim}{u}\cdot\underset{\sim}{v}+\underset{\sim}{v}\cdot\underset{\sim}{u}+\underset{\sim}{v}\cdot\underset{\sim}{v}\\ &= \|\underset{\sim}{u}\|^2+2\underset{\sim}{u}\cdot\underset{\sim}{v}+\|\underset{\sim}{v}\|^2 \quad \text{per (16a)}\\ &\leq \|\underset{\sim}{u}\|^2+2|\underset{\sim}{u}\cdot\underset{\sim}{v}|+\|\underset{\sim}{v}\|^2\\ &\leq \|\underset{\sim}{u}\|^2+2\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|+\|\underset{\sim}{v}\|^2 \quad \text{per Schwarz inequality (13) in Sec. 9.5}\\ &= (\|\underset{\sim}{u}\|+\|\underset{\sim}{v}\|)^2\end{aligned}$$
so $\|\underset{\sim}{u}+\underset{\sim}{v}\| \leq \|\underset{\sim}{u}\|+\|\underset{\sim}{v}\|$, and we're done. ✓

8. We cannot merely use the result of Exercise 7 because the taxicab norm is not a natural norm. Now, if $\|\underset{\sim}{u}\| \equiv |u_1|+\cdots+|u_n|$ then
$$\|\alpha\underset{\sim}{u}\| = |\alpha u_1|+\cdots+|\alpha u_n| = |\alpha||u_1|+\cdots+|\alpha||u_n| = |\alpha|\,\|\underset{\sim}{u}\| \ \checkmark$$
Next, $\|\underset{\sim}{u}\|\ \begin{aligned}&>0\ (\underset{\sim}{u}\neq\underset{\sim}{0})\\ &=0\ (\underset{\sim}{u}=\underset{\sim}{0})\end{aligned}$ is clearly satisfied since each $|u_j|\geq 0$.

Finally, $\|\underset{\sim}{u}+\underset{\sim}{v}\| = \|(u_1+v_1,\dots,u_n+v_n)\| = |u_1+v_1|+\cdots+|u_n+v_n|$.
Now, for scalars a,b it is easily seen that $|a+b|\leq|a|+|b|$, so
$$\begin{aligned}\|\underset{\sim}{u}+\underset{\sim}{v}\| &\leq |u_1|+|v_1|+\cdots+|u_n|+|v_n|\\ &= |u_1|+\cdots+|u_n|+|v_1|+\cdots+|v_n|\\ &= \|\underset{\sim}{u}\|+\|\underset{\sim}{v}\|. \ \checkmark\end{aligned}$$

9.(a) Is $\|\underset{\sim}{u}\| = \max\limits_{1\leq j\leq n}|u_j|$ okay?

Surely the maximum $|u_j|$ and the maximum $|\alpha u_j|$ occur at the same j's, so
$$\|\alpha\underset{\sim}{u}\| = \max_{1\leq j\leq n}|\alpha u_j| = |\alpha|\max_{1\leq j\leq n}|u_j| = |\alpha|\,\|\underset{\sim}{u}\|. \ \checkmark$$
Next, $\|\underset{\sim}{u}\| = \max\limits_{1\leq j\leq n}|u_j|\ \begin{aligned}&>0 \text{ if } \underset{\sim}{u}\neq\underset{\sim}{0}\\ &=0 \text{ if } \underset{\sim}{u}=\underset{\sim}{0}\end{aligned}$ is clearly satisfied ✓

Finally, $\|\underset{\sim}{u}+\underset{\sim}{v}\| = \max\limits_{1\leq j\leq n}|u_j+v_j| \leq \overbrace{\max\limits_{1\leq j\leq n}(|u_j|+|v_j|)}^{ⓐ}$ since $|a+b|\leq|a|+|b|$
$$\leq \overbrace{\max_{1\leq j\leq n}|u_j|}^{ⓑ} + \overbrace{\max_{1\leq j\leq n}|v_j|}^{ⓒ} = \|\underset{\sim}{u}\|+\|\underset{\sim}{v}\|$$

so this *is* a legitimate norm. Note that the maxima in ⓑ and ⓒ need not occur at the same j's, nor are one or both of those j's necessarily the same as that in ⓐ.

(b) Is $\|\underset{\sim}{u}\| = \min_{1\le j\le n} |u_j|$ okay? (17a) is satisfied but (17b) and (17c) are not. For example, let $\underset{\sim}{u} = (2,1,5)$ and $\underset{\sim}{v} = (0,3,2)$. Then $\|\underset{\sim}{u}\| = 1$, $\|\underset{\sim}{v}\| = 0$, $\|\underset{\sim}{u}+\underset{\sim}{v}\| = \|(2,4,7)\| = 2$ so $\|\underset{\sim}{v}\| = 0$ even though $\underset{\sim}{v} \neq \underset{\sim}{0}$, in violation of (17b), and $2 \le 0+1$ shows that (17c) is violated, in this example, as well.

11. If $\langle u,v\rangle \equiv \int_0^1 u(x)v(x)w(x)\,dx$ $(w(x)>0$ on $0\le x\le 1)$, then

$$\langle v,u\rangle = \int_0^1 v(x)u(x)w(x)\,dx = \int_0^1 u(x)v(x)w(x)\,dx = \langle u,v\rangle \ ✓$$

$$\langle u,u\rangle = \int_0^1 u^2(x)w(x)\,dx \begin{array}{l} >0 \text{ for } u(x)\neq \underset{\sim}{0} \ ✓ \\ =0 \text{ for } u(x)=\underset{\sim}{0} \end{array}$$

$$\begin{aligned}\langle(\alpha u+\beta v),w\rangle &= \int_0^1 [\alpha u(x)+\beta v(x)]w(x)W(x)\,dx \quad \swarrow \text{weight function} \\ &= \alpha\int_0^1 u(x)w(x)W(x)\,dx + \beta\int_0^1 v(x)w(x)W(x)\,dx \\ &= \alpha\langle u,w\rangle + \beta\langle v,w\rangle\end{aligned}$$

12. (a) $|\underset{\sim}{u}\cdot\underset{\sim}{v}| = |3+2-5| = 0$, $\|\underset{\sim}{u}\| = \sqrt{9+1+1} = \sqrt{11}$, $\|\underset{\sim}{v}\| = \sqrt{1+4+25+16} = \sqrt{46}$, $0 \le \sqrt{11}\sqrt{46}$ ✓

(b) $|\underset{\sim}{u}\cdot\underset{\sim}{v}| = |1(0)+5(8)+3(4)+2(-3)| = 46$, $\|\underset{\sim}{u}\| = \sqrt{1(1)+5(4)+3(16)+2(9)} = \sqrt{87}$,
$\|\underset{\sim}{v}\| = \sqrt{1(0)+5(16)+3(1)+2(1)} = \sqrt{85}$, $46 \le \sqrt{87}\sqrt{85} = 85.99$ ✓

(c) $|\underset{\sim}{u}\cdot\underset{\sim}{v}| = |1(2)+2(2)+3(2)+4(2)+5(2)| = 30$, $\|\underset{\sim}{u}\| = \sqrt{1(1)+2(1)+3(1)+4(1)+5(1)} = \sqrt{15}$,
$\|\underset{\sim}{v}\| = \sqrt{1(4)+2(4)+3(4)+4(4)+5(4)} = \sqrt{60}$, $30 \le \sqrt{15}\sqrt{60} = 30$ ✓

(d) $\|\underset{\sim}{u}\| = \sqrt{\int_0^1 (2+x)^2\,dx} = \sqrt{19/3}$, $\|\underset{\sim}{v}\| = \sqrt{\int_0^1 (3x^2)^2\,dx} = \sqrt{9/5}$,
$|\langle u,v\rangle| = \left|\int_0^1 (2+x)3x^2\,dx\right| = 11/4 \le \sqrt{19/3}\sqrt{9/5}$
$2.75 \le 3.376$ ✓

13. The system is of the form

$$\begin{array}{c} a_{11}x_1+\cdots+a_{1n}x_n=0 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1+\cdots+a_{mn}x_n=0 \end{array} \qquad ①$$

The set, S say, of all solution vectors is closed under addition since if $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are solutions of ① then so is $\underset{\sim}{u}+\underset{\sim}{v}$ because

$$\begin{array}{c} a_{11}(u_1+v_1)+\cdots+a_{1n}(u_n+v_n) = (a_{11}u_1+\cdots+a_{1n}u_n)+(a_{11}v_1+\cdots+a_{1n}v_n) = 0+0=0 \ ✓ \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}(u_1+v_1)+\cdots+a_{mn}(u_n+v_n) = (a_{m1}u_1+\cdots+a_{mn}u_n)+(a_{m1}v_1+\cdots+a_{mn}v_n) = 0+0=0 \ ✓ \end{array}$$

Also, it contains the zero vector $\underset{\sim}{u} = (0,\dots,0)$. It contains the negative inverse of each vector $\underset{\sim}{u}$ in S because if $\underset{\sim}{u} = (u_1,\dots,u_n)$ satisfies ① then surely $-\underset{\sim}{u} = (-u_1,\dots,-u_n)$ does too. And it is closed under scalar multiplication since if $\underset{\sim}{u} = (u_1,\dots,u_n)$ satisfies ① then so does $\alpha\underset{\sim}{u} = (\alpha u_1,\dots,\alpha u_n)$ since

$$\begin{array}{c} a_{11}(\alpha u_1)+\cdots+a_{1n}(\alpha u_n) = \alpha(a_{11}u_1+\cdots+a_{1n}u_n) = \alpha(0)=0 \\ \vdots \\ a_{m1}(\alpha u_1)+\cdots+a_{mn}(\alpha u_n) = \alpha(a_{m1}u_1+\cdots+a_{mn}u_n) = \alpha(0)=0 \end{array}$$

Section 9.7

1. (b) $\underset{\sim}{v} = \alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2 + \alpha_3 \underset{\sim}{u}_3 + \alpha_4 \underset{\sim}{u}_4$ gives

$$\begin{aligned} v_1 &= \alpha_4 \\ v_2 &= \alpha_3 + \alpha_4 \\ v_3 &= \alpha_2 + \alpha_3 + \alpha_4 \\ v_4 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{aligned}$$

Obviously, these equations are consistent for every $\underset{\sim}{v}$ in $\mathbb{R}^4$, so the set spans $\mathbb{R}^4$.

(c) $\underset{\sim}{v} = \alpha_1 \underset{\sim}{u}_1 + \cdots + \alpha_5 \underset{\sim}{u}_5$ gives

$$\begin{aligned} v_1 &= \alpha_1 + 2\alpha_2 + \alpha_5 \\ v_2 &= 2\alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_5 \\ v_3 &= \alpha_2 + 2\alpha_5 \\ v_4 &= 4\alpha_1 - \alpha_2 + \alpha_3 + 3\alpha_4 . \end{aligned}$$

Gauss elim. gives

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & v_1 \\ 2 & 3 & 1 & 0 & 1 & v_2 \\ 0 & 1 & 0 & 0 & 2 & v_3 \\ 4 & -1 & 1 & 0 & 3 & v_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & v_1 \\ 0 & -1 & 1 & 0 & -1 & v_2 - 2v_1 \\ 0 & 1 & 0 & 0 & 2 & v_3 \\ 0 & -9 & 1 & 0 & -1 & v_4 - 4v_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & v_1 \\ 0 & -1 & 1 & 0 & -1 & v_2 - 2v_1 \\ 0 & 0 & 1 & 0 & 1 & v_3 + v_2 - 2v_1 \\ 0 & 0 & -8 & 0 & 8 & v_4 - 9v_2 + 14v_1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & v_1 \\ 0 & -1 & 1 & 0 & -1 & v_2 - 2v_1 \\ 0 & 0 & 1 & 0 & 1 & v_3 + v_2 - 2v_1 \\ 0 & 0 & 0 & 0 & 16 & v_4 + 8v_3 - v_2 - 2v_1 \end{bmatrix}$$

which gives α_4 = arbitrary and unique values for α_1, $\alpha_2, \alpha_3, \alpha_5$. Thus, the system is consistent for every $\underset{\sim}{v}$ in $\mathbb{R}^4$; hence, $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_5\}$ spans $\mathbb{R}^4$.

(e) $\underset{\sim}{v} = \alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2 + \alpha_3 \underset{\sim}{u}_3$ gives

$$\begin{aligned} v_1 &= \alpha_1 + 2\alpha_2 + \alpha_3 \\ v_2 &= \alpha_2 + 2\alpha_3 \\ v_3 &= \alpha_1 - \alpha_2 - 5\alpha_3 . \end{aligned}$$

Gauss elim. gives $\begin{bmatrix} 1 & 2 & 1 & v_1 \\ 0 & 1 & 2 & v_2 \\ 1 & -1 & -5 & v_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & v_1 \\ 0 & 1 & 2 & v_2 \\ 0 & -3 & -6 & v_3 - v_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & v_1 \\ 0 & 1 & 2 & v_2 \\ 0 & 0 & 0 & v_3 + 3v_2 - v_1 \end{bmatrix}$

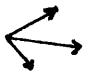
which is inconsistent—unless $v_3 + 3v_2 - v_1 = 0$. That is, $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$ do NOT span $\mathbb{R}^3$; they only span the subset of $\mathbb{R}^3$ consisting of the plane $v_3 + 3v_2 - v_1 = 0$

(f) Yes (h) Yes (i) Yes (j) No (k) No (l) No (m) Yes (n) No

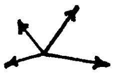
(o) Yes (p) Yes (q) Yes

2. (a)

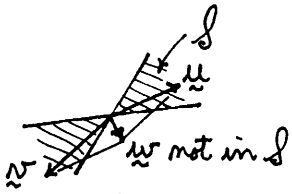
(b)



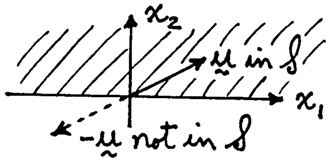
(c)



3. (b) No, it is not closed under vector addition. For example, $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{w}$ is not in the space:



(c) No, it is not true that each vector $\underset{\sim}{u}$ in $\mathcal{S}$ has a negative inverse $-\underset{\sim}{u}$ in $\mathcal{S}$:



4. (b) $x_1+x_2+x_3-x_4=0$. $x_4=\alpha, x_3=\beta, x_2=\gamma, x_1=\alpha-\beta-\gamma$,
so $\underset{\sim}{x}=(\alpha-\beta-\gamma, \gamma, \beta, \alpha)=\alpha(1,0,0,1)+\beta(-1,0,1,0)+\gamma(-1,1,0,0)$
Thus, the solution space (which is a "hyperplane" in $\mathbb{R}^4$) is
span$\{(1,0,0,1), (-1,0,1,0), (-1,1,0,0)\}$

(c) Gauss elim.: $\begin{bmatrix}1&-1&1&0\\1&1&1&0\end{bmatrix}\rightarrow\begin{bmatrix}1&-1&1&0\\0&2&0&0\end{bmatrix}$ gives $x_2=0, x_3=\alpha, x_1=-\alpha$, so $\underset{\sim}{x}=(-\alpha,0,\alpha)$ and solution space is span$\{(-1,0,1)\}$, which is a line in $\mathbb{R}^3$.

(e) Gauss elim.: $\begin{bmatrix}1&-1&1&-2&0&0\\1&-1&0&1&2&0\end{bmatrix}\rightarrow\begin{bmatrix}1&-1&1&-2&0&0\\0&0&-1&3&2&0\end{bmatrix}$ gives $x_5=\alpha, x_4=\beta, x_3=2\alpha+3\beta$, $x_2=\gamma, x_1=2\beta-(2\alpha+3\beta)+\gamma$
$=-2\alpha-\beta+\gamma$ so $\underset{\sim}{x}=(-2\alpha-\beta+\gamma, \gamma, 2\alpha+3\beta, \beta, \alpha)$
$=\alpha(-2,0,2,0,1)+\beta(-1,0,3,1,0)+\gamma(1,1,0,0,0)$, so the solution space is
span$\{(-2,0,2,0,1), (-1,0,3,1,0), (1,1,0,0,0)\}$

(f) Solution space is span$\{(-3,2,0,1), (2,-1,1,0)\}$

5. (b) $x_3=\alpha, x_2=\beta, x_1=3\alpha-\frac{1}{2}\beta$ so $\underset{\sim}{x}=(3\alpha-\frac{1}{2}\beta, \beta, \alpha)=\alpha(3,0,1)+\beta(-\frac{1}{2},1,0)$
so $(3,0,1)$ and $(-\frac{1}{2},1,0)$ span the plane.
NOTE: What if the equation is nonhomogeneous, say $2x_1+x_2-6x_3=4$?
Then $x_3=\alpha, x_2=\beta, x_1=2+3\alpha-\frac{1}{2}\beta$ so $\underset{\sim}{x}=(2,0,0)+\alpha(3,0,1)+\beta(-\frac{1}{2},1,0)$
so the answer is the same: $(3,0,1)$ and $(-\frac{1}{2},1,0)$ span the plane. The vector $(2,0,0)$ takes us from the origin to a point on the plane, and then $\alpha(\ )+\beta(\ )$ take us from there to any point in the plane.

This variation might make a nice examination question.

6. (b) The sets are identical iff

$$\alpha(1,2,3)+\beta(2,-1,1)=\gamma(1,2,3)+\delta(3,1,5)$$

is solvable (i.e., consistent) for α,β for any given γ,δ and visa versa. Let us solve (by Gauss elim.)

$$\begin{aligned}\alpha+2\beta&=\gamma+3\delta\\2\alpha-\beta&=2\gamma+\delta\\3\alpha+\beta&=3\gamma+5\delta\end{aligned}$$

first for α,β. Then, if it is found to be a consistent system in the unknowns α,β, let us solve for γ,δ. Gauss:

$$\begin{bmatrix}1&2&\gamma+3\delta\\2&-1&2\gamma+\delta\\3&1&3\gamma+5\delta\end{bmatrix}\rightarrow\begin{bmatrix}1&2&\gamma+3\delta\\0&-5&-5\delta\\0&-5&-4\delta\end{bmatrix}\rightarrow\begin{bmatrix}1&2&\gamma+3\delta\\0&-5&-5\delta\\0&0&\delta\end{bmatrix}$$

Thus, the system is not consistent in general, but only if $\delta=0$. Thus, no, the two sets are not identical.

(c) Same logic as in (b). First, let us solve (if possible)

$$\left.\begin{aligned}4\alpha+\beta&=\gamma+2\delta\\\alpha+\beta&=\gamma-\delta\\\beta&=\gamma-2\delta\end{aligned}\right\}\quad\star$$

for α,β. Gauss:

$$\begin{bmatrix} 4 & 1 & \gamma+2\delta \\ 1 & 1 & \gamma-\delta \\ 0 & 1 & \gamma-2\delta \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \gamma-\delta \\ 4 & 1 & \gamma+2\delta \\ 0 & 1 & \gamma-2\delta \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \gamma-\delta \\ 0 & -3 & -3\gamma+6\delta \\ 0 & 1 & \gamma-2\delta \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \gamma-\delta \\ 0 & -3 & -3\gamma+6\delta \\ 0 & 0 & 0 \end{bmatrix}$$

is consistent (for all choices of γ, δ), so now see if $\ast$ is also consistent in γ, δ.

Gauss: $$\begin{bmatrix} 1 & 2 & 4\alpha+\beta \\ 1 & -1 & \alpha+\beta \\ 1 & -2 & \beta \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4\alpha+\beta \\ 0 & -3 & -3\alpha \\ 0 & -4 & -4\alpha \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4\alpha+\beta \\ 0 & -3 & -3\alpha \\ 0 & 0 & 0 \end{bmatrix}$$

is consistent (for all choices of α, β), so the two sets <u>are</u> identical.

(d) Same logic as in (b). First, let us solve (if possible)

$$\begin{aligned} \alpha - 3\beta &= \gamma + \delta \\ 2\alpha &= 3\delta \\ -\alpha &= 0 \end{aligned}$$

Clearly, this system is not consistent, in the unknowns α, β, for all possible γ, δ, since the 3rd equation gives $\alpha = 0$ whereas the second gives $\alpha = 0$ <u>only</u> if $\delta = 0$. Thus, no, the two sets are <u>not</u> identical.

7. (b) Let $\underset{\sim}{u}_1 = (1,2,4)$ and $\underset{\sim}{u}_2 = (2,-1,3) + \alpha(1,2,4)$ such that $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 = 12 + 21\alpha = 0$ so $\alpha = -12/21 = -4/7$. Then $\underset{\sim}{u}_2 = (2,-1,3) - \frac{4}{7}(1,2,4) = (\frac{10}{7}, -\frac{15}{7}, \frac{5}{7})$, so an ON set in span$\{(1,2,4),(2,-1,3)\}$ is $\hat{\underset{\sim}{u}}_1 = \frac{1}{\sqrt{21}}(1,2,4)$, $\hat{\underset{\sim}{u}}_2 = \frac{1}{\sqrt{14}}(2,-3,1)$

8. The key points are these:

(i) Show that $\mathcal{S}' = \text{span}\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_k\}$ is closed under addition. Let $\underset{\sim}{U} = \alpha_1 \underset{\sim}{u}_1 + \cdots + \alpha_k \underset{\sim}{u}_k$ and $\underset{\sim}{V} = \beta_1 \underset{\sim}{u}_1 + \cdots + \beta_k \underset{\sim}{u}_k$ be any two vectors in $\mathcal{S}'$. Then*
$\underset{\sim}{U} + \underset{\sim}{V} = (\alpha_1 \underset{\sim}{u}_1 + \cdots + \alpha_k \underset{\sim}{u}_k) + (\beta_1 \underset{\sim}{u}_1 + \cdots + \beta_k \underset{\sim}{u}_k) = (\alpha_1 + \beta_1)\underset{\sim}{u}_1 + \cdots + (\alpha_k + \beta_k)\underset{\sim}{u}_k$, which is in $\mathcal{S}'$ because it is a linear combination of $\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_k$. ✓

(ii) $\mathcal{S}'$ does contain a zero vector because

$$\begin{aligned} 0\underset{\sim}{u}_1 + \cdots + 0\underset{\sim}{u}_k &= 0\underset{\sim}{u}_1 + (0\underset{\sim}{u}_2 + \cdots + 0\underset{\sim}{u}_k) \\ &= \underset{\sim}{0} + (\qquad \text{"} \qquad) \quad \text{by (15a) in Sec. 9.6} \\ &= \qquad 0\underset{\sim}{u}_2 + \cdots + 0\underset{\sim}{u}_k \quad \text{by (3) in Sec. 9.6} \\ &= \cdots \text{by repeating the argument} = 0\underset{\sim}{u}_k = \underset{\sim}{0}. \ ✓ \end{aligned}$$

---

* To illustrate the idea,

$$\begin{aligned} (2\underset{\sim}{u}+3\underset{\sim}{v})+(5\underset{\sim}{u}+4\underset{\sim}{v}) &= 2\underset{\sim}{u}+[3\underset{\sim}{v}+(5\underset{\sim}{u}+4\underset{\sim}{v})] && \text{by (2) in Sec. 9.6} \\ &= 2\underset{\sim}{u}+[3\underset{\sim}{v}+(4\underset{\sim}{v}+5\underset{\sim}{u})] && \text{by (1) "} \\ &= 2\underset{\sim}{u}+[(3\underset{\sim}{v}+4\underset{\sim}{v})+5\underset{\sim}{u}] && \text{by (2) "} \\ &= 2\underset{\sim}{u}+(7\underset{\sim}{v}+5\underset{\sim}{u}) && \text{by (6) "} \\ &= 2\underset{\sim}{u}+(5\underset{\sim}{u}+7\underset{\sim}{v}) && \text{by (1) "} \\ &= (2\underset{\sim}{u}+5\underset{\sim}{u})+7\underset{\sim}{v} && \text{by (2) "} \\ &= 7\underset{\sim}{u}+7\underset{\sim}{v} && \text{by (6) "} \end{aligned}$$

That is, $(2\underset{\sim}{u}+3\underset{\sim}{v})+(5\underset{\sim}{u}+4\underset{\sim}{v})$ is simply $= (2+5)\underset{\sim}{u} + (3+4)\underset{\sim}{v}$.

(iii) $\mathcal{S}'$ does contain a negative inverse $-\underset{\sim}{U}$ for each $\underset{\sim}{U}$ in $\mathcal{S}'$ since

$$\begin{aligned}\overbrace{(\alpha_1\underset{\sim}{u}_1+\cdots+\alpha_k\underset{\sim}{u}_k)}^{\underset{\sim}{U}}+\overbrace{(-\alpha_1\underset{\sim}{u}_1-\cdots-\alpha_k\underset{\sim}{u}_k)}^{-\underset{\sim}{U}} &= (\alpha_1-\alpha_1)\underset{\sim}{u}_1+\cdots+(\alpha_k-\alpha_k)\underset{\sim}{u}_k \quad [\text{as in (i)}]\\ &= 0\underset{\sim}{u}_1+\cdots+0\underset{\sim}{u}_k\\ &= \underset{\sim}{0}+\cdots+\underset{\sim}{0} = \underset{\sim}{0} \quad ✓\end{aligned}$$

(iv) $\mathcal{S}'$ is closed under scalar multiplication since

$$\begin{aligned}\alpha\underset{\sim}{U} &= \alpha(\alpha_1\underset{\sim}{u}_1+\cdots+\alpha_k\underset{\sim}{u}_k)\\ &= \alpha[\alpha_1\underset{\sim}{u}_1+(\alpha_2\underset{\sim}{u}_2+\cdots+\alpha_k\underset{\sim}{u}_k)]\\ &= \alpha(\alpha_1\underset{\sim}{u}_1)+\alpha(\alpha_2\underset{\sim}{u}_2+\cdots+\alpha_k\underset{\sim}{u}_k) && \text{by (7) in Sec. 9.6}\\ &= (\alpha\alpha_1)\underset{\sim}{u}_1+\alpha[(\alpha_2\underset{\sim}{u}_2)+(\alpha_3\underset{\sim}{u}_3+\cdots+\alpha_k\underset{\sim}{u}_k)] && \text{by (5) in Sec. 9.6}\\ &= (\alpha\alpha_1)\underset{\sim}{u}_1+\alpha(\alpha_2\underset{\sim}{u}_2)+\alpha(\alpha_3\underset{\sim}{u}_3+\cdots+\alpha_k\underset{\sim}{u}_k) && \text{by (7) in Sec. 9.6}\\ &= (\alpha\alpha_1)\underset{\sim}{u}_1+(\alpha\alpha_2\underset{\sim}{u}_2)+\quad \text{"} && \text{by (5) in Sec. 9.6}\\ &= \cdots = (\alpha\alpha_1)\underset{\sim}{u}_1+\cdots+(\alpha\alpha_k)\underset{\sim}{u}_k,\end{aligned}$$

which vector is in $\mathcal{S}'$. ✓

Section 9.8

2. (b) By inspection $(2,8) = 2(1,4)+0(3,-1)$.

(c) " " $(-3,3) = -3(1,-1)+0(4,2)$.

(d) Set $a(1,2,3)+b(3,2,1)+c(5,5,5) = \underset{\sim}{0}$, or,

$$\begin{aligned}a+3b+5c&=0\\ 2a+2b+5c&=0\\ 3a+b+5c&=0\end{aligned} \quad \text{Gauss:} \begin{bmatrix}1&3&5&0\\2&2&5&0\\3&1&5&0\end{bmatrix} \to \begin{bmatrix}1&3&5&0\\0&-4&-5&0\\0&-8&-10&0\end{bmatrix} \to \begin{bmatrix}1&3&5&0\\0&-4&-5&0\\0&0&0&0\end{bmatrix}$$

so $c=\alpha$, $b=-5\alpha/4$, $a=-5\alpha-3(-5\alpha/4) = -5\alpha/4$ so if we let $\alpha=4$, say, then $a=-5, b=-5, c=4$ so $-5(1,2,3)-5(3,2,1)+4(5,5,5)=\underset{\sim}{0}$. Solving for $(1,2,3)$, for example, $(1,2,3) = \frac{4}{5}(5,5,5)-(3,2,1)$.

3. (b) $a(1,3)+b(2,0)+c(1,2)+d(-1,5)=\underset{\sim}{0}$ gives

$$\begin{aligned}a+2b+c-d&=0\\ 3a\qquad +2c+5d&=0\end{aligned}$$

or, after Gauss elim., $\begin{pmatrix}1&2&1&-1&0\\0&-6&-1&6&0\end{pmatrix}$ so $d=\alpha$, $c=\beta$, $b=-\frac{1}{6}\beta+\alpha$, $a=\alpha-\beta+\frac{1}{3}\beta-2\alpha$ $=-\alpha-\frac{2}{3}\beta$. Let $\alpha=0$ and $\beta=6$, say. Then $d=0, c=6, b=-1, a=-4$ so $-4(1,3)-(2,0)+6(1,2)+0(-1,5)=\underset{\sim}{0}$. Solving for $(2,0)$, for example, gives $(2,0)=-4(1,3)+6(1,2)+0(-1,5)$. LD.

(c) Neither vector is a scalar multiple of the other so, by Thm 9.8.2, they are LI.

(e) LD because, by inspection, $(0,0,2)=\frac{2}{3}(0,0,3)+0(2,-1,5)+0(1,2,4)+0(7,9,1)+0(2,0,-4)$

(f) Write $a(2,3,0,0)+b(1,-5,0,2)+c(3,1,2,2)=\underset{\sim}{0}$, or,

$$\left.\begin{aligned}2a+b+3c&=0\\ 3a-5b+c&=0\\ 2c&=0\\ 2b+2c&=0\end{aligned}\right\} \text{ clearly gives } c=b=a=0, \text{ so the set is LI.}$$

(g) Write $a(1,3,2,0)+b(4,1,-2,-2)+c(0,2,0,3)+d(4,7,1,2)=\underset{\sim}{0}$, or,

$$\begin{aligned} a+4b \qquad +4d&=0\\ 3a+b+2c+7d&=0\\ 2a-2b \qquad +d&=0\\ -2b+3c+2d&=0 \end{aligned} \rightarrow \begin{bmatrix} 1&4&0&4&0\\ 0&-11&2&-5&0\\ 0&-10&0&-7&0\\ 0&-2&3&2&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&4&0&4&0\\ 0&-11&2&5&0\\ 0&0&-\frac{20}{11}&-\frac{27}{11}&0\\ 0&0&\frac{29}{11}&\frac{32}{11}&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&4&0&4&0\\ 0&-11&2&5&0\\ 0&0&20&27&0\\ 0&0&29&32&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&4&0&4&0\\ 0&-11&2&5&0\\ 0&0&20&27&0\\ 0&0&0&1&0 \end{bmatrix}$$

so $a=b=c=d=0$, hence LI

(h) Let's use Maple for this one:

```
with(linalg):
A:= array([[2,1,4],[0,2,-4],[1,0,3],[-1,3,-9],[0,1,-2]]);
b:= array([0,0,0,0,0]);
```

gives, for the solution of $a(2,0,1,-1,0)+b(1,2,0,3,1)+c(4,-4,3,-9,-2)=\underset{\sim}{0}$, or,

$$\begin{aligned} 2a+b+4c&=0\\ 0a+2b-4c&=0\\ 1a+0b+3c&=0\\ -1a+3b-9c&=0\\ 0a+1b-2c&=0, \end{aligned}$$

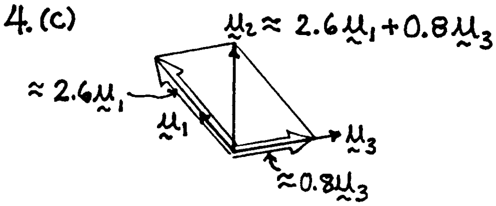
the result "[-3_t₁, 2_t₁, _t₁]". With $t_1=1$, say, we have $a=-3, b=2, c=1$, so $-3(2,0,1,-1,0)+2(1,2,0,3,1)+1(4,-4,3,-9,-2)=\underset{\sim}{0}$,

or, solving for one of the vectors, $(4,-4,3,-9,-2)=3(2,0,1,-1,0)-2(1,2,0,3,1)$. LD

(i) LD since $0(1,3,0)+0(0,1,-1)+5(0,0,0)=\underset{\sim}{0}$. Or, $(0,0,0)=0(1,3,0)+0(0,1,-1)$.

Of the remainder, (j), (m), (o), (q) are LI;
(k), (l), (n), (p), (r) are LD.

4.(c)



5. No. For example, let $\underset{\sim}{u}_1=(1,0,0,0)$, $\underset{\sim}{u}_2=(1,1,0,0)$, $\underset{\sim}{u}_3=(1,2,0,0)$. Then $\underset{\sim}{u}_1, \underset{\sim}{u}_2$ are LI (because neither is a multiple of the other); $\underset{\sim}{u}_1, \underset{\sim}{u}_3$ are LI; and $\underset{\sim}{u}_2, \underset{\sim}{u}_3$ are LI. Yet, $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3$ are LD because $(1,0,0,0)+(1,2,0,0)-2(1,1,0,0)=\underset{\sim}{0}$.

6.(a) False. For example, let $\underset{\sim}{u}_1=\cdots=\underset{\sim}{u}_k=\underset{\sim}{0}$. Then $\{\underset{\sim}{v},\underset{\sim}{0},\ldots,\underset{\sim}{0}\}$ is LD, whereas $\underset{\sim}{v}$ is not in span $\{\underset{\sim}{0},\ldots,\underset{\sim}{0}\}$ (assuming that $\underset{\sim}{v}\neq\underset{\sim}{0}$).

(b) True, because if $\underset{\sim}{v}$ were in span $\{\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ then $\{\underset{\sim}{v},\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ would have to be LD. (NOTE: $A\Rightarrow B$ implies not $B\Rightarrow$ not A.)

(c) False, because $\underset{\sim}{v}$ not in span $\{\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ does not imply that $\{\underset{\sim}{v},\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ is LI. For ex., let $\underset{\sim}{v}=(1,2)$ and let $\underset{\sim}{u}_1=\cdots=\underset{\sim}{u}_k=(0,0)$.

7.(b) $\{\underset{\sim}{0},\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_k\}$ is necessarily LD because $3\underset{\sim}{0}+0\underset{\sim}{u}_1+\cdots+0\underset{\sim}{u}_k=\underset{\sim}{0}$ without all of the coefficients $3,0,\ldots,0$ being zero.

(c) Surely it is sufficient. To see that it is necessary, write $(a_1-b_1)\underset{\sim}{u}_1+\cdots+(a_k-b_k)\underset{\sim}{u}_k=\underset{\sim}{0}$. Since the $\underset{\sim}{u}$'s are LI by assumption, we must have $a_1-b_1=0$ (so $a_1=b_1$), $\ldots$, $a_k-b_k=0$.

Section 9.9

1. (b) $(3,2), (-1,-5)$ is a basis for $\mathbb{R}^2$ iff $\alpha_1(3,2)+\alpha_2(-1,-5)=(v_1,v_2)$ admits a unique solution for α_1,α_2 for any values of v_1,v_2. Let's see: $(3\alpha_1-\alpha_2, 2\alpha_1-5\alpha_2)=(v_1,v_2)$,

$$\begin{aligned} 3\alpha_1-\alpha_2&=v_1 \\ 2\alpha_1-5\alpha_2&=v_2 \end{aligned} \qquad \text{Gauss: } \begin{pmatrix} 3 & -1 & v_1 \\ 2 & -5 & v_2 \end{pmatrix} \to \begin{pmatrix} 3 & -1 & v_1 \\ 0 & -\frac{13}{3} & -\frac{2}{3}v_1+v_2 \end{pmatrix}$$

does give unique solution for α_1,α_2 for each v_1,v_2. Hence, yes it is a basis for $\mathbb{R}^2$.

(c) We know from Definition 9.9.2 and Theorems 9.9.2 and 9.9.3 that every basis for $\mathbb{R}^n$ is composed of n vectors. Thus, the single vector $(1,1)$ cannot be a basis for $\mathbb{R}^2$.

(e) Same idea as in (b), above:

$$\begin{bmatrix} 5 & 2 & 1 & v_1 \\ -1 & 0 & -1 & v_2 \\ 2 & 1 & 1 & v_3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & v_2 \\ 5 & 2 & 1 & v_1 \\ 2 & 1 & 1 & v_3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & v_2 \\ 0 & 2 & -4 & 5v_2+v_1 \\ 0 & 1 & -1 & 2v_2+v_3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & v_2 \\ 0 & 1 & -1 & 2v_2+v_3 \\ 0 & 2 & -4 & 5v_2+v_1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & v_2 \\ 0 & 1 & -1 & 2v_2+v_3 \\ 0 & 0 & -2 & v_2-2v_3+v_1 \end{bmatrix}$$

does give unique solution, so the set *is* a basis for $\mathbb{R}^3$.

(f) No, a basis for $\mathbb{R}^4$ must contain 4 vectors [see (c), above].

(h) Same idea as in (b), above:

$$\begin{bmatrix} 4 & 1 & 5 & 0 & v_1 \\ 2 & 2 & -2 & -6 & v_2 \\ 0 & 3 & 3 & 0 & v_3 \\ 0 & 0 & 1 & 1 & v_4 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & -2 & -6 & v_2 \\ 4 & 1 & 5 & 0 & v_1 \\ 0 & 3 & 3 & 0 & v_3 \\ 0 & 0 & 1 & 1 & v_4 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & -2 & -6 & v_2 \\ 0 & -3 & 9 & 12 & -2v_2+v_1 \\ 0 & 3 & 3 & 0 & v_3 \\ 0 & 0 & 1 & 1 & v_4 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & -2 & -6 & v_2 \\ 0 & -3 & 9 & 12 & -2v_2+v_1 \\ 0 & 0 & 12 & 12 & -2v_2+v_1+v_3 \\ 0 & 0 & 1 & 1 & v_4 \end{bmatrix}$$

and at the next step we will obtain $0\ 0\ 0\ 0\ \frac{1}{6}v_2-\frac{1}{12}v_1-\frac{1}{12}v_3+v_4$, so there is no solution unless $\frac{1}{6}v_2-\frac{1}{12}v_1-\frac{1}{12}v_3+v_4$ happens to $=0$. Hence, no, the set is *not* a basis for $\mathbb{R}^4$.

(i) Yes (j) Yes (k) No (l) No, a basis for $\mathbb{R}^4$ must contain 4 vectors [see (c)]

(m) No for two reasons: a basis for $\mathbb{R}^4$ must contain 4 vectors; and the set is obviously LD due to the $(0,0,0,0)$ vector

(n) No, the set is LD due to the $(0,0,0,0)$ vector

(p) Yes iff $\alpha_1(1,1,2)+\alpha_2(4,-2,-1)=a(3,-5,-6)+b(1,2,1)$ has a unique solution for α_1,α_2 for every pair of numbers a,b.

$$\begin{aligned} \alpha_1+4\alpha_2&=3a+b \\ \alpha_1-2\alpha_2&=-5a+2b \\ 2\alpha_1-\alpha_2&=-6a+b. \end{aligned}$$

$$\text{Gauss: } \begin{bmatrix} 1 & 4 & 3a+b \\ 1 & -2 & -5a+2b \\ 2 & -1 & -6a+b \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3a+b \\ 0 & -6 & -8a+b \\ 0 & -9 & -12a-b \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3a+b \\ 0 & -6 & -8a+b \\ 0 & 0 & -\frac{5}{2}b \end{bmatrix}$$

has a unique solution for α_1,α_2 only if $b=0$. Thus, no, $(1,1,2),(4,-2,-1)$ is *not* a basis for span$\{(3,-5,-6),(1,2,1)\}$. Geometrically, the planes span$\{(1,1,2),(4,-2,-1)\}$ and span$\{(3,-5,-6),(1,2,1)\}$ do not coincide; they intersect along a line [namely, the line generated by $(3,-5,-6)$].

(q) Same method as in (p).

$$\begin{bmatrix} 1 & 1 & 2a+b \\ 1 & -1 & 4a+7b \\ 1 & 2 & a-2b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2a+b \\ 0 & -2 & 2a+6b \\ 0 & 1 & -a-3b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2a+b \\ 0 & -2 & 2a+6b \\ 0 & 0 & 0 \end{bmatrix}$$

does give unique solution for α_1, α_2 for any values of a, b so yes, the vectors $(1,1,1), (1,-1,2)$ <u>are</u> a basis for span $\{(2,4,1),(1,7,-2)\}$.

(r) It is not. NOTE: a simpler way to work parts (o), (p), (q), (r) is to see if the two cross products are colinear. In (q), for example,

$$(1,1,1)\times(1,-1,2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3\hat{\imath} - \hat{\jmath} - 2\hat{k} \text{ and } (2,4,1)\times(1,7,-2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 1 \\ 1 & 7 & -2 \end{vmatrix} = -15\hat{\imath} + 5\hat{\jmath} + 10\hat{k}$$

are indeed aligned. However, the cross product is not introduced until Section 14.2, so it will not be used here.

2. (b) $\underset{\sim}{u} = (1,0,0) = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_1}{\underset{\sim}{e}_1\cdot\underset{\sim}{e}_1}\underset{\sim}{e}_1 + \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_2}{\underset{\sim}{e}_2\cdot\underset{\sim}{e}_2}\underset{\sim}{e}_2 + \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_3}{\underset{\sim}{e}_3\cdot\underset{\sim}{e}_3}\underset{\sim}{e}_3 = \dfrac{2}{14}\underset{\sim}{e}_1 + \dfrac{1}{5}\underset{\sim}{e}_2 + \dfrac{6}{70}\underset{\sim}{e}_3$

3. First, normalize the $\underset{\sim}{e}$'s: $\hat{\underset{\sim}{e}}_1 = \frac{1}{\sqrt{14}}(2,1,3)$, $\hat{\underset{\sim}{e}}_2 = \frac{1}{\sqrt{5}}(1,-2,0)$, $\hat{\underset{\sim}{e}}_3 = \frac{1}{\sqrt{70}}(6,3,-5)$

(b) $\underset{\sim}{u} = (1,0,0) = (\underset{\sim}{u}\cdot\hat{\underset{\sim}{e}}_1)\hat{\underset{\sim}{e}}_1 + (\underset{\sim}{u}\cdot\hat{\underset{\sim}{e}}_2)\hat{\underset{\sim}{e}}_2 + (\underset{\sim}{u}\cdot\hat{\underset{\sim}{e}}_3)\hat{\underset{\sim}{e}}_3 = \frac{2}{\sqrt{14}}\hat{\underset{\sim}{e}}_1 + \frac{1}{\sqrt{5}}\hat{\underset{\sim}{e}}_2 + \frac{6}{\sqrt{70}}\hat{\underset{\sim}{e}}_3$

The latter does in fact agree with the result obtained in 2(b) since

$$\underset{\sim}{u} = \frac{2}{\sqrt{14}}\hat{\underset{\sim}{e}}_1 + \frac{1}{\sqrt{5}}\hat{\underset{\sim}{e}}_2 + \frac{6}{\sqrt{70}}\hat{\underset{\sim}{e}}_3 = \frac{2}{\sqrt{14}}\frac{1}{\sqrt{14}}(2,1,3) + \frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}}(1,-2,0) + \frac{1}{\sqrt{70}}\frac{1}{\sqrt{70}}(6,3,-5) \checkmark$$

4. (b) $\underset{\sim}{u} = (0,6,0,0) = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_1}{\underset{\sim}{e}_1\cdot\underset{\sim}{e}_1}\underset{\sim}{e}_1 + \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_2}{\underset{\sim}{e}_2\cdot\underset{\sim}{e}_2}\underset{\sim}{e}_2 + \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{e}_3}{\underset{\sim}{e}_3\cdot\underset{\sim}{e}_3}\underset{\sim}{e}_3 = 0\underset{\sim}{e}_1 + 0\underset{\sim}{e}_2 + \frac{6}{1}\underset{\sim}{e}_3 + 0\underset{\sim}{e}_4 = 6\underset{\sim}{e}_3$

(c) $\underset{\sim}{u} = (2,5,1,-3) =$ " " " $= \frac{18}{30}\underset{\sim}{e}_1 + 0\underset{\sim}{e}_2 + \frac{5}{1}\underset{\sim}{e}_3 + \frac{4}{5}\underset{\sim}{e}_4$

$= \frac{3}{5}\underset{\sim}{e}_1 + 5\underset{\sim}{e}_3 + \frac{4}{5}\underset{\sim}{e}_4.$

5. It is easily verified that they are orthogonal. Hence (Thm 9.8.5) they are LI (indeed, we can think of orthogonality as the <u>extreme</u> case of linear independence) and there are 4 of them, so they constitute a basis for $\mathbb{R}^4$. Also, let us solve (26) to show that our result agrees with (25):

$$\begin{bmatrix} 1 & 0 & -2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 1 & -3 \\ 0 & 0 & 5 & -1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 5 & 5 & -11 \\ 0 & 0 & 5 & -1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 5 & 5 & -11 \\ 0 & 0 & 0 & -6 & 17 \end{bmatrix}$$

gives $\alpha_4 = -17/6$, $\alpha_3 = 19/30$, $\alpha_2 = 3$, $\alpha_1 = -2/5$ ✓

6. (a) Yes iff (if and only if) $\dim \mathcal{S} = k$.

(b) Yes

7. If $\mathcal{S}$ contains only $\underset{\sim}{0}$, then the only basis possible is the single vector $\underset{\sim}{0}$. The latter does span $\mathcal{S}$ but it is not a LI set (see Thm 9.8.1) since $\alpha_1\underset{\sim}{0} = \underset{\sim}{0}$ even if $\alpha_1 \neq 0$.

8. (b) 2 (c) 3 (e) 2 (f) 3 (g) 2 (h) 2

9. (b) 2 (c) 2 (e) 2 (f) 2 (g) 1 (h) 3

11. Let us <u>derive</u> (11.1) for the representative case where $k = 3$.

Given 3 LI (but not orthog.) vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$, we can obtain 3 orthogonal vectors by "bootstrapping" as follows.

Let $\underset{\sim}{e}_1 = \underset{\sim}{v}_1$. ①

Then, let $\underset{\sim}{e}_2 = \underset{\sim}{v}_2 + \alpha \underset{\sim}{v}_1$ ②

such that $\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1 = \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 + \alpha \|\underset{\sim}{v}_1\|^2 = 0$

or, $\alpha = -\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 / \|\underset{\sim}{v}_1\|^2$

so $\underset{\sim}{e}_2 = \underset{\sim}{v}_2 - \dfrac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{\|\underset{\sim}{v}_1\|^2} \underset{\sim}{v}_1$

$= \underset{\sim}{v}_2 - (\hat{\underset{\sim}{e}}_1 \cdot \underset{\sim}{v}_2) \hat{\underset{\sim}{e}}_1$ ③

Next, let $e_3 = \underset{\sim}{v}_3 + \beta \underset{\sim}{v}_1 + \gamma \underset{\sim}{v}_2$ ④

such that $\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_1 = \underset{\sim}{v}_3 \cdot \underset{\sim}{v}_1 + \beta \|\underset{\sim}{v}_1\|^2 + \gamma (\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1) = 0$

and $\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_2 = \underset{\sim}{v}_3 \cdot \underset{\sim}{e}_2 + \beta \underset{\sim}{v}_1 \cdot \underset{\sim}{e}_2 + \gamma (\underset{\sim}{v}_2 \cdot \underset{\sim}{e}_2) = 0$

The latter two equations give

$$\beta = [(\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_2)(\underset{\sim}{v}_2 \cdot \underset{\sim}{e}_1) - (\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_1)(\underset{\sim}{v}_2 \cdot \underset{\sim}{e}_2)] / [\|\underset{\sim}{v}_1\|^2 \underset{\sim}{v}_2 \cdot \underset{\sim}{e}_2 - (\underset{\sim}{v}_2 \cdot \underset{\sim}{e}_1)(\underset{\sim}{v}_1 \cdot \underset{\sim}{e}_2)]$$

$$\gamma = [(\underset{\sim}{v}_1 \cdot \underset{\sim}{e}_2)(\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_1) - \|\underset{\sim}{v}_1\|^2 (\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_2)] / [\qquad " \qquad]$$

Putting these into ④ will give $\underset{\sim}{e}_3$, but this looks too messy. Let us return to ④, noting that the linear combination of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ is equivalent – more conveniently – to a linear combination of $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$. Thus, replace ④ by

$\underset{\sim}{e}_3 = \underset{\sim}{v}_3 + \beta \underset{\sim}{e}_1 + \gamma \underset{\sim}{e}_2$ ⑤

and set $\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_1 = \underset{\sim}{v}_3 \cdot \underset{\sim}{e}_1 + \beta \|\underset{\sim}{e}_1\|^2 + \gamma \overset{0}{\cancel{\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1}} = 0$

$\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_2 = \underset{\sim}{v}_3 \cdot \underset{\sim}{e}_2 + \beta \overset{0}{\cancel{\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2}} + \gamma \|\underset{\sim}{e}_2\|^2 = 0$

which (much more easily than above) gives $\beta = -\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_1 / \|\underset{\sim}{e}_1\|^2$

$\gamma = -\underset{\sim}{v}_3 \cdot \underset{\sim}{e}_2 / \|\underset{\sim}{e}_2\|^2$

and putting these into ⑤ gives

$\underset{\sim}{e}_3 = \underset{\sim}{v}_3 - (\underset{\sim}{v}_3 \cdot \hat{\underset{\sim}{e}}_1) \hat{\underset{\sim}{e}}_1 - (\underset{\sim}{v}_3 \cdot \hat{\underset{\sim}{e}}_2) \hat{\underset{\sim}{e}}_2$ ⑥

From ①, ③, ⑥ it is becoming clear that

$$\underset{\sim}{e}_j = \underset{\sim}{v}_j - (\underset{\sim}{v}_j \cdot \hat{\underset{\sim}{e}}_1) \hat{\underset{\sim}{e}}_1 - (\underset{\sim}{v}_j \cdot \hat{\underset{\sim}{e}}_2) \hat{\underset{\sim}{e}}_2 - \cdots - (\underset{\sim}{v}_j \cdot \hat{\underset{\sim}{e}}_{j-1}) \hat{\underset{\sim}{e}}_{j-1},$$

as in (11.1). The only step remaining is to normalize these $\underset{\sim}{e}_j$'s by dividing by their norms, as in (11.1).

Note the bootstrapping: $\underset{\sim}{e}_1$ is a linear combination of $\underset{\sim}{v}_1$, $\underset{\sim}{e}_2$ is a linear combination of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, $\underset{\sim}{e}_3$ is a " " of $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$, and so on.

12. (b) $\underset{\sim}{v}_1 = (1,-2)$, $\underset{\sim}{v}_2 = (3,4)$.

$\underset{\sim}{e}_1 = \underset{\sim}{v}_1 = (1,-2)$, $\hat{\underset{\sim}{e}}_1 = \frac{1}{\sqrt{5}}(1,-2)$

$\underset{\sim}{e}_2 = \underset{\sim}{v}_2 - (\underset{\sim}{v}_2 \cdot \hat{\underset{\sim}{e}}_1) \hat{\underset{\sim}{e}}_1 = (3,4) - (-\sqrt{5}) \frac{1}{\sqrt{5}} (1,-2) = (4,2)$, $\hat{\underset{\sim}{e}}_2 = \frac{1}{\sqrt{20}}(4,2) = \frac{1}{\sqrt{5}}(2,1)$

(c) $\underset{\sim}{v}_1 = (1,0,0)$, $\underset{\sim}{v}_2 = (1,1,0)$, $\underset{\sim}{v}_3 = (1,1,1)$

$\underset{\sim}{e}_1 = \underset{\sim}{v}_1 = (1,0,0)$, $\hat{\underset{\sim}{e}}_1 = (1,0,0)$

$\underset{\sim}{e}_2 = \underset{\sim}{v}_2 - (\underset{\sim}{v}_2 \cdot \hat{\underset{\sim}{e}}_1) \hat{\underset{\sim}{e}}_1 = (1,1,0) - (1)(1,0,0) = (0,1,0)$, $\hat{\underset{\sim}{e}}_2 = (0,1,0)$

$\underset{\sim}{e}_3 = \underset{\sim}{v}_3 - (\underset{\sim}{v}_3 \cdot \hat{\underset{\sim}{e}}_1) \hat{\underset{\sim}{e}}_1 - (\underset{\sim}{v}_3 \cdot \hat{\underset{\sim}{e}}_2) \hat{\underset{\sim}{e}}_2 = (1,1,1) - (1)(1,0,0) - (1)(0,1,0) = (0,0,1)$, $\hat{\underset{\sim}{e}}_3 = (0,0,1)$

(i) $\mathbf{v}_1 = (2,1,1,0)$, $\mathbf{v}_2 = (1,5,-1,2)$

$\mathbf{e}_1 = \mathbf{v}_1 = (2,1,1,0)$, $\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{6}}(2,1,1,0)$

$\mathbf{e}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 = (1,5,-1,2) - \sqrt{6}\,\frac{1}{\sqrt{6}}(2,1,1,0) = (-1,4,-2,2)$, $\hat{\mathbf{e}}_2 = \frac{1}{5}(-1,4,-2,2)$

$$13.\ (a)\quad \mathbf{u} = \sum_1^n \alpha_j \mathbf{e}_j, \quad \mathbf{u}\cdot\mathbf{e}_i^* = \sum_1^n \alpha_j \underset{=1 \text{ if } j=i,\ 0 \text{ otherwise}}{\boxed{\mathbf{e}_j \cdot \mathbf{e}_i^*}} = \alpha_i(1), \text{ so } \alpha_i = \mathbf{u}\cdot\mathbf{e}_i^*$$

and $\mathbf{u} = \sum_1^n (\mathbf{u}\cdot\mathbf{e}_j^*)\mathbf{e}_j$.

(e)
$$\left.\begin{aligned} \mathbf{e}_1^*\cdot\mathbf{e}_1 &= (a,b,c)\cdot(1,0,0) = a = 1 \\ \mathbf{e}_1^*\cdot\mathbf{e}_2 &= (a,b,c)\cdot(1,1,0) = a+b = 0 \\ \mathbf{e}_1^*\cdot\mathbf{e}_3 &= (a,b,c)\cdot(1,1,1) = a+b+c = 0 \end{aligned}\right\} \begin{array}{l} \text{these give 3 eqns in} \\ \text{the 3 unknowns } a,b,c. \\ \text{Solving, } a=1, b=-1, c=0, \text{ so } \mathbf{e}_1^* = (1,-1,0). \end{array}$$

Next,
$$\left.\begin{aligned} \mathbf{e}_2^*\cdot\mathbf{e}_1 &= (d,e,f)\cdot(1,0,0) = d = 0 \\ \mathbf{e}_2^*\cdot\mathbf{e}_2 &= (d,e,f)\cdot(1,1,0) = d+e = 1 \\ \mathbf{e}_2^*\cdot\mathbf{e}_3 &= (d,e,f)\cdot(1,1,1) = d+e+f = 0 \end{aligned}\right\} \text{these give } d=0, e=1, f=-1, \text{ so } \mathbf{e}_2^* = (0,1,-1)$$

Finally,
$$\left.\begin{aligned} \mathbf{e}_3^*\cdot\mathbf{e}_1 &= (g,h,i)\cdot(1,0,0) = g = 0 \\ \mathbf{e}_3^*\cdot\mathbf{e}_2 &= (g,h,i)\cdot(1,1,0) = g+h = 0 \\ \mathbf{e}_3^*\cdot\mathbf{e}_3 &= (g,h,i)\cdot(1,1,1) = g+h+i = 1 \end{aligned}\right\} \text{These give } g=0, h=0, i=1, \text{ so } \mathbf{e}_3^* = (0,0,1)$$

Then

$$\mathbf{u} = (4,-1,5) = (\mathbf{u}\cdot\mathbf{e}_1^*)\mathbf{e}_1 + (\mathbf{u}\cdot\mathbf{e}_2^*)\mathbf{e}_2 + (\mathbf{u}\cdot\mathbf{e}_3^*)\mathbf{e}_3 = 5\mathbf{e}_1 - 6\mathbf{e}_2 + 5\mathbf{e}_3$$
$$\mathbf{v} = (0,0,2) = (\mathbf{v}\cdot\mathbf{e}_1^*)\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2^*)\mathbf{e}_2 + (\mathbf{v}\cdot\mathbf{e}_3^*)\mathbf{e}_3 = 0\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3$$
$$\mathbf{w} = (5,-2,3) = (\mathbf{w}\cdot\mathbf{e}_1^*)\mathbf{e}_1 + (\mathbf{w}\cdot\mathbf{e}_2^*)\mathbf{e}_2 + (\mathbf{w}\cdot\mathbf{e}_3^*)\mathbf{e}_3 = 7\mathbf{e}_1 - 5\mathbf{e}_2 + 3\mathbf{e}_3$$

Section 9.10

1. $\|\mathbf{E}\|^2 = \sum_1^N (c_j - \alpha_j)^2 + \|\mathbf{u}\|^2 - \sum_1^N \alpha_j^2$

so $\partial\|\mathbf{E}\|^2/\partial c_k = 2(c_k - \alpha_k) = 0$ gives $c_k = \alpha_k = \mathbf{u}\cdot\hat{\mathbf{e}}_k$. ✓

2. (b) $\mathbf{u} = (0,0,0,2,1) \approx (\mathbf{u}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\mathbf{u}\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\mathbf{u}\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$
$= 0\hat{\mathbf{e}}_1 + \frac{1}{\sqrt{6}}\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3$

$\mathbf{E} = (0,0,0,2,1) - \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}(2,0,-1,0,1) - 2(0,0,0,1,0) = (0,0,0,2,1) - (\frac{1}{3},0,-\frac{1}{6},2,\frac{1}{6})$
$= (-\frac{1}{3},0,\frac{1}{6},0,\frac{5}{6})$,

$\|\mathbf{E}\| = \sqrt{\frac{1}{9}+\frac{1}{36}+\frac{25}{36}} = \sqrt{\frac{30}{36}} = \frac{\sqrt{30}}{6}$

4. (a) In span$\{\hat{\mathbf{e}}_1\}$: $\mathbf{u} = (4,1,0,-1) \approx (\mathbf{u}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 = \frac{6}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,1,0,-1) = (2,2,0,-2)$

$\mathbf{E} = (4,1,0,-1) - (2,2,0,-2) = (2,-1,0,1)$, $\|\mathbf{E}\| = \sqrt{6} = \underline{\underline{2.45}}$

In span$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$: $\tilde{\mathbf{u}} = (4,1,0,-1) \approx (\mathbf{u}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\mathbf{u}\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$
$= \frac{6}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,1,0,-1) + \frac{3}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,-1,-1,0) = (3,1,-1,-2)$

$\mathbf{E} = (4,1,0,-1) - (3,1,-1,-2) = (1,0,1,1)$, $\|\mathbf{E}\| = \sqrt{3} = \underline{\underline{1.73}}$

In span$\{\underset{\sim}{\hat e}_1, \underset{\sim}{\hat e}_2, \underset{\sim}{\hat e}_3\}$: $\underset{\sim}{u} = (4,1,0,-1) \approx (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_1)\underset{\sim}{\hat e}_1 + (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_2)\underset{\sim}{\hat e}_2 + (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_3)\underset{\sim}{\hat e}_3$

$$= \frac{6}{\sqrt3}\frac{1}{\sqrt3}(1,1,0,-1) + \frac{3}{\sqrt3}\frac{1}{\sqrt3}(1,-1,-1,0) + \frac{3}{\sqrt3}\frac{1}{\sqrt3}(1,0,1,1)$$

$$= (4,1,0,-1)$$

$\underset{\sim}{E} = \underset{\sim}{0}$; $\|\underset{\sim}{E}\| = \underset{=}{0}$. That is, $\underset{\sim}{u}$ happens to lie in span$\{\underset{\sim}{\hat e}_1, \underset{\sim}{\hat e}_2, \underset{\sim}{\hat e}_3\}$.

In span$\{\underset{\sim}{\hat e}_1, \underset{\sim}{\hat e}_2, \underset{\sim}{\hat e}_3, \underset{\sim}{\hat e}_4\}$: $\underset{\sim}{u} = (4,1,0,-1) = \underbrace{(\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_1)\underset{\sim}{\hat e}_1 + \cdots + (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_3)\underset{\sim}{\hat e}_3}_{\text{These will be the same as above, and will give } \underset{\sim}{u} \text{ exactly}} + \underbrace{(\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_4)}_{\text{this will be 0}}\underset{\sim}{\hat e}_4$

This time we can write =, rather than ≈ since $\underset{\sim}{\hat e}_1, \dots, \underset{\sim}{\hat e}_4$ are in fact a basis for $\mathbb{R}^4$.

5. Choosing the optimal c_j's, namely $c_j = \alpha_j = \underset{\sim}{u}\cdot\underset{\sim}{\hat e}_j$, (4) becomes $\|\underset{\sim}{E}\|^2 = 0 + \|\underset{\sim}{u}\|^2 - \sum_1^N \alpha_j^2 \geq 0$ because $\|\underset{\sim}{E}\| \geq 0$, as for any norm.

Thus,
$$\sum_{j=1}^N (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_j)^2 \leq \|\underset{\sim}{u}\|^2$$

6. (a) $\underset{\sim}{u}\cdot\underset{\sim}{v} \equiv 2u_1v_1 + u_2v_2$, $\|\underset{\sim}{u}\| = \sqrt{2u_1^2 + u_2^2}$

$\underset{\sim}{u} = (1,1) \approx (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_1)\underset{\sim}{\hat e}_1$

but what is $\underset{\sim}{\hat e}_1$? $\underset{\sim}{e}_1 = (12,5)$ and $\|\underset{\sim}{e}_1\| = \sqrt{\underset{\sim}{e}_1\cdot\underset{\sim}{e}_1} = \sqrt{2(12)^2 + 5^2} = \sqrt{313}$,

so $\underset{\sim}{\hat e}_1 = \frac{1}{\sqrt{313}}(12,5)$ and

$$\underset{\sim}{u} = (1,1) \approx (\underset{\sim}{u}\cdot\underset{\sim}{\hat e}_1)\underset{\sim}{\hat e}_1 = \frac{1}{\sqrt{313}}[2(12)+5]\frac{1}{\sqrt{313}}(12,5) = \frac{29}{313}(12,5)$$

8. (a) $\sum_{j=1}^N \delta_{ij} = \delta_{i1} + \delta_{i2} + \cdots + \delta_{iN}$. One of these terms $=1$ and all others $=0$, so $\Sigma = 1$.

(b) $\sum_{j=1}^N \delta_{ij}\delta_{jk}$. Each term is 0 unless $i=j$ and $j=k$ and that term is 1. If $i \neq k$ then we never have $i=j$ and $j=k$ so all terms are 0.

(c) $\sum_j \sum_k \delta_{ij}\delta_{jk}\delta_{kl} = \sum_j \delta_{ij}\left(\sum_k \delta_{jk}\delta_{kl}\right) = \sum_j \delta_{ij}\delta_{jl}$ per (b)

$= \delta_{il}$ per (b) again

CHAPTER 10

Section 10.2

1. $\underset{\sim}{A}=\underset{3\times2}{\begin{pmatrix}0&3\\2&-5\\1&10\end{pmatrix}}$, $\underset{\sim}{B}=\underset{2\times2}{\begin{pmatrix}5&-1\\0&2\end{pmatrix}}$, $\underset{\sim}{x}=\underset{2\times1}{\begin{pmatrix}4\\3\end{pmatrix}}$, $\underset{\sim}{y}=\underset{1\times2}{(-1,2)}$

$\underset{\sim}{A}\underset{\sim}{B}=\begin{pmatrix}0&3\\2&-5\\1&10\end{pmatrix}\begin{pmatrix}5&-1\\0&2\end{pmatrix}=\begin{pmatrix}0&6\\10&-12\\5&19\end{pmatrix}$, $\underset{2\times2\ 3\times2}{\underset{\sim}{B}\underset{\sim}{A}}$ does not exist, $\underset{\sim}{A}\underset{\sim}{x}=\begin{pmatrix}0&3\\2&-5\\1&10\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}9\\-7\\34\end{pmatrix}$, $\underset{\sim}{x}\underset{\sim}{A}$ does not exist,

$\underset{\sim}{B}\underset{\sim}{x}=\begin{pmatrix}5&-1\\0&2\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}17\\6\end{pmatrix}$, $\underset{\sim}{x}\underset{\sim}{B}$ does not exist, $\underset{\sim}{y}\underset{\sim}{B}=\underset{1\times2}{(-1,2)}\underset{2\times2}{\begin{pmatrix}5&-1\\0&2\end{pmatrix}}=\underset{1\times2}{(-5,5)}$, $\underset{\sim}{A}^2$ does not exist,

$\underset{\sim}{B}^2=\begin{pmatrix}5&-1\\0&2\end{pmatrix}\begin{pmatrix}5&-1\\0&2\end{pmatrix}=\begin{pmatrix}25&-7\\0&4\end{pmatrix}$, $\underset{\sim}{x}^2$ does not exist, $\underset{\sim}{x}\underset{\sim}{y}=\begin{pmatrix}4\\3\end{pmatrix}(-1,2)=\begin{pmatrix}-4&8\\-3&6\end{pmatrix}$, $\underset{\sim}{y}\underset{\sim}{x}=(-1,2)\begin{pmatrix}4\\3\end{pmatrix}=2$

2. (a) ND (not defined) (b) 4×4 (c) 6×3 (d) 6×1 (e) ND
(f) $4\times1+3\times1=$ ND (g) 4×1 (h) ND (i) ND (j) ND

3. $\begin{pmatrix}1&2&3\\8&10&12\\21&24&27\end{pmatrix}$ and $\begin{pmatrix}1&4&9\\4&10&18\\7&16&27\end{pmatrix}$

4. No; $\underset{\sim}{A}-\underset{\sim}{c}=n\times n-1\times1=$ not defined

5. (a) $(\underset{\sim}{A}+\underset{\sim}{B})^2=(\underset{\sim}{A}+\underset{\sim}{B})(\underset{\sim}{A}+\underset{\sim}{B})=\underset{\sim}{A}^2+\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}+\underset{\sim}{B}^2=\underset{\sim}{A}^2+2\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}^2$ only in the exceptional case when $\underset{\sim}{A},\underset{\sim}{B}$ commute.

(b) $(\underset{\sim}{A}+\underset{\sim}{B})(\underset{\sim}{A}-\underset{\sim}{B})=\underset{\sim}{A}^2-\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}-\underset{\sim}{B}^2=\underset{\sim}{A}^2-\underset{\sim}{B}^2$ only in the exceptional case when $\underset{\sim}{A},\underset{\sim}{B}$ commute

(c) $(\underset{\sim}{A}\underset{\sim}{B})^2=\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{B}=\underset{\sim}{A}\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{B}=\underset{\sim}{A}^2\underset{\sim}{B}^2$ " " " " " " " "

(d) Same idea as in (c).

6. (a) $\underset{\sim}{A}^p=\underset{m\times n}{A}\cdot\underset{m\times n}{A}\cdots A$ p times

so we need m=n. Otherwise, $\underset{\sim}{A}^p$ is not defined

(b) $(\underset{\sim}{A}\underset{\sim}{B})^2=\underset{m\times n}{\underset{\sim}{A}}\,\underset{p\times q}{\underset{\sim}{B}}\underset{\sim}{A}\underset{\sim}{B}$ so for $\underset{\sim}{A}\underset{\sim}{B}$ to exist we need $n=p$. Then $\underset{\sim}{A}\underset{\sim}{B}=(m\times n)(n\times q)=m\times q$. Next, for $\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{B}=(m\times q)(m\times q)$ to exist we need $q=m$. Thus, we need $n=p$ and $q=m$. Note that we do not need $m=n=p=q$. For ex., let $n=p=2$ and $q=m=3$. Then

$(\underset{\sim}{A}\underset{\sim}{B})^2=\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{B}=\underbrace{(3\times2)(2\times3)}_{3\times3}\underbrace{(3\times2)(2\times3)}_{3\times3}=3\times3$ does exist.

7. (b) $(\underset{\sim}{A}+\underset{\sim}{B})\underset{\sim}{C}(\underset{\sim}{D}+\underset{\sim}{E})=(\underset{\sim}{A}+\underset{\sim}{B})[\underset{\sim}{C}(\underset{\sim}{D}+\underset{\sim}{E})]$ by (30b) ; i.e., we can group in any way
$=(\underset{\sim}{A}+\underset{\sim}{B})[\underset{\sim}{C}\underset{\sim}{D}+\underset{\sim}{C}\underset{\sim}{E}]$ by (30d)
$=A(CD+CE)+B(CD+CE)$ by (30c)
$=\underset{\sim}{A}\underset{\sim}{C}\underset{\sim}{D}+\underset{\sim}{A}\underset{\sim}{C}\underset{\sim}{E}+\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{D}+\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{E}$ by (30d)

(d) $(\underset{\sim}{A}-3\underset{\sim}{I})(2\underset{\sim}{A}+\underset{\sim}{I})=\underset{\sim}{A}(2\underset{\sim}{A}+\underset{\sim}{I})-(3\underset{\sim}{I})(2\underset{\sim}{A}+\underset{\sim}{I})$ by (30c)
$=\underset{\sim}{A}(2\underset{\sim}{A})+\underset{\sim}{A}\underset{\sim}{I}-(3\underset{\sim}{I})(2\underset{\sim}{A})-(3\underset{\sim}{I})(\underset{\sim}{I})$ by (30d)
$=2\underset{\sim}{A}^2+\underset{\sim}{A}-3(\underset{\sim}{I})(2\underset{\sim}{A})-3(\underset{\sim}{I})(\underset{\sim}{I})$ by (30a) and (27)
$=2\underset{\sim}{A}^2+\underset{\sim}{A}-6(\underset{\sim}{I}\underset{\sim}{A})-3\underset{\sim}{I}$ by (30a) and (27)

$$= 2\underset{\sim}{A}^2 + \underset{\sim}{A} - 6\underset{\sim}{A} - 3\underset{\sim}{I} \quad \text{by } (27)$$

$$= (2\underset{\sim}{A}^2 + (\underset{\sim}{A} - 6\underset{\sim}{A})) + (-3\underset{\sim}{I}) \quad \text{by } (9b)$$

$$= (2\underset{\sim}{A}^2 + (-3\underset{\sim}{A})) + (-3\underset{\sim}{I})$$

$$= 2\underset{\sim}{A}^2 - 3\underset{\sim}{A} - 3\underset{\sim}{I}$$

NOTE: We do <u>not</u> advise such fussiness in applications, however, just as we do not get bogged down in justifying the steps in the arithmetic calculation of $(2+3)5(6-4)$ as $(5)(5)(2) = 50$.

8. (b) $\underset{\sim}{B}^2 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}$, $\underset{\sim}{B}^3 = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix} = \begin{pmatrix}1&3\\0&1\end{pmatrix}$, $\cdots$, $\underset{\sim}{B}^{100} = \begin{pmatrix}1&100\\0&1\end{pmatrix}$

   (c) $\underset{\sim}{C}^2 = \begin{pmatrix}0&3\\0&0\end{pmatrix}\begin{pmatrix}0&3\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}, \ldots, C^{100} = \underset{\sim}{0}$

   (e) $\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C} = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&3\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}0&3\\0&0\end{pmatrix} = \begin{pmatrix}0&3\\0&3\end{pmatrix}$, $(\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C})^2 = \begin{pmatrix}0&3\\0&3\end{pmatrix}\begin{pmatrix}0&3\\0&3\end{pmatrix} = \begin{pmatrix}0&9\\0&9\end{pmatrix}$, $(ABC)^3 = \begin{pmatrix}0&9\\0&9\end{pmatrix}\begin{pmatrix}0&3\\0&3\end{pmatrix} = \begin{pmatrix}0&27\\0&27\end{pmatrix}$

10. (b) Since $\underbrace{A}_{m\times n}\underbrace{x}_{4\times 1} = 2\times 1$ we see that $n=4$ and $m=2$. Thus $\underset{\sim}{A}$ is 2×4 and

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4\\a_{21}x_1+a_{22}x_2+a_{23}x_3+a_{24}x_4\end{pmatrix} = \begin{pmatrix}x_2+x_3-x_4\\x_1+5x_3\end{pmatrix}$$

Since the latter must hold for all $\underset{\sim}{x}$ vectors (i.e., for arbitrary x_1, x_2, x_3, x_4) it follows that $a_{11}=0,\ a_{12}=1,\ a_{13}=1,\ a_{14}=-1$

$a_{21}=1,\ a_{22}=0,\ a_{23}=5,\ a_{24}=0$

and so

$$A = \begin{pmatrix}0&1&1&-1\\1&0&5&0\end{pmatrix}$$

   (c) $(m\times n)(4\times 1) = 3\times 1 \Rightarrow n=4, m=3$. Thus, proceeding as in (b), $\underset{\sim}{A} = \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}$

   (e) $(m\times n)(4\times 1) = 4\times 1 \Rightarrow n=4, m=4$, and $\underset{\sim}{A} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$

11. (b) $\underset{\sim}{A}\underset{\sim}{B} = \begin{pmatrix}0&2\\0&3\\0&-1\\0&4\\0&8\end{pmatrix}\begin{pmatrix}6&4\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\\0&0\\0&0\\0&0\end{pmatrix}$

12. (a) $\left(\begin{array}{cc|c}2&0&-1\\1&-1&0\\\hline 5&2&4\end{array}\right)\left(\begin{array}{cc|c}3&6&2\\0&1&0\\\hline 3&-4&7\end{array}\right) = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ where

$a = \begin{pmatrix}2&0\\1&-1\end{pmatrix}\begin{pmatrix}3&6\\0&1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}(3\ -4) = \begin{pmatrix}6&12\\3&5\end{pmatrix} + \begin{pmatrix}-3&4\\0&0\end{pmatrix} = \begin{pmatrix}3&16\\3&5\end{pmatrix}$

$b = \begin{pmatrix}2&0\\1&-1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}(7) = \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-7\\0\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$

$c = (5\ 2)\begin{pmatrix}3&6\\0&1\end{pmatrix} + (4)(3\ -4) = (15\ 32) + (12\ -16) = (27\ 16)$

$d = (5\ 2)\begin{pmatrix}2\\0\end{pmatrix} + (4)(7) = (10) + (28) = (38)$

so the result is equal to $\left(\begin{array}{cc|c}3&16&-3\\3&5&2\\\hline 27&16&38\end{array}\right) = \begin{pmatrix}3&16&-3\\3&5&2\\27&16&38\end{pmatrix}$

13. $$\underbrace{(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n)}_{1\times n}\underbrace{\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}}_{n\times 1} = \underbrace{(c_1\underset{\sim}{x}_1 + \cdots + c_n\underset{\sim}{x}_n)}_{1\times 1}$$

14. (a) We can easily get the partitioned matrices to be of the same form — for ex.,

$$\left(\begin{array}{cc|c}1&2&3\\4&5&6\\\hline 7&8&9\end{array}\right)+\left(\begin{array}{c|c}4&3\\\hline 2&1\end{array}\right) = 2\times2 + 2\times2 \checkmark$$

but the a_{11} element of the sum matrix is $= \overbrace{\begin{pmatrix}1&2\\4&5\end{pmatrix}}^{2\times2}+\overbrace{(4)}^{1\times1}$, which is undefined. Similarly, in this example, for a_{12} and a_{21}. Thus, the answer is no, one cannot render noncomformable matrices conformable by partitioning. Similarly for part (b).

15.(a) $\underset{\sim}{A}^2=\begin{pmatrix}0&a\\0&0\end{pmatrix}\begin{pmatrix}0&a\\0&a\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}=\underset{\sim}{0}$, so $\underset{\sim}{A}^2=\underset{\sim}{0}$.

$$\underset{\sim}{A}^2=\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&ac\\0&0&0\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&ac\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=\underset{\sim}{0},\text{ so }\underset{\sim}{A}^3=\underset{\sim}{0}.$$

$$\underset{\sim}{A}^2=\begin{pmatrix}0&a&b&c\\0&0&d&e\\0&0&0&f\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&a&b&c\\0&0&d&e\\0&0&0&f\\0&0&0&0\end{pmatrix}=\begin{pmatrix}0&0&ad&ae+bf\\0&0&0&df\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \underset{\sim}{A}^3=\text{etc}=\begin{pmatrix}0&0&0&adf\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \underset{\sim}{A}^4=\underset{\sim}{0}.$$

(b) Proposition: If $\underset{\sim}{A}$ is $n\times n$ upper triangular (with null main diagonal), then $\underset{\sim}{A}$ is nilpotent. Specifically, $\underset{\sim}{A}^n=\underset{\sim}{0}$ [which was seen, in part (a), to hold for $n=2,3,4$]. Proof by induction: It holds for $n=1$ [since the only 1×1 upper triangular matrix with null main diagonal is $\underset{\sim}{A}=(0)=\underset{\sim}{0}$]. Now assume it holds for $n=k$. We wish to show that its truth for $n=k$ implies its truth for $n=k+1$.

$$\text{Partition }\underset{\sim}{A}=\left(\begin{array}{ccccc|c}0&a_{12}&a_{13}&\cdots&a_{1,k-1}&a_{1k}\\0&0&a_{23}&\cdots&a_{2,k-1}&a_{2k}\\\vdots&&&&\vdots&\vdots\\0&0&\cdots&&0&a_{k-1,k}\\\hline 0&0&\cdots&&0&0\end{array}\right)=\begin{pmatrix}\overset{(k-1)\times(k-1)}{\underset{\sim}{A}_1}&\overset{(k-1)\times1}{\underset{\sim}{A}_2}\\\underset{1\times(k-1)}{\underset{\sim}{0}}&\underset{1\times1}{\underset{\sim}{0}}\end{pmatrix}$$

$$\underset{\sim}{A}^2=\begin{pmatrix}\underset{\sim}{A}_1&\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}\begin{pmatrix}\underset{\sim}{A}_1&\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}=\begin{pmatrix}\underset{\sim}{A}_1^2&\underset{\sim}{A}_1\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix},\ \underset{\sim}{A}^3=\begin{pmatrix}\underset{\sim}{A}_1^2&\underset{\sim}{A}_1\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}\begin{pmatrix}\underset{\sim}{A}_1&\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}=\begin{pmatrix}\underset{\sim}{A}_1^3&\underset{\sim}{A}_1^2\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix},$$

$$\cdots\ \underset{\sim}{A}^{k+1}=\begin{pmatrix}\underset{\sim}{A}_1^{k+1}&\underset{\sim}{A}_1^{k}\underset{\sim}{A}_2\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}=\begin{pmatrix}\underset{\sim}{0}&\underset{\sim}{0}\\\underset{\sim}{0}&\underset{\sim}{0}\end{pmatrix}=\underset{\sim}{0}\ \checkmark$$

(c) Yes ...

(d)
$$\begin{aligned}(\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\cdots+\underset{\sim}{A}^{p-1})(\underset{\sim}{I}-\underset{\sim}{A})&=\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\cdots+\underset{\sim}{A}^{p-1}\\&\quad-\underset{\sim}{A}-\underset{\sim}{A}^2-\cdots-\underset{\sim}{A}^{p-1}-\underset{\sim}{A}^p=\underset{\sim}{I}-\underset{\sim}{A}^p=\underset{\sim}{I}-\underset{\sim}{0}=\underset{\sim}{I}.\end{aligned}$$

$$\begin{aligned}(\underset{\sim}{I}-\underset{\sim}{A})(\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\cdots+\underset{\sim}{A}^{p-1})&=\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\cdots+\underset{\sim}{A}^{p-1}\\&\quad-\underset{\sim}{A}-\underset{\sim}{A}^2-\cdots-\underset{\sim}{A}^{p-1}-\underset{\sim}{A}^p=\underset{\sim}{I}-\underset{\sim}{A}^p=\underset{\sim}{I}-\underset{\sim}{0}=\underset{\sim}{I}.\end{aligned}$$

16. (a) If $\underset{\sim}{A}^2=\underset{\sim}{I}$ then
$$\begin{aligned}(\underset{\sim}{I}-\underset{\sim}{A})(\underset{\sim}{I}+\underset{\sim}{A}) &= (\underset{\sim}{I}-\underset{\sim}{A})\underset{\sim}{I}+(\underset{\sim}{I}-\underset{\sim}{A})\underset{\sim}{A}\\ &= \underset{\sim}{I}-\underset{\sim}{A}+(\underset{\sim}{I}+(-1)\underset{\sim}{A})\underset{\sim}{A}\\ &= \underset{\sim}{I}-\underset{\sim}{A}+\underset{\sim}{I}\underset{\sim}{A}+((-1)A)(\underset{\sim}{A})\\ &= \underset{\sim}{I}-\underset{\sim}{A}+\underset{\sim}{A}-\underset{\sim}{A}^2\\ &= \underset{\sim}{I}-\underset{\sim}{A}^2=\underset{\sim}{I}-\underset{\sim}{I}=\underset{\sim}{0}. \checkmark\end{aligned}$$

Conversely, if $(\underset{\sim}{I}-\underset{\sim}{A})(\underset{\sim}{I}+\underset{\sim}{A})=\underset{\sim}{0}$, then
$$\begin{aligned}\underset{\sim}{0} &= (\underset{\sim}{I}-\underset{\sim}{A})\underset{\sim}{I}+(\underset{\sim}{I}-\underset{\sim}{A})\underset{\sim}{A}\\ &= \underset{\sim}{I}-\cancel{\underset{\sim}{A}}+\cancel{\underset{\sim}{A}}-\underset{\sim}{A}^2, \text{ so } \underset{\sim}{A}^2=\underset{\sim}{I}. \checkmark\end{aligned}$$

(b) $\underset{\sim}{A}=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$, because $(\underset{\sim}{I}-\underset{\sim}{A})(\underset{\sim}{I}+\underset{\sim}{A})=\begin{pmatrix}0&-1\\0&2\end{pmatrix}\begin{pmatrix}2&1\\0&0\end{pmatrix}=\underset{\sim}{0}$.

17. (a) $\underset{\sim}{A}+\underset{\sim}{B}=\{a_{ij}+b_{ij}\}\overset{\text{by the commutativity of scalar addition}}{=}\{b_{ij}+a_{ij}\}=\underset{\sim}{B}+\underset{\sim}{A}$ ✓
$$\begin{aligned}(\underset{\sim}{A}+\underset{\sim}{B})+\underset{\sim}{C} &= \{a_{ij}+b_{ij}\}+\{c_{ij}\}=\{(a_{ij}+b_{ij})+c_{ij}\}\\ &= \{a_{ij}+(b_{ij}+c_{ij})\} \text{ by the associativity of } \underline{\text{scalar}} \text{ addition}\\ &= \underset{\sim}{A}+(\underset{\sim}{B}+\underset{\sim}{C}) \checkmark\end{aligned}$$
$\underset{\sim}{A}+\underset{\sim}{0}=\{a_{ij}+0\}=\{a_{ij}\}=A$ ✓

(b) $\underset{\sim}{A}+(-\underset{\sim}{A})=\underset{\sim}{A}+(-1)\underset{\sim}{A}=\{a_{ij}\}+\{-a_{ij}\}=\{a_{ij}-a_{ij}\}=\{0\}=\underset{\sim}{0}$. ✓

$\alpha(\beta\underset{\sim}{A})=\alpha(\beta\{a_{ij}\})=\alpha\{\beta a_{ij}\}=\{\alpha\beta a_{ij}\}=(\alpha\beta)\{a_{ij}\}=(\alpha\beta)\underset{\sim}{A}$ ✓
$$\begin{aligned}(\alpha+\beta)\underset{\sim}{A} &= (\alpha+\beta)\{a_{ij}\}=\{(\alpha+\beta)a_{ij}\}\\ &= \{\alpha a_{ij}+\beta a_{ij}\} \text{ by the distributivity of } \underline{\text{scalar}} \text{ multiplic.}\\ &= \{\alpha a_{ij}\}+\{\beta a_{ij}\} \text{ by the definition (4)}\\ &= \alpha\{a_{ij}\}+\beta\{a_{ij}\} \text{ by the definition (7)}\\ &= \alpha\underset{\sim}{A}+\beta\underset{\sim}{A} \checkmark\end{aligned}$$

18. To prove that a proposition is not true in <u>general</u>, it suffices to put forward a single counterexample.

(a) For example, $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\3&0\end{pmatrix}$, whereas $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\0&0\end{pmatrix}$

(b) For example, $\begin{pmatrix}1&3\\1&3\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$, yet $\begin{pmatrix}1&3\\1&3\end{pmatrix}\neq\begin{pmatrix}1&1\\1&0\end{pmatrix}$

(c) For example, $\begin{pmatrix}0&2\\0&3\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\underset{\sim}{0}$, even though $\begin{pmatrix}0&2\\0&3\end{pmatrix}\neq\underset{\sim}{0}$ and $\begin{pmatrix}1&0\\0&0\end{pmatrix}\neq\underset{\sim}{0}$

(d) For example, $\begin{pmatrix}1/2&0\\0&1/3\end{pmatrix}\begin{pmatrix}2&0\\0&3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\underset{\sim}{I}$.

20. $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$
$$\left.\begin{array}{ll}\text{1,1 element:} & a+3b=a+2c\\ \text{1,2 element:} & 2a+4b=b+2d\\ \text{2,1 element:} & c+3d=3a+4c\\ \text{2,2 element:} & 2c+4d=3b+4d\end{array}\right\}\begin{array}{l}\text{Solving these by}\\ \text{Gauss elim. gives}\\ d=\alpha,\ c=\beta,\\ b=\frac{2\beta}{3},\ a=\alpha-\beta,\\ \text{where } \alpha,\beta \text{ are arbitrary}\end{array}$$

21. (a) $\underset{\sim}{A}\underset{\sim}{B}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+2c&b+2d\\3a+4c&3b+4d\end{pmatrix}$ and
$$\left.\begin{array}{r}a+2c=0\\ 3a+4c=0\\ b+2d=0\\ 3b+4d=0\end{array}\right\}\text{Gives } a=b=c=d=0, \text{ so } \underset{\sim}{B}=\underset{\sim}{0}$$

(b) $\underset{\sim}{A}\underset{\sim}{B} = \begin{pmatrix}2&3\\0&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}2a+3c & 2b+3d\\0&0\end{pmatrix}$ so $\begin{matrix}2a+3c=0\\2b+3d=0\end{matrix}$ so $c = \text{arb.} = -\frac{2}{3}\alpha$, say. Then $a=\alpha$. and $d = \text{arb.} = -\frac{2}{3}\beta$, say, so $b=\beta$

Thus, $\underset{\sim}{B} = \begin{pmatrix}\alpha & \beta\\ -\frac{2}{3}\alpha & -\frac{2}{3}\beta\end{pmatrix}$

(c) $\underset{\sim}{B} = \begin{pmatrix}0&0\\ \alpha&\beta\end{pmatrix}$

(d) $\underset{\sim}{B} = \begin{pmatrix}\alpha & \beta\\ -\frac{2}{3}\alpha & -\frac{2}{3}\beta\end{pmatrix}$

22. See bottom half of pg 471 and top half of page 472. $c_{ij} = a_{ij}b_{ij}$ would have the major advantage of commutativity ($\underset{\sim}{A}\underset{\sim}{B}$ defined if $\underset{\sim}{A} = m\times n$ and $\underset{\sim}{B} = m\times n$; then we would have $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{B}\underset{\sim}{A}$ for all such $\underset{\sim}{A}$'s and $\underset{\sim}{B}$'s), but the general system of linear algebraic equations

$$\begin{matrix}a_{11}x_1 + \cdots + a_{1n}x_n = c_1\\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = c_m\end{matrix}$$

could not be expressed as $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{c}$.

23. What is $p_{ij}^{(2)}$? We can get from i to j in 2 steps via states 1,2,3,4,5 as shown in the figure at the right. Thus,

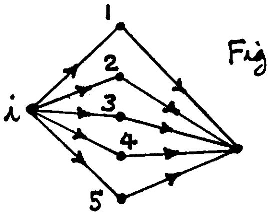


Fig. A

$$\begin{aligned} p_{ij}^{(2)} = {} & \text{prob. of moving from } i \text{ to } 1 \text{ to } j\\ & + \text{ " " " " " } 2 \text{ " "}\\ & + \text{ " " " " " } 3 \text{ " "}\\ & + \text{ " " " " " } 4 \text{ " "}\\ & + \text{ " " " " " } 5 \text{ " "} = p_{i1}^{(1)}p_{1j}^{(1)} + p_{i2}^{(1)}p_{2j}^{(1)} + p_{i3}^{(1)}p_{3j}^{(1)} + p_{i4}^{(1)}p_{4j}^{(1)} + p_{i5}^{(1)}p_{5j}^{(1)}\end{aligned}$$

where we've used two basic ideas from probability theory:

(1) If we can get from one state to another by any of n mutually exclusive routes and the probabilities associated with these routes are $p_1, \cdots, p_n$ then $p_{ab} = p_1 + p_2 + \cdots + p_n$. In circuit terminology the routes are in "parallel."

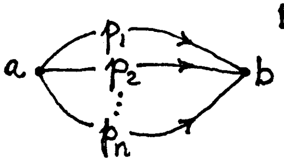


Fig. B

(2) If, instead, we can pass from a to b only by successfully accomplishing n successive steps, with associated probabilities $p_1, \cdots, p_n$, then $p_{ab} = p_1 p_2 \cdots p_n$. Here the routes are in series.

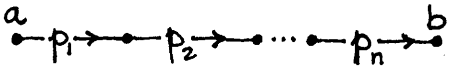


Fig. C

For the case shown in Fig. A we have a combination of parallel and series: 5 parallel routes, each of which is a series of 2 routes.

Thus,

$$p_{ij}^{(2)} = \sum_{k=1}^{5} p_{ik}^{(1)} p_{kj}^{(1)} \quad \text{or, in matrix notation,} \quad \underset{\sim}{P}^{(2)} = [\underset{\sim}{P}^{(1)}]^2.$$

Likewise, $\underset{\sim}{P}^{(3)} = [\underset{\sim}{P}^{(1)}]^3$, etc. Thus, given $\underset{\sim}{P}^{(1)}$ we can now compute

$$\underset{\sim}{P}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{P}^{(3)} = \underset{\sim}{P}^{(2)}\underset{\sim}{P}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/8 & 0 & 1/4 & 0 & 1/8 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/8 & 0 & 1/4 & 0 & 5/8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the probability that A is bankrupt after 3 matches, starting with #2, is $p_{31}^{(3)} = 1/4$,
" " " " " " " " " " " \$3 " $p_{41}^{(3)} = 1/8$,
and " " " " " " " " " " " \$1 " $p_{21}^{(3)} = 5/8$.

24. (b) The Maple commands

```
with(linalg):
A:= array([[2,-1],[3,0],[1,4]]);
B:= array([[5,3,25],[2,0.1,-6]]);
C:= array([[9,1,-1],[2,0,7],[0,4,6]]);
evalm((A&*B)^3+5*C&*C);
```

gives the result

$$\begin{bmatrix} 24124.0 & 11054.35 & 72017.0 \\ 36967.5 & 16633.5 & 102507.5 \\ 15836.5 & 5453.10 & 19014.5 \end{bmatrix}$$

Section 10.3

1. (b) $\underset{\sim}{x} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$, $\underset{\sim}{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\underset{\sim}{x}^T\underset{\sim}{y} = (4\ -1\ 0)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4-2+0 = 2$, $\underset{\sim}{x}\underset{\sim}{y}^T = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}(1\ 2\ 3) = \begin{pmatrix} 4 & 8 & 12 \\ -1 & -2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

2. Repeated use of (3d) gives $(\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{D})^T = (\underset{\sim}{A}(\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{D}))^T = (BCD)^T A^T = (B(CD))^T A^T$
$= (CD)^T B^T A^T = D^T C^T B^T A^T$

3. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} \text{etc} & ae+bf \\ \text{etc} & \text{etc} \end{pmatrix}$
$\begin{pmatrix} d & e \\ e & f \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \text{etc} & db+ce \\ \text{etc} & \text{etc} \end{pmatrix}$ different, so the symmetry of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is not sufficient.

6. $\underset{\sim}{A}_1^T = \frac{1}{2}(\underset{\sim}{A}+\underset{\sim}{A}^T)^T = \frac{1}{2}(\underset{\sim}{A}^T+\underset{\sim}{A}^{TT}) = \frac{1}{2}(\underset{\sim}{A}^T+\underset{\sim}{A}) = \underset{\sim}{A}_1$, so $\underset{\sim}{A}_1$ is symmetric
$\underset{\sim}{A}_2^T = \frac{1}{2}(\underset{\sim}{A}-\underset{\sim}{A}^T)^T = \frac{1}{2}(\underset{\sim}{A}^T-\underset{\sim}{A}^{TT}) = \frac{1}{2}(\underset{\sim}{A}^T-\underset{\sim}{A}) = -\underset{\sim}{A}_2$, so $\underset{\sim}{A}_2$ is skew-symmetric.

7. (e) Using (6.1), $\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}\right] + \frac{1}{2}\left[\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}\right]$

$$= \begin{pmatrix} 9 & 7 & 5 \\ 7 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

8. (b) For $n=2$, $\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} = (x_1, x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1, x_2)\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{12}x_1 + a_{22}x_2 \end{pmatrix}$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

$$= x_1^2 - 3x_2^2 + 6x_1x_2$$

requires that $a_{11}=1$, $a_{22}=-3$, $a_{12}=3$ ($\underline{\underline{\text{not}}}$ 6), so $\underset{\sim}{A} = \begin{pmatrix} 1 & 3 \\ 3 & -3 \end{pmatrix}$.

(c) Proceeding as in (b) we find, for $n=3$, that

$$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

$$= 4x_1^2 + x_2^2 - x_3^2 + 8x_1x_2 + 3x_1x_3 - 2x_2x_3$$

gives $a_{11}=4$, $a_{22}=1$, $a_{33}=-1$, $a_{12}=4$, $a_{13}=3/2$, $a_{23}=-1$ so $\underset{\sim}{A} = \begin{pmatrix} 4 & 4 & 3/2 \\ 4 & 1 & -1 \\ 3/2 & -1 & -1 \end{pmatrix}$

9. $(\underset{\sim}{A}\underset{\sim}{A}^T)^T = (\underset{\sim}{A}^T)^T\underset{\sim}{A}^T = \underset{\sim}{A}\underset{\sim}{A}^T$, so $\underset{\sim}{A}\underset{\sim}{A}^T$ is symmetric.

10. $(\underset{\sim}{A}\underset{\sim}{B})^T = \underset{\sim}{B}^T\underset{\sim}{A}^T = \underset{\sim}{B}\underset{\sim}{A}$ (since $\underset{\sim}{A}, \underset{\sim}{B}$ assumed symmetric) not necessarily $= \underset{\sim}{A}\underset{\sim}{B}$. For ex., $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$,

whereas $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$.

11. (e) The maple commands

```
with(linalg):
A: = array([[4,1,2],[0,5,7]]);
B: = array([[1,-4,2],[8,1,4]]);
evalm((transpose(B) &* A)^8);
```

gives

```
[-38231919608   78538578773   104219222341]
[ 39512373176       etc            etc    ]
[-35221910032       etc            etc    ]
```

Section 10.4

2. (b) About 1st row, $\begin{vmatrix} 2 & -3 & 0 \\ 1 & 4 & 2 \\ -6 & 1 & 5 \end{vmatrix} = 2(18) - (-3)(17) + 0(25) = 87$

About 3rd row it $= (-6)(-6) - (1)(4) + 5(11) = 87$ ✓

About 3rd column it $= 0(25) - 2(-16) + 5(11) = 87$ ✓

(i) $\det = af(be - cd)$

4. (b) $\begin{vmatrix} 2 & -3 & 0 \\ 1 & 4 & 2 \\ -6 & 1 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 2 \\ 2 & -3 & 0 \\ -6 & 1 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 2 \\ 0 & -11 & -4 \\ 0 & 25 & 17 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 2 \\ 0 & -11 & -4 \\ 0 & 0 & \frac{87}{11} \end{vmatrix} = -(1)(-11)(\tfrac{87}{11}) = 87$

(g) $\begin{vmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & -1 \\ 0 & 4 & 5 & 0 \\ 1 & 2 & 3 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 6 \\ 0 & 3 & 1 & -1 \\ 0 & 4 & 5 & 0 \\ 2 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 6 \\ 0 & 3 & 1 & -1 \\ 0 & 4 & 5 & 0 \\ 0 & -4 & -5 & -12 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 6 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & \frac{11}{3} & \frac{4}{3} \\ 0 & 0 & -\frac{11}{3} & -\frac{40}{3} \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 & 6 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & \frac{11}{3} & \frac{4}{3} \\ 0 & 0 & 0 & -12 \end{vmatrix} = -(1)(3)(\tfrac{11}{3})(-12)$

$= 132$.

5. (g) The Maple commands

```
with(linalg):
A:=array([[2,0,1,0],[0,3,1,-1],[0,4,5,0],[1,2,3,6]]);
det(A);
```

gives 132, as obtained in 4(g).

6. (a) $\begin{vmatrix} 1&2&3&4\\ 2&3&4&5\\ 3&4&5&6\\ 0&1&-3&5 \end{vmatrix} \overset{\underset{\sim}{r}_2 \to \underset{\sim}{r}_2+(-1)\underset{\sim}{r}_1}{=} \begin{vmatrix} 1&2&3&4\\ 1&1&1&1\\ 3&4&5&6\\ 0&1&-3&5 \end{vmatrix} \overset{\underset{\sim}{r}_3 \to \underset{\sim}{r}_3+(-1)\underset{\sim}{r}_1}{=} \begin{vmatrix} 1&2&3&4\\ 1&1&1&1\\ 2&2&2&2\\ 0&1&-3&5 \end{vmatrix} = 0$ by property D5

(c) By cofactor expansion, det = − abd. The minus sign seems to disagree with property D3, but note that this matrix is NOT triangular since it is not zero above or below the MAIN diagonal (upper left to lower right).

(d) By cofactor expansion, det = − cef.

NOTE: If $\underset{\sim}{A}$ is triangular (upper or lower) then, by property D3, $\det|\underset{\sim}{A}| = a_{11}a_{22}\cdots a_{nn}$. But if $\underset{\sim}{A}$ is triangular about the "wrong" diagonal (lower left to upper right), then

$$\det \underset{\sim}{A} = \underline{\underline{(-1)}}^n a_{n1}\, a_{n-1,2}\, a_{n-2,3} \cdots a_{1n}$$

9. (a) Let an nxn matrix $\underset{\sim}{A} = \{a_{ij}\}$ be of the form shown. We can use properties D1 and D2 to triangularize the block $\underset{\sim}{A}_1$, then $\underset{\sim}{A}_2, \dots$, then $\underset{\sim}{A}_m$. When done, the modified $\underset{\sim}{A}$ matrix will be triangular, so we will have, by property D3,

$$\det \underset{\sim}{A} = a'_{11}a'_{22}a'_{33}a'_{44}a'_{55}\cdots a'_{nn}$$

But, by properties D1-D3 the latter product can be grouped as

$$(\det \underset{\sim}{A}_1)(\det \underset{\sim}{A}_2)\cdots(\det \underset{\sim}{A}_m).$$

(b) and (c) Yes, the same reasoning, as in (a), holds in these cases.

(d) In (2c), $\left|\begin{array}{cc|c} -4&1&0\\ 3&2&0\\ \hline 1&5&7 \end{array}\right| = \begin{vmatrix} -4&1\\ 3&2 \end{vmatrix}(7) = -77$

In (2i), $\left|\begin{array}{c|cc|c} a&0&0&0\\ \hline 0&b&c&0\\ 0&d&e&0\\ \hline 0&0&0&f \end{array}\right| = (a)\begin{vmatrix} b&c\\ d&e \end{vmatrix}(f) = a(be-cd)f$

In (2j), $\left|\begin{array}{ccc|c} a&b&c&0\\ d&e&f&0\\ g&h&i&0\\ \hline 0&0&0&k \end{array}\right| = \begin{vmatrix} a&b&c\\ d&e&f\\ g&h&i \end{vmatrix}(k) = $ etc.

10. $\det(\underset{\sim}{A}_1\underset{\sim}{A}_2\cdots\underset{\sim}{A}_k) = \det(\underset{\sim}{A}_1(\underset{\sim}{A}_2\cdots\underset{\sim}{A}_k)) = (\det \underset{\sim}{A}_1)(\det(\underset{\sim}{A}_2\cdots\underset{\sim}{A}_k))$

$= (\det \underset{\sim}{A}_1)(\det((\underset{\sim}{A}_2)(\underset{\sim}{A}_3\cdots\underset{\sim}{A}_k)))$

$= (\det \underset{\sim}{A}_1)(\det \underset{\sim}{A}_2)(\det(\underset{\sim}{A}_3\cdots\underset{\sim}{A}_k))$

$=$ etc. $= (\det \underset{\sim}{A}_1)(\det \underset{\sim}{A}_2)\cdots(\det \underset{\sim}{A}_k)$

11. (b) Each row contributes a scale factor α, and there are n rows, so $\det(\alpha\underset{\sim}{A}) = \alpha^n \det \underset{\sim}{A}$.

12. If any row is a linear combination of the others then repeated use of property D1 can reduce that row to a row of zeros. Then, of course, a cofactor expansion about that row gives 0. To illustrate, suppose $\underset{\sim}{A}$ is 3x3 and that $\underset{\sim}{r}_3 = \underset{\sim}{r}_1 + 2\underset{\sim}{r}_2$. Then

$$\det\begin{pmatrix} r_1\\ r_2\\ r_3 \end{pmatrix} \to \begin{vmatrix} r_1\\ r_2\\ r_3+(-1)r_1 \end{vmatrix} \to \begin{vmatrix} r_1\\ r_2\\ r_3+(-1)r_1+(-2)r_2 \end{vmatrix} = \begin{pmatrix} r_1\\ r_2\\ \underset{\sim}{0} \end{pmatrix} = 0$$

14. (b) $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$, so in this case $n=4$ and $a_1=2, a_2=7, a_3=4, a_4=8$. Thus,

$$\Delta_1 = a_1 = 2,\quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} = 10,\quad \Delta_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 4 & 7 & 2 \\ 0 & 0 & 4 \end{vmatrix} = 40,$$

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 4 & 7 & 2 & 1 \\ 0 & 8 & 4 & 7 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 8.$$

Each Δ_j is >0 and each a_j is >0; hence, stable (In fact, Maple gives 4 complex roots, each with negative real part.)

(d) $a_2 = 0$, so unstable (Maple gives 4 complex rts, 2 with negative real part, 2 with pos. real part

(e) $n=5$, $a_1 = a_2 = \cdots = a_5 = 1$. Thus, $\Delta_1 = 1$, $\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$. Can stop here since we need all the Δ_j's >0 for stability. $\Delta_2 = 0 \Rightarrow$ unstable. (Maple gives

[The Maple command

```
fsolve(x^5+x^4+x^3+x^2+x+1, x);
```

gives the single root -1. To obtain complex roots as well, use the complex option:

```
fsolve(x^5+x^4+x^3+x^2+x+1, x, complex);
```

and obtain the roots $-1, -.5 \pm .866i, +.5 \pm .866i$

↑ hence, unstable ✓]

15. (a)

$$\begin{vmatrix} 3 & 2 & -5 & 0 \\ 0 & 3 & 2 & -5 \\ 3 & 3 & -2 & 0 \\ 0 & 3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 2 & -5 & 0 \\ 0 & 3 & 2 & -5 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 2 & -5 & 0 \\ 0 & 3 & 2 & -5 \\ 0 & 0 & 7/3 & 5/3 \\ 0 & 0 & 1 & 3 \end{vmatrix} \neq 0.$$

Hence, the equations have no common roots.

16. (a) (16.1) follows readily from a cofactor expansion of $\det \underset{\sim}{A}$, about either the ith row or the jth column. Then, by chain differentiation

$$\frac{d}{dt}\det \underset{\sim}{A}\,[a_{11}(t), a_{12}(t), \ldots, a_{nn}(t)] = \left(\frac{\partial}{\partial a_{11}}\det \underset{\sim}{A}\right)\left(\frac{da_{11}}{dt}\right) + \cdots + \left(\frac{\partial}{\partial a_{nn}}\det \underset{\sim}{A}\right)\left(\frac{da_{nn}}{dt}\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\frac{da_{ij}}{dt}$$

(b)

$$\frac{d}{dt}(\det \underset{\sim}{A}) = \begin{vmatrix} 2t & 1 & 0 \\ 0 & 3t & 1 \\ 4 & 0 & \sin t \end{vmatrix} + \begin{vmatrix} t^2 & t & 2 \\ 0 & 3 & 0 \\ 4 & 0 & \sin t \end{vmatrix} + \begin{vmatrix} t^2 & t & 2 \\ 0 & 3t & 1 \\ 0 & 0 & \cos t \end{vmatrix} = 9t^2\sin t + 3t^3\cos t - 20$$

17. For $n=2$: $\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1 - x_2$

$$\prod_{1 \le i < j \le 2} (x_i - x_j) = x_1 - x_2 \ \checkmark$$

For $n=3$:

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_1(x_2^2 - x_3^2) + x_2x_3(x_2 - x_3)$$
$$= (x_2 - x_3)(x_1^2 - x_1x_2 - x_1x_3 + x_2x_3)$$
$$\prod_{1 \le i < j \le 3} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

same ✓

18. (a) $N(n) = n$ multiplications $+ (n-1)$ additions and subtractions $+ nN(n-1)$ calculations for the n cofactor determinants
$= nN(n-1) + 2n - 1$.

Then $P_n - nP_{n-1} = 2n - 1 \quad (n \geq 2)$

which is a first-order (because the difference between the subscripted indices is 1) difference equation (with initial condition $P_2 = 3$), analogous to the first-order differential equation $y' + M(x)y = N(x)$. To solve, first seek a

"homogeneous solution", i.e., to $P_n - nP_{n-1} = 0$, or, $P_n = nP_{n-1}$:
$$P_2 = 2P_1$$
$$P_3 = 3P_2 = 3!\,P_1$$
$$P_4 = 4P_3 = 4!\,P_1$$
so $P_n = n!\,P_1$,

i.e., some constant times $n!$. Thus, by "variation of parameters" seek the solution to the nonhomogeneous difference equation in the form
$$P_n = n!\,Q_n.$$
Plugging this into $P_n - nP_{n-1} = 2n-1$ gives
$$n!\,Q_n - \underbrace{n(n-1)!}_{n!}\,Q_{n-1} = 2n-1$$
or,
$$Q_n - Q_{n-1} = \frac{2n-1}{n!}, \qquad ☆$$
which is simpler than 🯅 in that ★ has constant coefficients whereas 🯅 has nonconstant coefficients. First, observe that $P_2 = 3$ (2 multipications and 1 subtraction) so $P_2 = 3 = 2!\,Q_2$ so $Q_2 = 3/2$. Then ★ gives
$$Q_2 = 3/2$$
$$Q_3 = 3/2 + \frac{2(3)-1}{3!}$$
$$Q_4 = \left(\frac{3}{2} + \frac{2(3)-1}{3!}\right) + \frac{2(4)-1}{4!}$$
$$\vdots$$
$$Q_n = \frac{3}{2} + 2\sum_{j=2}^{n-1}\frac{1}{j!} - \sum_{j=3}^{n}\frac{1}{j!}$$
so we have the exact solution for $P_n = n!\,Q_n =$ etc. Here, however, we're more interested in the asymptotic behavior as $n \to \infty$. We have
$$Q_n \sim \frac{3}{2} + 2\sum_{2}^{\infty}\frac{1}{j!} - \sum_{3}^{\infty}\frac{1}{j!} = \frac{3}{2} + 2\left[\sum_{0}^{\infty}\frac{1}{j!} - 2\right] - \left[\sum_{0}^{\infty}\frac{1}{j!} - \frac{5}{2}\right]$$
$$= \frac{3}{2} + 2e - 4 - e + \frac{5}{2} = e.$$
Thus
$$P_n = N(n) = n!\,Q_n \sim e\,n! \quad \text{as } n \to \infty. \ ✓$$

(b) For the first "indentation" we need $2n(n-1)$ calculations, for the second we need $2(n-1)(n-2)$, and so on. Thus, to triangularize we need
$$N(n) \approx 2n(n-1) + 2(n-1)(n-2) + \cdots + 0$$
$$\overset{*}{=} \frac{2n}{3}(n^2-1) \sim \frac{2n^3}{3} \text{ as } n \to \infty. \ ✓$$

---

* Recall the identity $\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + x^n$, which holds for all $x \neq 1$, and also for $x = 1$ if we use l'Hôpital's rule for the 0/0. Taking d/dx twice and then setting $x = 1$ on both sides (with the help of l'Hôpital's rule for the left-hand side) gives the result $n(n^2-1)/3 = 0 + 2 \cdot 1 + \cdots + (n-1)(n-2) + n(n-1)$ that was given in the HINT.

NOTE: This application of difference equations, studied in Chap. 6, might be suitable for discussion in class. The result is striking, that the number of calculations should grow asymptotically proportional to the transcendental number e.

Section 10.5

1. (b) $r=1$, nullity $=2$, #LI rows $=1$, #LI columns $=1$
   (c) $r=2$, nullity $=0$, " $=2$, " $=2$
   (d) $r=1$, " $=2$, " $=1$, " $=1$
   (f) $r=2$, " $=1$, " $=2$, " $=2$
   (g) $r=2$, " $=1$, " $=2$, " $=2$
   (h) $r=1$, " $=2$, " $=1$, " $=1$
   (j) $r=3$, " $=1$, " $=3$, " $=3$
   (k) $r=3$, " $=0$, " $=3$, " $=3$
   (l) $r=2$, " $=2$, " $=2$, " $=2$
   (m) $r=4$, " $=0$, " $=4$, " $=4$
   (n) $r=3$, " $=1$, " $=3$, " $=3$

2. (d) The maple command

```
with(linalg):
A: =array([[4,8,0],[3,6,0]]);
rank(A);
```

gives 1.

3. (b) $(1\ 2\ 3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=(1)$. $r(\underset{\sim}{A})=1$, $r(\underset{\sim}{A}|\underset{\sim}{c})=1$, $n-r=3-1=2$-parameter family of solutions

   (c) $\begin{pmatrix}5&7\\4&9\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$. $r(\underset{\sim}{A})=2$, $r(\underset{\sim}{A}|\underset{\sim}{c})=2$, $n-r=2-2=0$ so unique solution

   (d) $\begin{pmatrix}4&8&0\\3&6&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$. $r(\underset{\sim}{A})=1$, $r(\underset{\sim}{A}|\underset{\sim}{c})=2$. $r(\underset{\sim}{A})\neq r(\underset{\sim}{A}|\underset{\sim}{c})$ so no soln.; inconsistent

   (f) $\begin{pmatrix}3&2&1\\-1&1&4\\1&4&9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$. $r(\underset{\sim}{A})=2$, $r(\underset{\sim}{A}|\underset{\sim}{c})=3$. $r(\underset{\sim}{A})\neq r(\underset{\sim}{A}|\underset{\sim}{c})$ so no solution; inconsistent

   (g) $\begin{pmatrix}5&0&0\\3&0&4\\2&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $r(\underset{\sim}{A})=2$, $r(\underset{\sim}{A}|\underset{\sim}{c})=3$. " " " " "

   (j) $\begin{pmatrix}1&0&3&0\\0&1&0&3\\1&0&3&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$. $r(\underset{\sim}{A})=3$, $r(\underset{\sim}{A}|\underset{\sim}{c})=3$. $n-r=4-3=1$-parameter family of solutions

4. No. For example, (1 0) and (1 1) have same form and rank, but we cannot obtain one from the other by row operations.

5. $$\begin{pmatrix}1&3&-3&0\\2&1&0&4\end{pmatrix}\xrightarrow[r_1\rightleftarrows r_2]{}\begin{pmatrix}2&1&0&4\\1&3&-3&0\end{pmatrix}\xrightarrow[r_2\to r_2+r_1]{}\begin{pmatrix}2&1&0&4\\3&4&-3&4\end{pmatrix}\xrightarrow[r_1\to 2r_1]{}\begin{pmatrix}4&2&0&8\\3&4&-3&4\end{pmatrix}$$

6. (a) They can't be row equivalent because their ranks are different; 2 an 1, respectively. That is, row equivalent matrices are related through elementary row operations, and the latter leave rank unchanged.

7. (a) Although I didn't give the Maple commands, back in Chapter 8, we can use Maple to do Gauss elimination (See ?gauss for instructions) and also Gauss-Jordan reduction (See ?Gaussjord). Let's call these matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, respectively. Using the Maple commands (or of course we could do Gauss-Jordan by hand)

```
with(linalg):
A:= array([[1,2,3,-1],[2,4,-1,1],[0,5,6,3],[4,-2,0,6]]);
B:= array([[1,-1,2,1],[3,0,5,1],[2,2,1,3],[3,1,3,4]]);
gaussjord(A);
gaussjord(B);
```

we obtain the reduced forms

$$A \to \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \quad \text{and} \quad B \to \begin{pmatrix} 1&0&0&4 \\ 0&1&0&-7/5 \\ 0&0&1&-11/5 \\ 0&0&0&0 \end{pmatrix}$$

Clearly, these matrices are not row equivalent (they are not even of the same rank!) so $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are not row equivalent either.

(b) Again using Gauss-Jordan elimination we obtain

$$A \to \begin{pmatrix} 1&0&0&10/11 \\ 0&1&0&17/11 \\ 0&0&1&5/11 \end{pmatrix} \quad \text{and} \quad B \to \begin{pmatrix} 1&0&0&10/11 \\ 0&0&1&5/11 \\ 0&0&0&0 \end{pmatrix}$$

Again, it is evident that these matrices are not row equivalent, so $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are not row equivalent either. Suppose Gauss-Jordan reduction of $\underset{\sim}{B}$ had given

$$B \to \begin{pmatrix} 1&0&0&3 \\ 0&1&0&7 \\ 0&0&1&2 \end{pmatrix}$$

instead. Our conclusion would still be the same. In short, the idea is that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are row equivalent iff their Gauss-Jordan reduced forms are identical. (But I have not proved that claim here.)

8. (b) Obviously this set is LD because there are at most 3 LI vectors in $\mathbb{R}^3$. Nevertheless, Theorem 10.5.2 gives

$$\#\text{LI vectors} = \text{rank of} \begin{pmatrix} 4&1&2 \\ 2&2&1 \\ 2&-1&1 \\ 4&7&2 \\ 0&1&0 \end{pmatrix}, \text{ which} = 2$$

(c) 2 (d) 3 (e) 2

9. Theorem 10.5.2 says #LI row vectors = #LI column vectors = rank. Since the transpose merely exchanges row and column vectors, $r(\underset{\sim}{A}^T) = r(\underset{\sim}{A})$.

10. Clearly false. For ex., if $\underset{\sim}{A}, \underset{\sim}{B}$ are $n \times n$ then $r(\underset{\sim}{A}\underset{\sim}{B}) \leq n$, yet $r(\underset{\sim}{A})r(\underset{\sim}{B})$ can be as large as n^2.

11. Remember, it takes only a counterexample to show a proposition false. For ex., if $\underset{\sim}{A} = \begin{pmatrix} 1&-1 \\ 1&1 \end{pmatrix}$ and $\underset{\sim}{B} = \begin{pmatrix} -1&1 \\ -1&-1 \end{pmatrix}$, then $r(\underset{\sim}{A}+\underset{\sim}{B}) = 0$ whereas $r(\underset{\sim}{A}) + r(\underset{\sim}{B}) = 2+2 = 4$.

12. The rank of $\underset{\sim}{A}\underset{\sim}{B}$ equals the number of LI row vectors in $\begin{pmatrix} a_{11}\underset{\sim}{r}_1+\cdots+a_{1n}\underset{\sim}{r}_n \\ \vdots \\ a_{m1}\underset{\sim}{r}_1+\cdots+a_{mn}\underset{\sim}{r}_n \end{pmatrix}$. Since each row vector is a linear combination of $\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_n$, it follows that $r \le n$.

13.(a) As hinted, the essential idea of a proof can be found in Example 6. If the system $\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{c}$ is indeed consistent, then the Gauss-Jordan reduced row-echelon form of the augmented matrix $\underset{\sim}{A}|\underset{\sim}{c}$ will be of the form

$$\begin{array}{c|cccccc|l}
 & 1 & 2 & 3 & \cdots & n & n+1 & \\
1 & 1 & \longrightarrow & & & & & \\
2 & 0 & 1 & \longrightarrow & & & & \\
3 & 0 & 0 & 1 & \longrightarrow & & & \leftarrow ♀ \\
\vdots & \vdots & & & \ddots & & & \\
k & 0\;0\;0 & & & 1 & \longrightarrow & & \\
\vdots & 0\;0\;0 & & & 0 & \cdots & 0 & \\
m & 0\;0\;0 & & & 0 & \cdots & 0 &
\end{array}$$

where $k \le m$.* Then the solution will be of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \text{linear combin. of } \alpha_1,\ldots,\alpha_{n-k} \\ '' \quad '' \quad '' \quad '' \quad '' \\ '' \quad '' \quad '' \quad '' \quad '' \\ \alpha_{n-k} \\ \vdots \\ \alpha_1 \end{bmatrix}$

as for example in (13) in Example 6 where $n=6$ and $k=3$. Then

$$[\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_{n-k}] = \begin{array}{c|cccc|c}
 & x & x & \cdots & x & n \\
 & \vdots & \vdots & & \vdots & \vdots \\
 & x & x & & x & \vdots \\
n-k & 0 & 0 & \cdots & 1 & \vdots \\
\vdots & \vdots & \vdots & \iddots & 0 & \vdots \\
2 & 0 & 1 & & \vdots & 2 \\
1 & 1 & 0 & \cdots & 0 & 1 \\
 & 1 & 2 & \cdots & n-k &
\end{array}$$

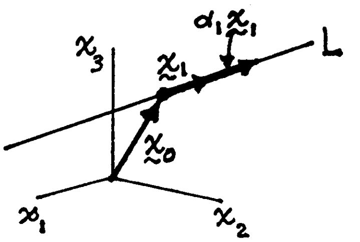
(where the x's denote numbers that, in general, are not zero; see, for ex., (16), where $n=6$ and $k=3$) is of rank $n-k$ because the bottom $(n-k)\times(n-k)$ matrix in the right-hand side of the latter equation. Thus, the $n-k$ $\underset{\sim}{x}_j$'s (which are the homogeneous solutions, i.e., of $\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{0}$) are LI.

---

* What if $\underset{\sim}{A}|\underset{\sim}{c} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, say, since the "indentation" pattern is not the same as shown in ♀, above? We can obtain the ♀ form merely by renaming the unknowns. For instance, let $x_3 \to x_2$, $x_5 \to x_3$, $x_2 \to x_5$. Then the revised matrix is $\begin{bmatrix} 1 & 3 & 5 & 4 & 2 & 6 \\ 0 & 1 & 8 & 7 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is of the form ♀.

(b) $\underset{\sim}{x}_0$ can't be in span$\{\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_{n-k}\}$ because if $\underset{\sim}{x}_0 = c_1\underset{\sim}{x}_1+\cdots+c_{n-k}\underset{\sim}{x}_{n-k}$ then $\underset{\sim}{A}\underset{\sim}{x}_0 = c_1\underset{\sim}{A}\underset{\sim}{x}_1+\cdots+c_{n-k}\underset{\sim}{A}\underset{\sim}{x}_{n-k} = c_1\underset{\sim}{0}+\cdots+c_{n-k}\underset{\sim}{0} = 0$, whereas we were to have $\underset{\sim}{A}\underset{\sim}{x}_0=\underset{\sim}{c}$.

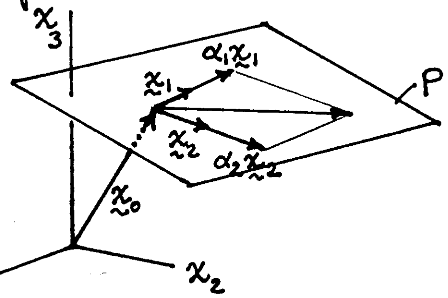
15.(a) Since $\underset{\sim}{A} = 3\times 3$ and $\underset{\sim}{x} = 3\times 1$, we can draw pictures in 3-space. If $\underset{\sim}{x} = \underset{\sim}{x}_0 + \alpha_1 \underset{\sim}{x}_1$ then



$$x_1 = x_{01} + \alpha_1 x_{11}$$
$$x_2 = x_{02} + \alpha_1 x_{12}$$
$$x_3 = x_{03} + \alpha_1 x_{13}.$$

The latter are parametric equations of a straight line in x_1, x_2, x_3 space, α_1 being the parameter and $x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{13}$ being constants. $\underset{\sim}{x}_0$ gets us from the origin to a point on that line (say the line "L"), and $\underset{\sim}{x}_1$ is a vector along L so that $\alpha_1 \underset{\sim}{x}_1$ can take us to any point on L, as sketched above.

(b) Same idea as in (a); $\underset{\sim}{x}_0$ gets us to some point in the plane and then $\alpha_1 \underset{\sim}{x}_1 + \alpha_2 \underset{\sim}{x}_2$ takes us to any desired point in that plane, denoted as P in the sketch.



16.(a) $H_2 + O_2 - 2OH = 0$
$H_2 + \frac{1}{2}O_2 - H_2O = 0$
$H + OH - H_2 - O = 0$
$H_2 - 2H = 0$
$O_2 - 2O = 0$

or,

$$\begin{array}{cccccc} H_2 & O_2 & OH & H_2O & H & O \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & -2 & 0 & 0 & 0 \\ 1 & \tfrac{1}{2} & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

We could do Gauss elimination by hand, but let us use Maple, for convenience:

```
with(linalg):
A:= array([[1,1,-2,0,0,0],[1,1/2,0,-1,0,0],[-1,0,1,0,1,-1],
     [1,0,0,0,-2,0],[0,1,0,0,0,-2]]);
gaussjord(A);
```

gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

or,
$H_2 - 2H = 0$
$O_2 - 2O = 0$
$OH - H - O = 0$
$H_2O - 2H - O = 0$

or,
$H_2 \rightleftharpoons 2H$
$O_2 \rightleftharpoons 2O$
$OH \rightleftharpoons H + O$
$H_2O \rightleftharpoons 2H + O$

, i.e., these 4 LI reactions

(b) As in (a),

$$\begin{array}{ccccc} H_2 & Cl_2 & HCl & Cl & H \end{array}$$
$$\begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so
$H_2 \rightleftharpoons 2H$
$Cl_2 \rightleftharpoons 2Cl$
$HCl \rightleftharpoons Cl + H$

is an equivalent set of 3 LI reactions

(c) Same idea:

$$\begin{array}{ccccccccc} O_2 & CH_3 & CH_3OO & CH_4 & CH_3OOH & CO & H_2 & O & OH \\ \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & -2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have this equivalent set of 4 reactions:

$$O_2 + CH_4 \rightleftharpoons CO + 2H_2 + O$$
$$CH_3 + OH \rightleftharpoons CH_4 + O$$
$$CH_3OO + OH \rightleftharpoons CO + 2H_2 + 2O$$
$$CH_3OOH \rightleftharpoons CO + 2H_2 + O$$

17. Seek all possible dimensionless products of the form $R^a d^b V^c \rho^e \mu^f g^h \lambda^i D^j$. That is, we seek exponents a,b,c,e,f,h,i,j such that

$$(L)^a(L)^b(LT^{-1})^c(ML^{-3})^e(ML^{-1}T^{-1})^f(LT^{-2})^h(L)^i(MLT^{-2})^j = M^0L^0T^0$$

so M: $e+f+j=0$

L: $a+b+c-3e-f+h+i+j=0$ or,

T: $-c-f-2h-2j=0$

$$\begin{array}{c} \begin{array}{cccccccc} a & b & c & e & f & h & i & j \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & -3 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Hence $j=\alpha_1$, $i=\alpha_2$, $h=\alpha_3$, $f=\alpha_4$, $e=-\alpha_1-\alpha_4$, $c=-\alpha_4-2\alpha_3-2\alpha_1$, $b=\alpha_5$, $a=-\alpha_1-\alpha_2-\alpha_3+\alpha_4-3(\alpha_1+\alpha_4)+\alpha_4+2\alpha_3+2\alpha_1-\alpha_5$, so

$$\begin{bmatrix} a \\ b \\ c \\ e \\ f \\ h \\ i \\ j \end{bmatrix} = \alpha_1 \begin{bmatrix} -2 \\ 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The choice $\alpha_1=\cdots=\alpha_4=0$, $\alpha_5=1$ gives the nondimensional parameter d/R;

" " $\alpha_1=\alpha_2=\alpha_3=\alpha_5=0$, $\alpha_4=-1$ " " "Reynolds number" $Re=\rho RV/\mu$;

" " $\alpha_1=\alpha_2=\alpha_4=\alpha_5=0$, $\alpha_3=-1$ " " "Froude number" $Fr=V^2/Rg$;

" " $\alpha_1=\alpha_3=\alpha_4=\alpha_5=0$, $\alpha_2=-1$ " " parameter λ/R;

" " $\alpha_2=\cdots=\alpha_5=0$, $\alpha_1=-1$ " " "drag coefficient" $D/\rho V^2R^2$.

Section 10.6

1\. (b) $M_{11}=2$, $A_{11}=2$, $\det \underset{\sim}{A}=-2$, so $\underset{\sim}{A}^{-1}=\frac{1}{-2}\begin{pmatrix}2&-4\\-3&5\end{pmatrix}=\begin{pmatrix}-1&2\\3/2&-5/2\end{pmatrix}$

$M_{12}=3$, $A_{12}=-3$

$M_{21}=4$, $A_{21}=-4$ Check: $\underset{\sim}{A}^{-1}\underset{\sim}{A}=\begin{pmatrix}-1&2\\3/2&-5/2\end{pmatrix}\begin{pmatrix}5&4\\3&2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. ✓

$M_{22}=5$, $A_{22}=5$

(c) $\underset{\sim}{A}^{-1}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

(f) $\det \underset{\sim}{A}=0$ so $\underset{\sim}{A}$ is singular; not invertible

(h) $\det \underset{\sim}{A}=1$, $M_{11}=1, M_{21}=1, M_{31}=0$ so $A_{11}=1, A_{21}=-1, A_{31}=0$ so

$M_{12}=0, M_{22}=1, M_{32}=1$ $\quad A_{12}=0, A_{22}=1, A_{32}=-1$ $\quad \underset{\sim}{A}^{-1}=\frac{1}{1}\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}$

$M_{13}=0, M_{23}=0, M_{33}=1$ $\quad A_{13}=0, A_{23}=0, A_{33}=1$

(i) $\det \underset{\sim}{A}=2$, $M_{11}=0, M_{12}=0, M_{13}=1$ so $A_{11}=0, A_{12}=0, A_{13}=1$ so

$M_{21}=0, M_{22}=-2, M_{23}=0$ $\quad A_{21}=0, A_{22}=-2, A_{23}=0$ $\quad \underset{\sim}{A}^{-1}=\frac{1}{2}\begin{pmatrix}0&0&2\\0&-2&0\\1&0&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&-1&0\\1/2&0&0\end{pmatrix}$

$M_{31}=2, M_{32}=0, M_{33}=0$ $\quad A_{31}=2, A_{32}=0, A_{33}=0$

(s) $\det \underset{\sim}{A}=\cos^2\theta+\sin^2\theta=1$. Let $c\equiv\cos\theta$ and $s\equiv\sin\theta$, for brevity:

$M_{11}=c, M_{12}=0, M_{13}=-s$ so $A_{11}=c, A_{12}=0, A_{13}=-s$ so

$M_{21}=0, M_{22}=1, M_{23}=0$ $\quad A_{21}=0, A_{22}=1, A_{23}=0$ $\quad \underset{\sim}{A}^{-1}=\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix}$

$M_{31}=s, M_{32}=0, M_{33}=c$ $\quad A_{31}=s, A_{32}=0, A_{33}=c$

2\. (c) $\left(\begin{array}{cc|cc}0&1&1&0\\1&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{cc|cc}1&0&0&1\\0&1&1&0\end{array}\right)$ so $\underset{\sim}{A}^{-1}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

(h) $\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\0&0&1&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&1&0&1&0&-1\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right)$ so $\underset{\sim}{A}^{-1}=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}$

(i) $\left(\begin{array}{ccc|ccc}0&0&2&1&0&0\\0&-1&0&0&1&0\\1&0&0&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&-1&0&0&1&0\\0&0&2&1&0&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&0&-1&0\\0&0&1&1/2&0&0\end{array}\right)$ so $\underset{\sim}{A}^{-1}=\begin{pmatrix}0&0&1\\0&-1&0\\1/2&0&0\end{pmatrix}$

(s) $\left(\begin{array}{ccc|ccc}c&0&-s&1&0&0\\0&1&0&0&1&0\\s&0&c&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}c&0&-s&1&0&0\\0&1&0&0&1&0\\0&0&1/c&-s/c&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}c&0&-s&1&0&0\\0&1&0&0&1&0\\0&0&1&-s&0&c\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}c&0&0&c^2&0&sc\\0&1&0&0&1&0\\0&0&1&-s&0&c\end{array}\right)$

$\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&c&0&s\\0&1&0&0&1&0\\0&0&1&-s&0&c\end{array}\right)$ so $\underset{\sim}{A}^{-1}=\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix}$

3\. (s) The Maple commands

```
with(linalg):
A:= array([[cos(x),0,-sin(x)],[0,1,0],[sin(x),0,cos(x)]]);
inverse(A);
```

gives the matrix $\begin{bmatrix}\frac{\cos(x)}{\%1}&0&\frac{\sin(x)}{\%1}\\0&1&0\\-\frac{\sin(x)}{\%1}&0&\frac{\cos(x)}{\%1}\end{bmatrix}$, $\%1=\cos(x)^2+\sin(x)^2$

but $\cos(x)^2+\sin(x)^2=1$, so this result agrees with the result in 2(s), above.

4.(c) $\begin{pmatrix}2&1\\1&2\end{pmatrix}^{-1}=\frac{1}{3}\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$, $\begin{pmatrix}-1&0&3\\2&5&1\\3&1&0\end{pmatrix}^{-1}=\frac{1}{38}\begin{pmatrix}1&-3&15\\-3&9&-7\\13&-1&5\end{pmatrix}$, $(4)^{-1}=(\frac{1}{4})$

so $$\begin{bmatrix}2&1&0&0&0&0\\1&2&0&0&0&0\\0&0&-1&0&3&0\\0&0&2&5&1&0\\0&0&3&1&0&0\\0&0&0&0&0&4\end{bmatrix}^{-1}=\begin{bmatrix}2/3&-1/3&0&0&0&0\\-1/3&2/3&0&0&0&0\\0&0&\frac{1}{38}&-\frac{3}{38}&\frac{15}{38}&0\\0&0&-\frac{3}{38}&\frac{9}{38}&-\frac{7}{38}&0\\0&0&\frac{13}{38}&-\frac{1}{38}&\frac{5}{38}&0\\0&0&0&0&0&\frac{1}{4}\end{bmatrix}$$

5. (b) $x_1=\dfrac{\begin{vmatrix}c&b\\f&e\end{vmatrix}}{\begin{vmatrix}a&b\\d&e\end{vmatrix}}=\dfrac{ce-bf}{ae-bd}$, $x_2=\dfrac{\begin{vmatrix}a&c\\d&f\end{vmatrix}}{\begin{vmatrix}a&b\\d&e\end{vmatrix}}=\dfrac{af-cd}{ae-bd}$

(c) $x_1=\dfrac{\begin{vmatrix}4&-2&1\\-7&3&1\\0&1&2\end{vmatrix}}{\begin{vmatrix}1&-2&1\\2&3&1\\4&1&2\end{vmatrix}}=\dfrac{-15}{-5}$, $x_2=\dfrac{\begin{vmatrix}1&-4&1\\2&7&1\\4&0&2\end{vmatrix}}{-5}=\dfrac{-14}{-5}=\dfrac{14}{5}$

6. (a) It is necessarily true, if $\underset{\sim}{A}^{-1}$ exists, that $(\underset{\sim}{A}^{-1})^{-1}=A$. But $\det\begin{bmatrix}1&2&0\\0&1&0\\3&0&0\end{bmatrix}=0$ so this $\underset{\sim}{A}^{-1}$ matrix is not invertible. Thus, our calculations must have been incorrect; $\begin{bmatrix}1&2&0\\0&1&0\\3&0&0\end{bmatrix}$ is not the inverse of any matrix.

(b) As in (a), $\begin{bmatrix}1&1&1\\2&2&2\\3&0&0\end{bmatrix}^{-1}$ must give us back the original $\underset{\sim}{A}$ matrix, but that inverse does not exist because $\det\begin{vmatrix}1&1&1\\2&2&2\\3&0&0\end{vmatrix}=0$. Thus, $\begin{bmatrix}1&1&1\\2&2&2\\3&0&0\end{bmatrix}$ is not the inverse of any matrix.

7. (b) If $\underset{\sim}{A}^{-1}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, then $\underset{\sim}{A}=(\underset{\sim}{A}^{-1})^{-1}=\begin{pmatrix}1&2\\3&4\end{pmatrix}^{-1}=\frac{1}{-2}\begin{pmatrix}4&-2\\-3&1\end{pmatrix}=\begin{pmatrix}-2&1\\3/2&-1/2\end{pmatrix}$

(c) If $\underset{\sim}{A}^{-1}=\begin{pmatrix}1&2&1\\0&3&1\\1&1&0\end{pmatrix}$, then $\underset{\sim}{A}=(\underset{\sim}{A}^{-1})^{-1}=\begin{pmatrix}1&2&1\\0&3&1\\1&1&0\end{pmatrix}^{-1}=\begin{pmatrix}1/2&-1/2&1/2\\-1/2&1/2&1/2\\3/2&-1/2&-3/2\end{pmatrix}$

8. If the vectors $\underset{\sim}{x}_1=\begin{pmatrix}2\\5\\-1\end{pmatrix}$, $\underset{\sim}{x}_2=\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\underset{\sim}{x}_3=\begin{pmatrix}3\\0\\4\end{pmatrix}$ are unique solutions corresponding to $\underset{\sim}{c}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underset{\sim}{c}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underset{\sim}{c}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$, then there is a unique solution corresponding to $\underset{\sim}{c}=\begin{pmatrix}4\\3\\-1\end{pmatrix}=4\underset{\sim}{c}_1+3\underset{\sim}{c}_2-\underset{\sim}{c}_3$

given by $\underset{\sim}{x}=\underset{\sim}{A}^{-1}\underset{\sim}{c}=\underset{\sim}{A}^{-1}(4\underset{\sim}{c}_1+3\underset{\sim}{c}_2-\underset{\sim}{c}_3)=4\underset{\sim}{A}^{-1}\underset{\sim}{c}_1+3\underset{\sim}{A}^{-1}\underset{\sim}{c}_2-\underset{\sim}{A}\underset{\sim}{c}_3=4\begin{pmatrix}2\\5\\-1\end{pmatrix}+3\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}8\\23\\-2\end{pmatrix}$

9. $\underset{\sim}{A}$ is invertible if and only if $\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{c}$ admits a unique solution, assuming that $\underset{\sim}{A}$ is square (in this case 6x6). Since the solution is not unique (due to α_1,α_2), it follows that $\underset{\sim}{A}$ must be singular, i.e., not invertible.

10. (a) Since $\underset{\sim}{A}^p=\underset{\sim}{0}$ for some positive integer p, it follows that
$$\det(\underset{\sim}{A}^p)=\det(\underbrace{\underset{\sim}{A}\underset{\sim}{A}\cdots\underset{\sim}{A}}_{p})=\underbrace{(\det\underset{\sim}{A})(\det\underset{\sim}{A})\cdots(\det\underset{\sim}{A})}_{p}=(\det\underset{\sim}{A})^p=\det\underset{\sim}{0}=0$$

Thus, $\det\underset{\sim}{A}=0$ so $\underset{\sim}{A}$ is singular.

(b) $(\underset{\sim}{I}-\underset{\sim}{A})(\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\cdots+\underset{\sim}{A}^{p-1})=\underset{\sim}{I}+\cancel{\underset{\sim}{A}}+\cancel{\underset{\sim}{A}^2}+\cdots+\cancel{\underset{\sim}{A}^{p-1}}-\cancel{\underset{\sim}{A}}-\cancel{\underset{\sim}{A}^2}-\cdots-\cancel{\underset{\sim}{A}^{p-1}}-\cancelto{\underset{\sim}{0}}{\underset{\sim}{A}^p}=\underset{\sim}{I}.$ ✓

11. (b) $\begin{pmatrix}1&0&0\\-3&1&0\\2&7&1\end{pmatrix}=\overset{\underset{\sim}{I}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}-\overset{\underset{\sim}{A}}{\begin{pmatrix}0&0&0\\3&0&0\\-2&-7&0\end{pmatrix}}$. $\underset{\sim}{A}^2=\begin{pmatrix}0&0&0\\0&0&0\\-21&0&0\end{pmatrix}$, $\underset{\sim}{A}^3=\underset{\sim}{0}$,

so $(\underset{\sim}{I}-\underset{\sim}{A})^{-1}=\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\3&0&0\\-2&-7&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\-21&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\3&1&0\\-23&-7&1\end{pmatrix}$

(d) $\begin{pmatrix}1&0&0&0\\-4&1&0&0\\0&3&1&0\\2&1&6&1\end{pmatrix}=\underset{\sim}{I}-\overset{\underset{\sim}{A}}{\begin{pmatrix}0&0&0&0\\4&0&0&0\\0&-3&0&0\\-2&-1&-6&0\end{pmatrix}}$. $\underset{\sim}{A}^2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-12&0&0&0\\-4&18&0&0\end{pmatrix}$, $\underset{\sim}{A}^3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\72&0&0&0\end{pmatrix}$, $\underset{\sim}{A}^4=\underset{\sim}{0}$,

so $(\underset{\sim}{I}-\underset{\sim}{A})^{-1}=\underset{\sim}{I}+\underset{\sim}{A}+\underset{\sim}{A}^2+\underset{\sim}{A}^3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&0\\4&0&0&0\\0&-3&0&0\\-2&-1&-6&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\-12&0&0&0\\-4&18&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\72&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\4&1&0&0\\-12&-3&1&0\\66&17&-6&1\end{pmatrix}$

12. (b) Follow the HINT in part (a): $\begin{pmatrix}3&0&0\\4&-2&0\\10&0&2\end{pmatrix}^{-1}=\left[\begin{pmatrix}3&0&0\\0&-2&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\5&0&1\end{pmatrix}\right]^{-1}=\begin{pmatrix}1&0&0\\-2&1&0\\5&0&1\end{pmatrix}^{-1}\begin{pmatrix}1/3&0&0\\0&-1/2&0\\0&0&1/2\end{pmatrix}$

Now, $\begin{pmatrix}1&0&0\\-2&1&0\\5&0&1\end{pmatrix}=\underset{\sim}{I}-\overset{\underset{\sim}{A}}{\begin{pmatrix}0&0&0\\2&0&0\\-5&0&0\end{pmatrix}}$. $\underset{\sim}{A}^2=\underset{\sim}{0}$, so $\begin{pmatrix}1&0&0\\-2&1&0\\5&0&1\end{pmatrix}^{-1}=\underset{\sim}{I}+\underset{\sim}{A}=\begin{pmatrix}1&0&0\\2&1&0\\-5&0&1\end{pmatrix}$ and

$\begin{pmatrix}3&0&0\\4&-2&0\\10&0&2\end{pmatrix}^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\-5&0&1\end{pmatrix}\begin{pmatrix}1/3&0&0\\0&-1/2&0\\0&0&1/2\end{pmatrix}=\begin{pmatrix}1/3&0&0\\2/3&-1/2&0\\-5/3&0&1/2\end{pmatrix}$

13. (a) $\underset{\sim}{x}=\underset{\sim}{A}^{-1}\underset{\sim}{c}=\begin{pmatrix}1&1/2&1/3\\1/2&1/3&1/4\\1/3&1/4&1/5\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}9&-36&30\\-36&192&-180\\30&-180&180\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-3\\-12\\30\end{pmatrix}$ is exact solution.

(b) Using a rounded off $\underset{\sim}{A}$ matrix, instead, gives

$\underset{\sim}{x}=\begin{pmatrix}1&.5&.33\\.5&.33&.25\\.33&.25&.2\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}55.56&-277.8&255.56\\-277.8&1446.0&-1349.2\\255.56&-1349.2&1269.8\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}11.1\\-84.1\\96.8\end{pmatrix}$ is quite inaccurate

$|\det\underset{\sim}{A}|\Big/\sqrt{\sum_1^n\sum_1^n a_{ij}^2}=\frac{1}{2160}\Big/\sqrt{1.998}=0.000328$ (for the $\underset{\sim}{A}$ matrix in (13.1))

(c) $\underset{\sim}{x}=\begin{pmatrix}1&.5&.333\\.5&.333&.25\\.333&.25&.2\end{pmatrix}^{-1}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}9.67&-39.51&33.28\\-39.51&210.19&-196.95\\33.28&-196.95&195.77\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-2.78\\-13.04\\30.92\end{pmatrix}$ is much better, but still not very accurate.

14. $(\underset{\sim}{A}^{-1})^{-1}\underset{\sim}{A}^{-1}=\underset{\sim}{I}$. Post-multiplying by $\underset{\sim}{A}$ gives $(\underset{\sim}{A}^{-1})^{-1}\underset{\sim}{A}^{-1}\underset{\sim}{A}=\underset{\sim}{I}\underset{\sim}{A}$,

$(\underset{\sim}{A}^{-1})^{-1}\underset{\sim}{I}=\underset{\sim}{A}$,

$(\underset{\sim}{A}^{-1})^{-1}=\underset{\sim}{A}$. ✓

15. If $\underset{\sim}{A}\underset{\sim}{B}=\underset{\sim}{A}\underset{\sim}{C}$ and $\underset{\sim}{A}$ is invertible, then we can premultiply both sides by $\underset{\sim}{A}^{-1}$:

$$\underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{B}=\underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{C}\text{, so }\underset{\sim}{I}\underset{\sim}{B}=\underset{\sim}{I}\underset{\sim}{C}\text{, so }\underset{\sim}{B}=\underset{\sim}{C}.$$

If $\underset{\sim}{B}\underset{\sim}{A}=\underset{\sim}{C}\underset{\sim}{A}$ and $\underset{\sim}{A}$ is invertible, then we can postmultiply both sides by $\underset{\sim}{A}^{-1}$:

$$\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{A}^{-1}=\underset{\sim}{C}\underset{\sim}{A}\underset{\sim}{A}^{-1}\text{, so }\underset{\sim}{B}\underset{\sim}{I}=\underset{\sim}{C}\underset{\sim}{I}\text{, so }\underset{\sim}{B}=\underset{\sim}{C}.$$

Similarly, if $\underset{\sim}{B}\underset{\sim}{A}=\underset{\sim}{O}$ then $\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{A}^{-1}=\underset{\sim}{O}\underset{\sim}{A}^{-1}$ gives $\underset{\sim}{B}=\underset{\sim}{O}$
and if $\underset{\sim}{A}\underset{\sim}{B}=\underset{\sim}{O}$ then $\underset{\sim}{A}^{-1}\underset{\sim}{A}\underset{\sim}{B}=\underset{\sim}{A}^{-1}\underset{\sim}{O}$ gives $\underset{\sim}{B}=\underset{\sim}{O}$.

16. If $k_{12}=k_{23}=0$ then (32) becomes
$$\begin{bmatrix} k_1+k_{13} & 0 & -k_{13} \\ 0 & 0 & 0 \\ -k_{13} & 0 & k_{13}+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

or, in scalar form,
$$(k_1+k_{13})x_1-k_{13}x_3=f_1 \qquad ⓐ$$
$$-k_{13}x_1+(k_{13}+k_3)x_3=f_3 \qquad ⓑ$$
$$0x_2=f_2. \qquad ⓒ$$

ⓐ and ⓑ give a unique* solution for x_1 and x_3, but ⓒ has no solution if $f_2\neq 0$ and a nonunique solution ($x_2=$ arbitrary) if $f_2=0$. Physically, this makes sense because if $k_{12}=k_{23}=0$ (i.e., those springs are removed), then the mass #2 has only one force on it: f_2. Thus, if $f_2\neq 0$ then $m_2x_2''\neq 0$ after all, so (32) has this physical contradiction built into it, and if $f_2=0$ then x_2 is arbitrary because, insofar as force balance is concerned, it is equally happy to sit anywhere.

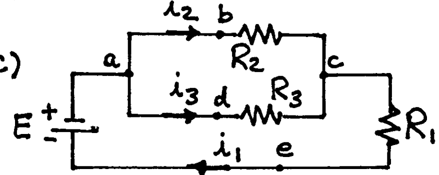
---

* Unique because $\begin{vmatrix} k_1+k_{13} & -k_{13} \\ -k_{13} & k_{13}+k_3 \end{vmatrix} = k_1k_{13}+k_1k_3+\cancel{k_{13}^2}+k_{13}k_3-\cancel{k_{13}^2}>0.$

17.
(a) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & R_2 & -R_3 \\ R_1 & R_2 & 0 \end{pmatrix}\equiv\underset{\sim}{R}$ has $\det\underset{\sim}{R}=R_2R_3+R_1R_2+R_1R_3\neq 0$

(b) $$\underset{\sim}{i}=\underset{\sim}{R}^{-1}\underset{\sim}{e}=\frac{1}{R_2R_3+R_1R_2+R_1R_3}\begin{pmatrix} R_3R_2 & -R_2 & R_2+R_3 \\ -R_1R_3 & R_1 & R_3 \\ -R_1R_2 & -R_1-R_2 & R_2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}=\begin{bmatrix} E(R_2+R_3)/(R_2R_3+R_1R_2+R_1R_3) \\ ER_3/(R_2R_3+R_1R_2+R_1R_3) \\ ER_2/(R_2R_3+R_1R_2+R_1R_3) \end{bmatrix}$$

(c)



Let us respond "in words". We've seen that $\det\underset{\sim}{R}=R_1R_3+R_1R_2+R_2R_3\neq 0$, so there is a unique solution for i_1, i_2, i_3, unless two (or three) of the R_j's are zero. Consider those cases in physical terms.

$R_1=R_2=0$: Then there is no resistance in the loop abcea so we have a "short circuit". That is, $i_1=i_2=$"∞" (i.e., no solution) unless $E=0$, in which case $i_3=0$ and $i_1=i_2=$ arbitrary (i.e., nonunique soln.)

$R_2=R_3=0$: Then $i_1=E/R_1$, but $i_2+i_3=E/R_1$ so there is a nonunique soln. (namely, $i_1=E/R_1$, $i_2=\alpha=$ arbitrary, $i_3=E/R_1-\alpha$)

$R_1=R_3=0$: Then there is no resistance in the loop adcea so we have a "short circuit". That is, $i_1=i_3=$ "∞" (i.e., no soln.) unless $E=0$, in which case $i_2=0$ and $i_1=i_3=$ arbitrary (i.e., nonunique solution)

$R_1=R_2=R_3=0$: No solution unless $E=0$. If $E=0$ then $i_3=\alpha$ (arbitrary), $i_2=\beta$ (arbitrary), $i_1=\alpha+\beta$ (nonunique soln.).

To see these results mathematically we dont really need Gauss elimination, we can easily see them from (17.1).

19. (b) $\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}7\\ 13\end{pmatrix}$. $\underset{\sim}{A}=\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}=\underset{\sim}{L}\underset{\sim}{U}=\begin{pmatrix}1 & 0\\ l_{21} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12}\\ 0 & u_{22}\end{pmatrix}=\begin{pmatrix}u_{11} & u_{12}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22}\end{pmatrix}$

gives $u_{11}=2$, $u_{12}=-1$, $l_{21}=1$, $u_{22}=2$

Then, $\underset{\sim}{L}\underset{\sim}{y}=\underset{\sim}{c}$ is $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}7\\ 13\end{pmatrix}$, which gives $y_1=7$, $y_2=6$.

Then, $\underset{\sim}{U}\underset{\sim}{x}=\underset{\sim}{y}$ is $\begin{pmatrix}2 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}7\\ 6\end{pmatrix}$, which gives $x_2=3$, $x_1=5$

(c) $\begin{pmatrix}2 & 5 & 1\\ 2 & 8 & 0\\ 8 & 2 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ -7\\ 10\end{pmatrix}$. $\underset{\sim}{A}=\begin{pmatrix}2 & 5 & 1\\ 2 & 8 & 0\\ 8 & 2 & 2\end{pmatrix}=\underset{\sim}{L}\underset{\sim}{U}=\begin{pmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{pmatrix}$

so $u_{11}=2$

$u_{12}=5$

$u_{13}=1$

$l_{21}u_{11}=2$ so $l_{21}=1$

$l_{21}u_{12}+u_{22}=8$ so $u_{22}=3$

$l_{21}u_{13}+u_{23}=0$ so $u_{23}=-1$

$l_{31}u_{11}=8$ so $l_{31}=4$

$l_{31}u_{12}+l_{32}u_{22}=2$ so $l_{32}=-6$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2$ so $u_{33}=-8$

Then $\underset{\sim}{L}\underset{\sim}{y}=\underset{\sim}{c}$ is $\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 4 & -6 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}0\\ -7\\ 10\end{pmatrix}$, which gives $y_1=0$, $y_2=-7$, $y_3=-32$.

Then $\underset{\sim}{U}\underset{\sim}{x}=\underset{\sim}{y}$ is $\begin{pmatrix}2 & 5 & 1\\ 0 & 3 & -1\\ 0 & 0 & -8\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ -7\\ -32\end{pmatrix}$, which gives $x_3=4$, $x_2=-1$, $x_1=1/2$

20. Let the $\underset{\sim}{e}_j$'s be the rows of a matrix $\underset{\sim}{B}$. Since the $\underset{\sim}{e}_j$'s are LI, the nxn matrix $\underset{\sim}{B}$ must be invertible. If we denote the columns of $\underset{\sim}{B}^{-1}$ as $\underset{\sim}{c}_j$, then

$$\underset{\sim}{B}\underset{\sim}{B}^{-1}=\begin{bmatrix}\overline{\quad e_1 \quad}\\ \vdots\\ \overline{\quad e_n \quad}\end{bmatrix}\begin{bmatrix}\begin{pmatrix}c_1\end{pmatrix}\cdots\begin{pmatrix}c_n\end{pmatrix}\end{bmatrix}=\begin{bmatrix}\underset{\sim}{e}_1\cdot\underset{\sim}{c}_1 & \cdots & \underset{\sim}{e}_1\cdot\underset{\sim}{c}_n\\ \vdots & & \vdots\\ \underset{\sim}{e}_n\cdot\underset{\sim}{c}_1 & \cdots & \underset{\sim}{e}_n\cdot\underset{\sim}{c}_n\end{bmatrix}=\underset{\sim}{I},$$

and since $\underset{\sim}{e}_i\cdot\underset{\sim}{c}_j=\delta_{ij}$, it follows that the columns $\underset{\sim}{c}_1,\dots,\underset{\sim}{c}_n$ are the desired dual vectors $\underset{\sim}{e}_1^*,\dots,\underset{\sim}{e}_n^*$. Since $\underset{\sim}{B}^{-1}$ is itself invertible, its n columns are necessarily LI. Thus, the n LI $\underset{\sim}{e}_j^*$ vectors form a basis for $\mathbb{R}^n$.

Section 10.7

2. $\underset{\sim}{Q} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. Not orthogonal matrix because its column vectors are not ON.

$$[\underset{\sim}{x}]_{B'} = \underset{\sim}{Q}[\underset{\sim}{x}]_{\underset{\sim}{B}} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix}, \quad [\underset{\sim}{x}]_B = \underset{\sim}{Q}^{-1}[\underset{\sim}{x}]_{B'} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ 1 \end{pmatrix}$$

3. (a) Both bases are ON so (10) applies:

$$q_{11} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_1 = 2/\sqrt{5}, \quad q_{12} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_2 = 1/\sqrt{5}$$
$$q_{21} = \hat{\underset{\sim}{e}}_2' \cdot \hat{\underset{\sim}{e}}_1 = 1/\sqrt{5}, \quad q_{22} = \hat{\underset{\sim}{e}}_2' \cdot \hat{\underset{\sim}{e}}_2 = -2/\sqrt{5} \qquad \text{so } \underset{\sim}{Q} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

The columns of $\underset{\sim}{Q}$ are ON, so $\underset{\sim}{Q}$ is orthogonal.

(b) From (6),

$$[\underset{\sim}{x}]_{B'} = \underset{\sim}{Q}[\underset{\sim}{x}]_B = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 8 \\ -6 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

(c) $[\underset{\sim}{x}]_B = \underset{\sim}{Q}^{-1}[\underset{\sim}{x}]_{B'} = \underset{\sim}{Q}^T[\underset{\sim}{x}]_{B'}$ (since $\underset{\sim}{Q}$ is orthogonal) $= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ -\sqrt{5} \end{pmatrix}$

4. (a) Both bases are ON so (10) applies:

$$q_{11} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_1 = 1/\sqrt{2}, \quad q_{12} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_2 = 1/\sqrt{2}, \quad q_{13} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_3 = 0, \quad q_{14} = \hat{\underset{\sim}{e}}_1' \cdot \hat{\underset{\sim}{e}}_4 = 0, \text{ etc., so}$$

$$\underset{\sim}{Q} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & 0 & -2/\sqrt{6} \end{pmatrix}, \text{ which is orthogonal}$$

(b) From (6),

$$[\underset{\sim}{x}]_{B'} = \underset{\sim}{Q}[\underset{\sim}{x}]_B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & 0 & -2/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2 \\ 5/\sqrt{3} \\ -10/\sqrt{6} \end{pmatrix}$$

(c)

$$[\underset{\sim}{x}]_B = \underset{\sim}{Q}^{-1}[\underset{\sim}{x}]_{B'} = \underset{\sim}{Q}^T[\underset{\sim}{x}]_{B'} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} + 2/\sqrt{3} + 5/\sqrt{6} \\ 1/\sqrt{2} - 2/\sqrt{3} - 5/\sqrt{6} \\ 1 \\ 2/\sqrt{3} - 10/\sqrt{6} \end{pmatrix}$$

5. Are the columns ON?

(a) No; the columns are orthogonal but not ON. (b) Yes (c) No (d) No
(e) Yes (f) Yes

6. Yes. Recall that $\underset{\sim}{Q}$ was the orthogonal matrix $\underset{\sim}{Q} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \underset{\sim}{Q}(\theta)$, and that $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \underset{\sim}{Q}(-\theta)$, so that $\underset{\sim}{Q}^{-1}$ corresponds to a rotation $-\theta$ in the opposite direction.

7. (a) Let $\underset{\sim}{Q} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Since the columns are normalized, $a^2+b^2=1$ and $c^2+d^2=1$. Thus, we can set $a = \cos\theta$ and then $b = \sqrt{1-a^2} = \sqrt{1-\cos^2\theta} = \sin\theta$. Thus far we have

$$\underset{\sim}{Q} = \begin{pmatrix} \cos\theta & c \\ \sin\theta & d \end{pmatrix}.$$

Since $c^2+d^2=1$ we can express $c=\sin\theta_0$, $d=\cos\theta_0$. Orthogonality of the columns requires that $\sin\theta_0\cos\theta+\cos\theta_0\sin\theta=0$,

or, $\sin(\theta_0+\theta)=0$, so $\theta_0=n\pi-\theta$.

Then $c=\sin(n\pi-\theta)=-\cos n\pi\sin\theta$

$d=\cos(\;\text{"}\;)=\cos n\pi\cos\theta$.

n even gives $c=-\sin\theta, d=\cos\theta$ and n odd gives $c=\sin\theta, d=-\cos\theta$. Thus, we have these two forms for $\underset{\sim}{Q}$:

$$\underset{\sim}{Q}_1=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\quad \underset{\sim}{Q}_2=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}.$$

Actually, $\underset{\sim}{Q}_1$ differs from the one given in the exercise and also by (15) by two signs, but we obtain agreement if we change θ to $-\theta$.

(b) Thus, $\underset{\sim}{Q}_1$ given above corresponds to a <u>clock</u>wise rotation θ (and the $\underset{\sim}{Q}_1$ given by (15) and in the exercise " " " <u>counter</u>clockwise rotation θ), and we see that $\det Q_1=+1$. How about $\underset{\sim}{Q}_2$? Observe that $\underset{\sim}{Q}_2$ results from the calculations in (14) if we change $\hat{\underset{\sim}{e}}{}_2'$ to $-\hat{\underset{\sim}{e}}{}_2'$. Thus, $\underset{\sim}{Q}_2$ amounts to a counterclockwise rotation θ (clockwise if we change θ to $-\theta$), then a reflection about the $\hat{\underset{\sim}{e}}{}_1'$ axis. Whether the rotation is cw or ccw, $\det\underset{\sim}{Q}_2=-1$.

8. (a) $(\underset{\sim}{Q}^T)^T=(Q^{-1})^T$ since $\underset{\sim}{Q}$ is orthogonal by assumption

$=(Q^T)^{-1}$ by property I2 on page 514, so $\underset{\sim}{Q}^T$ is orthogonal.

(b) $(\underset{\sim}{Q}^{-1})^T=(\underset{\sim}{Q}^T)^{-1}$ " " " " " "

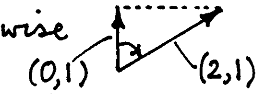
$=(\underset{\sim}{Q}^{-1})^{-1}$ since $\underset{\sim}{Q}$ is orthogonal by assumption, so $\underset{\sim}{Q}^{-1}$ is orthogonal.

9. Since
$$\underset{\sim}{Q}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$
is the transformation matrix corresponding to a ccw rotation θ in a plane, then that matrix to the n power must correspond to n successive rotations. Since the total rotation will be $n\theta$, it must be true that $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^n=\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$

Section 10.8

1. (a) $\underset{\sim}{A}\underset{\sim}{x}=\begin{pmatrix}3&2\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}$. Clearly, no rotation, and a dilation (scaling) of 3.

(b) $\underset{\sim}{A}\underset{\sim}{x}=\begin{pmatrix}3&2\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$. $\theta=\cos^{-1}\frac{1}{\sqrt{5}}\approx 1.1$ rad clockwise



dilation $=\|\underset{\sim}{A}\underset{\sim}{x}\|/\|\underset{\sim}{x}\|=\sqrt{5}/1=\sqrt{5}$

(c) $\underset{\sim}{A}\underset{\sim}{x}=\begin{pmatrix}3&2\\0&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$. Clearly, $\theta=0$, dilation $=1$.

2. (a) $$\underset{\sim}{F}(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})-\alpha\underset{\sim}{F}(\underset{\sim}{u})-\beta\underset{\sim}{F}(\underset{\sim}{v})=\begin{pmatrix}(\alpha u_1+\beta v_1)^2\\(\alpha u_1+\beta v_1)+(\alpha u_2+\beta v_2)\end{pmatrix}-\alpha\begin{pmatrix}u_1^2\\u_1+u_2\end{pmatrix}-\beta\begin{pmatrix}v_1^2\\v_1+v_2\end{pmatrix}$$
$$=\begin{pmatrix}(\alpha^2-\alpha)u_1^2+(\beta^2-\beta)v_1^2+2\alpha\beta u_1v_1\\0\end{pmatrix}$$

is not necessarily $\underset{\sim}{0}$ (e.g., it $=\begin{pmatrix}2\\0\end{pmatrix}$ if $\alpha=0,\beta=2,v_1=1$), so $\underset{\sim}{F}$ is <u>non</u>linear.

(b) $\mathbf{F}(\alpha\mathbf{u}+\beta\mathbf{v})-\alpha\mathbf{F}(\mathbf{u})-\beta\mathbf{F}(\mathbf{v})=\begin{pmatrix}3(\alpha u_1+\beta v_1)\\ u_1+u_2+v_1+v_2\end{pmatrix}-\alpha\begin{pmatrix}3u_1\\ u_1+u_2\end{pmatrix}-\beta\begin{pmatrix}3v_1\\ v_1+v_2\end{pmatrix}=\mathbf{0}$, so linear.

(c)
$$\begin{aligned}\mathbf{F}(\alpha\mathbf{u}+\beta\mathbf{v})-\alpha\mathbf{F}(\mathbf{u})-\beta\mathbf{F}(\mathbf{v})&=\begin{pmatrix}(\alpha u_1+\beta v_1)(\alpha u_2+\beta v_2)\\ \alpha u_3+\beta v_3\end{pmatrix}-\alpha\begin{pmatrix}u_1u_2\\ u_3\end{pmatrix}-\beta\begin{pmatrix}v_1v_2\\ v_3\end{pmatrix}\\ &=\begin{pmatrix}\alpha^2u_1u_2+\alpha\beta(u_1v_2+u_2v_1)+\beta^2v_1v_2-\alpha u_1u_2-\beta v_1v_2\\ \alpha u_3+\beta v_3-\alpha u_3-\beta v_3\end{pmatrix}\\ &=\begin{pmatrix}(\alpha^2-\alpha)u_1u_2+(\beta^2-\beta)v_1v_2+\alpha\beta(u_1v_2+u_2v_1)\\ 0\end{pmatrix}\neq\mathbf{0},\end{aligned}$$

in general, so nonlinear.

(d) Linear

(e) Nonlinear

(f) $\mathbf{F}(\alpha\mathbf{u}+\beta\mathbf{v})-\alpha\mathbf{F}(\mathbf{u})-\beta\mathbf{F}(\mathbf{v})=\begin{pmatrix}\alpha u_1+\beta v_1+1\\ \alpha u_2+\beta v_2+1\end{pmatrix}-\alpha\begin{pmatrix}u_1+1\\ u_2+1\end{pmatrix}-\beta\begin{pmatrix}v_1+1\\ v_2+1\end{pmatrix}=\begin{pmatrix}1-\alpha-\beta\\ 1-\alpha-\beta\end{pmatrix}\neq\mathbf{0}$,

in general, so nonlinear.

3.(a) Given any $\mathbf{x}$ in V we can expand $\mathbf{x}=\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n$.

Then $\mathbf{F}(\mathbf{x})=\mathbf{F}(\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n)=\alpha_1\mathbf{F}(\mathbf{v}_1)+\cdots+\alpha_n\mathbf{F}(\mathbf{v}_n)$ by the linearity of $\mathbf{F}$

$=\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n$

$=\mathbf{x}$, so $\mathbf{F}=\mathbf{I}$.

(b) $\mathbf{\Phi}(\mathbf{x})=\mathbf{A}\mathbf{x}=\mathbf{x}\rightarrow\mathbf{A}=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$. (nxn)

4. (a) Given any $\mathbf{x}$ in V we can expand $\mathbf{x}=\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n$.

Then $\mathbf{F}(\mathbf{x})=\mathbf{F}(\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n)=\alpha_1\mathbf{F}(\mathbf{v}_1)+\cdots+\alpha_n\mathbf{F}(\mathbf{v}_n)$

$=\alpha_1\mathbf{0}+\cdots+\alpha_n\mathbf{0}$

$=\mathbf{0}$, so $\mathbf{F}=\mathbf{0}$.

(b) $\mathbf{\Phi}(\mathbf{x})=\mathbf{A}\mathbf{x}=\mathbf{0}\rightarrow\mathbf{A}=\begin{pmatrix}0&\cdots&0\\ \vdots & & \vdots\\ 0&\cdots&0\end{pmatrix}$. (mxn)

5. (a) $\mathbf{F}(\mathbf{x})=\mathbf{A}\mathbf{x}$, $\mathbf{A}=\begin{pmatrix}2&1&1\\1&1&1\end{pmatrix}$.

$\begin{pmatrix}2&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}$. $\left(\begin{array}{ccc|c}2&1&1&c_1\\1&1&1&c_2\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}2&1&1&c_1\\0&1&1&2c_2-c_1\end{array}\right)$

The Gauss elimination, above, tells the whole story:

$\dim R=r(\mathbf{A})=2$ since $\begin{vmatrix}2&1\\0&1\end{vmatrix}\neq 0$

$\dim K=1$ since $\mathbf{c}=\mathbf{0}$ gives $\mathbf{x}=\begin{pmatrix}0\\-\alpha\\ \alpha\end{pmatrix}=\alpha\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\dim V=\dim\mathbb{R}^3=3$

$\mathbf{F}$ is onto because for every $\mathbf{c}$ in $W=\mathbb{R}^2$ there is an $\mathbf{x}$ such that $\mathbf{F}(\mathbf{x})=\mathbf{c}$, namely, $\mathbf{x}=\begin{pmatrix}c_1-c_2\\ 2c_2-c_1-\alpha\\ \alpha\end{pmatrix}=\begin{pmatrix}c_1-c_2\\ 2c_2-c_1\\ 0\end{pmatrix}+\alpha\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

However, it is not one-to-one because that $\mathbf{x}$ is not unique (since α is arbitrary). Since it is not one-to-one it is not invertible. A basis for R is what? Well, $\mathbf{F}(\mathbf{x})=\begin{pmatrix}2&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2x_1+x_2+x_3\\ x_1+x_2+x_3\end{pmatrix}=x_1\begin{pmatrix}2\\1\end{pmatrix}+x_2\begin{pmatrix}1\\1\end{pmatrix}+x_3\begin{pmatrix}1\\1\end{pmatrix}$, so R is the span of the column vectors of $\mathbf{A}$: span$\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$, a basis for which is $\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$, for instance — or, for that matter $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ since R is all of $\mathbb{R}^2$. If $\mathbf{c}=\mathbf{0}$, then $\mathbf{x} = \alpha\begin{pmatrix}0\\-1\\1\end{pmatrix}$, so a basis for the kernel K is $\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

(d) $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\mathbf{A}=\begin{pmatrix}4&1\\3&2\\0&-1\end{pmatrix}$.

$$\begin{pmatrix}4&1\\3&2\\0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}. \quad \begin{pmatrix}4&1&c_1\\3&2&c_2\\0&-1&c_3\end{pmatrix}\rightarrow\begin{pmatrix}4&1&c_1\\0&5&4c_2-3c_1\\0&-1&c_3\end{pmatrix}\rightarrow\begin{pmatrix}4&1&c_1\\0&5&4c_2-3c_1\\0&0&4c_2-3c_1+5c_3\end{pmatrix}$$

From the Gauss-eliminated form we can see that

dim R = rank $\mathbf{A}$ = 2 since $\begin{vmatrix}4&1\\0&5\end{vmatrix}\neq 0$

dim K = 0 since $\mathbf{c}=\mathbf{0}$ gives only $\mathbf{x}=\mathbf{0}$

dim V = dim $\mathbb{R}^2$ = 2

F is not onto because dim R=2 whereas dim W=3. (R is only the plane $4c_2-3c_1+5c_3=0$

$\mathbf{F}$ is one-to-one because $\mathbf{A}\mathbf{x}=\mathbf{c}$ has a unique solution for each $\mathbf{c}$ in R, but it is not invertible because — even though it is one-to-one — it is not onto.

A basis for R? R = span$\left\{\begin{pmatrix}4\\3\\0\end{pmatrix},\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$ so a basis for R is $\left\{\begin{pmatrix}4\\3\\0\end{pmatrix},\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$.

K has no basis because it is the zero vector space.

(g) $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\mathbf{A}=\begin{pmatrix}2&-1&1\\0&0&3\\1&1&1\\1&2&3\end{pmatrix}$.

$$\begin{pmatrix}2&-1&1\\0&0&3\\1&1&1\\1&2&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}. \quad \begin{pmatrix}2&-1&1&c_1\\0&0&3&c_2\\1&1&1&c_3\\1&2&3&c_4\end{pmatrix}\rightarrow\begin{pmatrix}2&-1&1&c_1\\0&3&1&c_3-c_1\\0&6&5&c_4-c_1\\0&0&3&c_2\end{pmatrix}\rightarrow\begin{pmatrix}2&-1&1&c_1\\0&3&1&c_3-c_1\\0&0&3&c_4-2c_3+c_1\\0&0&0&c_4-2c_3-c_2+c_1\end{pmatrix}$$

From the Gauss-eliminated form we see that

dim R = $r(\mathbf{A})$ = 3

dim K = 0 because $\mathbf{c}=\mathbf{0}$ gives only the unique solution $\mathbf{x}=\mathbf{0}$

dim V = dim $\mathbb{R}^3$ = 3

$\mathbf{F}$ is not onto because dim R=3 whereas dim W = dim $\mathbb{R}^4$=4

$\mathbf{F}$ is one-to-one because $\mathbf{A}\mathbf{x}=\mathbf{c}$ has a unique solution for each $\mathbf{c}$ in R, but it is not invertible because — even though it is one-to-one — it is not onto.

R = span of the column vectors in $\mathbf{A}$; since $r(\mathbf{A})=3$, a basis for R is simply the columns of $\mathbf{A}$: $\{(2,0,1,1)^T,(-1,0,1,2)^T,(1,3,1,3)^T\}$.

K has no basis because it is the zero vector space.

6.(a) m=2, n=2.

To be one-to-one we need $r(\mathbf{A})=2$. Also, to be onto we need $r(\mathbf{A})=2$. If $r(\mathbf{A})=2$ then F is both one-to-one and onto [e.g., $\mathbf{A}=\begin{pmatrix}1&2\\0&3\end{pmatrix}$]; if $r(\mathbf{A})<2$ then it is neither [e.g., $\mathbf{A}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$]. Can't be one and not the other.

(b) $m=2, n=3$.

The only possibilities are these: if $r(\utilde{A})=2$ (e.g., $\utilde{A}=\begin{pmatrix}1&2&0\\0&1&0\end{pmatrix}$) then $\utilde{F}$ is onto, but not one-to-one; if $r(\utilde{A})<2$ (e.g., $\utilde{A}=\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}$) then it is neither onto nor one-to-one.

(c) $m=3, n=2$.

One-to-one but not onto: $\utilde{A}=\begin{pmatrix}1&2\\0&1\\0&0\end{pmatrix}$, as seen from $\begin{pmatrix}1&2\\0&1\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1+2x_2\\x_2\\0\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$

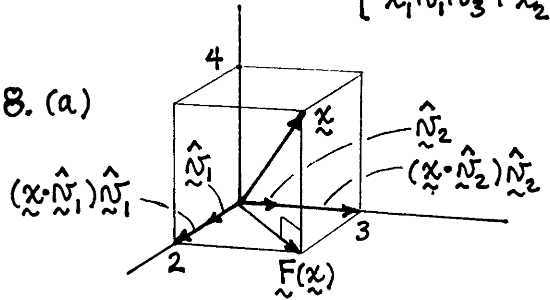
Onto but not one-to-one:
Onto and one-to-one:
} Both impossible because $\utilde{F}$ can't be onto since the maximum dimension of the range is 2 (since the maximum possible rank of $\utilde{A}$ is 2), whereas $\dim W = \dim \mathbb{R}^3 = 3$.

Neither onto nor one-to-one: $\utilde{A}=\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$, as seen from $\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1\\0\\0\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$

7. $$\begin{aligned}\utilde{F}(\alpha\utilde{x}+\beta\utilde{y}) &= [(\alpha\utilde{x}+\beta\utilde{y})\cdot\hat{\utilde{v}}]\hat{\utilde{v}}\\ &= [\alpha(\utilde{x}\cdot\hat{\utilde{v}})+\beta(\utilde{y}\cdot\hat{\utilde{v}})]\hat{\utilde{v}} \text{ by the linearity of the dot product}\\ &= \alpha(\utilde{x}\cdot\hat{\utilde{v}})\hat{\utilde{v}}+\beta(\utilde{y}\cdot\hat{\utilde{v}})\hat{\utilde{v}}\\ &= \alpha\utilde{F}(\utilde{x})+\beta\utilde{F}(\utilde{y}) \text{, so } \utilde{F} \text{ is linear.}\end{aligned}$$

$$\begin{aligned}\utilde{F}(\utilde{x}) = (\utilde{x}\cdot\hat{\utilde{v}})\hat{\utilde{v}} &= (x_1v_1+x_2v_2+x_3v_3)\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\\ &= \begin{bmatrix}x_1v_1^2+x_2v_1v_2+x_3v_1v_3\\ x_1v_1v_2+x_2v_2^2+x_3v_3v_2\\ x_1v_1v_3+x_2v_2v_3+x_3v_3^2\end{bmatrix} = \underbrace{\begin{bmatrix}v_1^2 & v_1v_2 & v_1v_3\\ v_2v_1 & v_2^2 & v_2v_3\\ v_3v_1 & v_3v_2 & v_3^2\end{bmatrix}}_{\utilde{A}\text{ matrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\end{aligned}$$

8. (a)



(b) $$\begin{aligned}\utilde{F}(\alpha\utilde{x}+\beta\utilde{y}) &= [(\alpha\utilde{x}+\beta\utilde{y})\cdot\hat{\utilde{v}}_1]\hat{\utilde{v}}_1+[(\alpha\utilde{x}+\beta\utilde{y})\cdot\hat{\utilde{v}}_2]\hat{\utilde{v}}_2\\ &= \alpha[(\utilde{x}\cdot\hat{\utilde{v}}_1)\hat{\utilde{v}}_1+(\utilde{x}\cdot\hat{\utilde{v}}_2)\hat{\utilde{v}}_2]+\beta[(\utilde{y}\cdot\hat{\utilde{v}}_1)\hat{\utilde{v}}_1+(\utilde{y}\cdot\hat{\utilde{v}}_2)\hat{\utilde{v}}_2]\\ &= \alpha\utilde{F}(\utilde{x})+\beta\utilde{F}(\utilde{y}) \text{, so } \utilde{F} \text{ is linear.}\end{aligned}$$

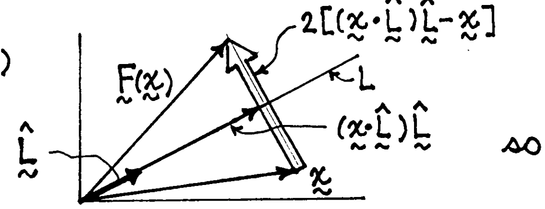
$$\utilde{F}(\utilde{x}) = (\utilde{x}\cdot\hat{\utilde{v}}_1)\hat{\utilde{v}}_1+(\utilde{x}\cdot\hat{\utilde{v}}_2)\hat{\utilde{v}}_2 = (x_1v_{11}+x_2v_{12}+x_3v_{13})\begin{bmatrix}v_{11}\\v_{12}\\v_{13}\end{bmatrix}+(x_1v_{21}+x_2v_{22}+x_3v_{23})\begin{bmatrix}v_{21}\\v_{22}\\v_{23}\end{bmatrix}$$

$$= \begin{bmatrix}x_1v_{11}^2+x_2v_{11}v_{12}+x_3v_{11}v_{13}\\ x_1v_{11}v_{12}+x_2v_{12}^2+x_3v_{12}v_{13}\\ x_1v_{11}v_{13}+x_2v_{12}v_{13}+x_3v_{13}^2\end{bmatrix}+\begin{bmatrix}\text{etc.}\end{bmatrix}$$

$$= \overbrace{\begin{bmatrix}v_{11}^2+v_{21}^2 & v_{11}v_{12}+v_{21}v_{22} & v_{11}v_{13}+v_{21}v_{23}\\ v_{11}v_{12}+v_{21}v_{22} & v_{12}^2+v_{22}^2 & v_{12}v_{13}+v_{22}v_{23}\\ v_{11}v_{13}+v_{21}v_{23} & v_{12}v_{13}+v_{22}v_{23} & v_{13}^2+v_{23}^2\end{bmatrix}}^{\utilde{A}\text{ matrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

9. $\alpha=\beta=1$ gives $\underset{\sim}{F}(\underset{\sim}{u}+\underset{\sim}{v})=\underset{\sim}{F}(\underset{\sim}{u})+\underset{\sim}{F}(\underset{\sim}{v})$ and $\beta=0$ gives $\underset{\sim}{F}(\alpha\underset{\sim}{u})=\alpha\underset{\sim}{F}(\underset{\sim}{u})$. Now we need to show that the reverse is true: that the latter two imply that $\underset{\sim}{F}(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})$ $=\alpha\underset{\sim}{F}(\underset{\sim}{u})+\beta\underset{\sim}{F}(\underset{\sim}{v})$. Well, $\underset{\sim}{F}(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})=\underset{\sim}{F}(\alpha\underset{\sim}{u})+F(\beta\underset{\sim}{v})$ per the first
$=\alpha\underset{\sim}{F}(\underset{\sim}{u})+\beta\underset{\sim}{F}(\underset{\sim}{v})$ per the second. Done.

10. (a)



so $\underset{\sim}{F}(\underset{\sim}{x})=\underset{\sim}{x}+2[(\underset{\sim}{x}\cdot\hat{\underset{\sim}{L}})\hat{\underset{\sim}{L}}-\underset{\sim}{x}]$

(b)
$$\begin{aligned}\underset{\sim}{F}(\alpha\underset{\sim}{x}+\beta\underset{\sim}{y}) &= (\alpha\underset{\sim}{x}+\beta\underset{\sim}{y})+2\{[(\alpha\underset{\sim}{x}+\beta\underset{\sim}{y})\cdot\hat{\underset{\sim}{L}}]\hat{\underset{\sim}{L}}-(\alpha\underset{\sim}{x}+\beta\underset{\sim}{y})\}\\ &= \alpha\{\underset{\sim}{x}+2[(\underset{\sim}{x}\cdot\hat{\underset{\sim}{L}})\hat{\underset{\sim}{L}}-\underset{\sim}{x}]\}+\beta\{\underset{\sim}{y}+2[(\underset{\sim}{y}\cdot\hat{\underset{\sim}{L}})\hat{\underset{\sim}{L}}-\underset{\sim}{y}]\}\\ &= \alpha\underset{\sim}{F}(\underset{\sim}{x})+\beta\underset{\sim}{F}(\underset{\sim}{y}),\end{aligned}$$

so $\underset{\sim}{F}$ is linear.

$$\underset{\sim}{F}(\underset{\sim}{x})=\begin{pmatrix}x_1\\x_2\end{pmatrix}+2\left[(x_1L_1+x_2L_2)\begin{pmatrix}L_1\\L_2\end{pmatrix}-\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]=\begin{pmatrix}2x_1L_1^2+2x_2L_2L_1-x_1\\2x_1L_1L_2+2x_2L_2^2-x_2\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}2L_1^2-1 & 2L_1L_2\\2L_1L_2 & 2L_2^2-1\end{pmatrix}}_{\underset{\sim}{A}}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

(c)
$$\underset{\sim}{A}^2=\begin{pmatrix}2L_1^2-1 & 2L_1L_2\\2L_1L_2 & 2L_2^2-1\end{pmatrix}\begin{pmatrix}2L_1^2-1 & 2L_1L_2\\2L_1L_2 & 2L_2^2-1\end{pmatrix}=\left[\begin{array}{c|c}4L_1^4-4L_1^2+1 & 4L_1^3L_2-2L_1L_2\\ +4L_1^2L_2^2 & +4L_1L_2^3-2L_1L_2\\ \hline 4L_1^3L_2-2L_1L_2 & 4L_1^2L_2^2+4L_2^4\\ +4L_1L_2^3-2L_1L_2 & -4L_2^2+1\end{array}\right]$$

$$=\begin{pmatrix}4L_1^2\cancel{(L_1^2+L_2^2)}-\cancel{4L_1^2}+1 & \cancel{4L_1L_2(L_1^2+L_2^2)}-\cancel{4L_1L_2}\\ 4L_1\cancel{L_2(L_1^2+L_2^2)}-\cancel{4L_1L_2} & 4\cancel{L_2^2(L_1^2+L_2^2)}\cancel{-4L_2^2}+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
since $\hat{\underset{\sim}{L}}$ = Unit vector (so $L_1^2+L_2^2=1$)

as is obvious even without working out $\underset{\sim}{A}^2$ since a double action of $\underset{\sim}{F}$ gives <u>two</u> reflections of $\underset{\sim}{x}$ about L, and hence gives $\underset{\sim}{x}$ again.

11. (a) $(\underset{\sim}{G}\underset{\sim}{F})(\underset{\sim}{x})=\begin{pmatrix}2&0\\4&0\end{pmatrix}\begin{pmatrix}1&5&3&-2\\2&5&0&0\end{pmatrix}\begin{pmatrix}3\\1\\-2\\6\end{pmatrix}=\begin{pmatrix}2&0\\4&0\end{pmatrix}\begin{pmatrix}-10\\11\end{pmatrix}=\begin{pmatrix}-20\\-40\end{pmatrix}$

$\underset{\sim}{G}\underset{\sim}{F}$ transformation matrix $=\begin{pmatrix}2&0\\4&0\end{pmatrix}\begin{pmatrix}1&5&3&-2\\2&5&0&0\end{pmatrix}=\begin{pmatrix}2&10&6&-4\\4&20&12&-8\end{pmatrix}$

(b) $\underset{\sim}{F}^2(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})=\underset{\sim}{F}(\underset{\sim}{F}(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v}))=\underset{\sim}{F}(\alpha\underset{\sim}{F}(\underset{\sim}{u})+\beta\underset{\sim}{F}(\underset{\sim}{v}))$
$=\alpha\underset{\sim}{F}(\underset{\sim}{F}(\underset{\sim}{u}))+\beta\underset{\sim}{F}(\underset{\sim}{F}(\underset{\sim}{v}))=\alpha\underset{\sim}{F}^2(\underset{\sim}{u})+\beta\underset{\sim}{F}^2(\underset{\sim}{v})$, so $\underset{\sim}{F}^2$ is linear too.

12. If $\underset{\sim}{F}(\underset{\sim}{x})=\underset{\sim}{x}+\underset{\sim}{c}$ then $\underset{\sim}{F}(\alpha\underset{\sim}{x}+\beta\underset{\sim}{y})-\alpha\underset{\sim}{F}(\underset{\sim}{x})-\beta\underset{\sim}{F}(\underset{\sim}{y})=(\alpha\underset{\sim}{x}+\beta\underset{\sim}{y})+\underset{\sim}{c}-\alpha(\underset{\sim}{x}+\underset{\sim}{c})-\beta(\underset{\sim}{y}+\underset{\sim}{c})$
$=(1-\alpha-\beta)\underset{\sim}{c}\neq 0,$

so $\underset{\sim}{F}$ is <u>nonlinear</u>.

13. (a) Easy, per the hint.

(b)
$$\underset{\sim}{T}\underset{\sim}{R}_z\underset{\sim}{X} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_z & -S_z & 0 & 0 \\ S_z & C_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} C_z & -S_z & 0 & \Delta x \\ S_z & C_z & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underset{\sim}{T}\underset{\sim}{R}_z}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

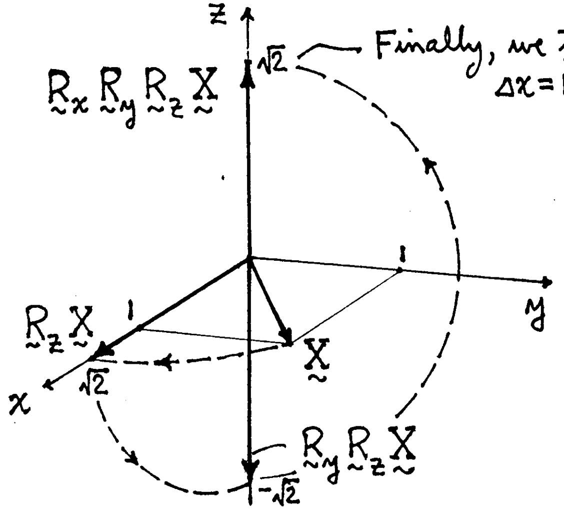
and

$$\underset{\sim}{R}_z\underset{\sim}{T} = \begin{bmatrix} C_z & -S_z & 0 & 0 \\ S_z & C_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_z & -S_z & 0 & (\Delta x C_z - \Delta y S_z) \\ S_z & C_z & 0 & (\Delta x S_z + \Delta y C_z) \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so in general $\underset{\sim}{T}\underset{\sim}{R}_z \neq \underset{\sim}{R}_z\underset{\sim}{T}$.

(c)
$$\underset{\sim}{F}(\underset{\sim}{X}) = \underset{\underset{\sim}{T}}{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\underset{\underset{\sim}{R}_x}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\underset{\underset{\sim}{R}_y}{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\underset{\underset{\sim}{R}_z}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\underset{\underset{\sim}{X}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{2}-3 \\ 1 \end{bmatrix}$$



Finally, we translate this point, $(0,0,\sqrt{2},1)$, by $\Delta x=1, \Delta y=1, \Delta z=-3$, giving the final point $(x,y,z,1)=(1,1,\sqrt{2}-3,1)$.

(d)
$$\underset{\sim}{F}(\underset{\sim}{X}_p) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .980 & -.1987 & 0 \\ 0 & .1987 & .980 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .9553 & 0 & -.2955 & 0 \\ 0 & 1 & 0 & 0 \\ .2955 & 0 & .9553 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .8253 & .5646 & 0 & 0 \\ -.5646 & .8253 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \underset{\underset{\sim}{A}\text{, say}\rightarrow}{\begin{bmatrix} .7884 & .5394 & -.2955 & 1 \\ .6018 & .7757 & -.1898 & 3 \\ .1268 & .3275 & .9362 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2439 \\ 3.5859 \\ 2.2637 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{F}(\underset{\sim}{X}_e) = \underset{\sim}{A}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \quad \begin{bmatrix}1.5394\\3.7757\\1.3275\\1\end{bmatrix}$$

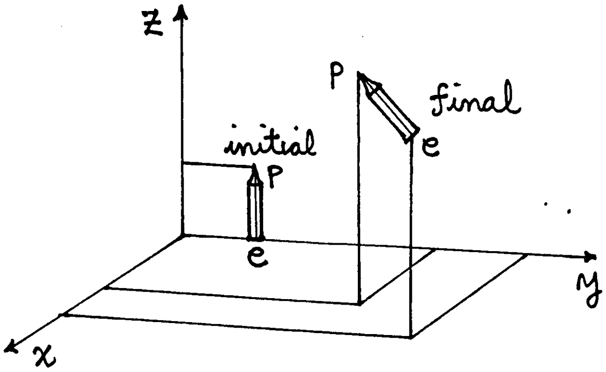
Initial length $= \|(0,1,1)^T - (0,1,0)^T\| = \|(0,0,1)^T\| = 1$

Final length $= \|(1.2439, 3.5859, 2.2637)^T - (1.5394, 3.7757, 1.3275)^T\|$

$= \sqrt{(1.2439-1.5394)^2 + (3.5859-3.7757)^2 + (2.2637-1.3275)^2} = 0.9998$

≈ 1 to the accuracy of the calculation. ✓

Not to scale:



CHAPTER 11

Section 11.2

1. (a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. $|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2-4=0$ gives $\lambda = -1, 3$.

$\lambda_1 = 3$:

$$(\mathbf{A}-3\mathbf{I})\mathbf{x} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\text{Gauss elim.}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ so } x_2 = \alpha,\ x_1 = 2\alpha,$$

so $\mathbf{e}_1 = \alpha(2,1)^T$.

$\lambda_2 = -1$:

$$(\mathbf{A}+\mathbf{I})\mathbf{x} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ so } x_2 = \beta,\ x_1 = -2\beta, \text{ so } \mathbf{e}_2 = \beta(-2,1)^T$$

(b) Putting $x(t) = q_1 e^{\lambda_1 t}$, $y(t) = q_2 e^{\lambda_2 t}$ into (15) gives

$$(\lambda_1 - 1) q_1 e^{\lambda_1 t} - 4 q_2 e^{\lambda_2 t} = 0 \qquad ①$$
$$q_1 e^{\lambda_1 t} - (\lambda_2 - 1) q_2 e^{\lambda_2 t} = 0 \qquad ②$$

If $\lambda_1 \neq \lambda_2$ then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are LI so ① $\Rightarrow (\lambda_1 - 1) q_1 = 0$ and $q_2 = 0$

and ② $\Rightarrow q_1 = 0$ and $(\lambda_2 - 1) q_2 = 0$

Thus, $q_1 = q_2 = 0$ and we obtain only the trivial solution $x(t) = 0$, $y(t) = 0$.

2. NOTE: This problem makes an important point, one that is worth emphasizing in class in connection with the Markov population example.

$$\mathbf{p}_0 = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3 \qquad ①$$
$$\mathbf{p}_1 = \mathbf{A}\mathbf{p}_0 = c_1 \mathbf{A}\mathbf{e}_1 + c_2 \mathbf{A}\mathbf{e}_2 + \mathbf{A}\mathbf{e}_3 \qquad ②$$

If the $\mathbf{e}$'s are the eigenvectors of $\mathbf{A}$, then ② becomes

$$\mathbf{p}_1 = c_1\lambda_1 \mathbf{e}_1 + c_2\lambda_2 \mathbf{e}_2 + c_3\lambda_3 \mathbf{e}_3,$$
$$\mathbf{p}_2 = c_1\lambda_1^2 \mathbf{e}_1 + c_2\lambda_2^2 \mathbf{e}_2 + c_3\lambda_3^2 \mathbf{e}_3,$$

and so on. However, if the base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are <u>not</u> eigenvectors of $\mathbf{A}$ then we need to expand

$$\mathbf{A}\mathbf{e}_1 = \alpha_{11}\mathbf{e}_1 + \alpha_{12}\mathbf{e}_2 + \alpha_{13}\mathbf{e}_3$$
$$\mathbf{A}\mathbf{e}_2 = \alpha_{21}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{23}\mathbf{e}_3$$
$$\mathbf{A}\mathbf{e}_3 = \alpha_{31}\mathbf{e}_1 + \alpha_{32}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3$$

and we have the <u>much</u> more cumbersome result

$$\mathbf{p}_0 = \sum_j c_j \mathbf{e}_j,$$
$$\mathbf{p}_1 = \sum_j c_j \mathbf{A}\mathbf{e}_j = \sum_j \sum_k c_j \alpha_{jk} \mathbf{e}_k,$$
$$\mathbf{p}_2 = \sum_j \sum_k c_j \alpha_{jk} \mathbf{A}\mathbf{e}_k = \sum_j \sum_k \sum_l c_j \alpha_{jk} \alpha_{kl} \mathbf{e}_l,$$

and so on. Thus, in this application, and others that we will meet, it is most convenient to use, as a basis, the eigenvectors generated by the $\mathbf{A}$ matrix contained within the problem (if that matrix does indeed generate a basis).

3. (b) $|\underset{\sim}{A}-\lambda\underset{\sim}{I}| = \begin{vmatrix}1-\lambda & -3\\ 0 & -\lambda\end{vmatrix} = \lambda^2-\lambda=0,\ \lambda=0,1$

$\lambda_1=1$: $(\underset{\sim}{A}-\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}0 & -3 & 0\\ 0 & 0 & 0\end{pmatrix} \to x_2=0, x_1=\alpha$ so $\underset{\sim}{e}_1=\alpha\begin{pmatrix}1\\0\end{pmatrix}$; basis is $\begin{pmatrix}1\\0\end{pmatrix}$

$\lambda_2=0$: $(\underset{\sim}{A}-\underset{\sim}{0})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}1 & -3 & 0\\ 0 & 0 & 0\end{pmatrix} \to x_2=\beta, x_1=3\beta$, $\underset{\sim}{e}_2=\beta\begin{pmatrix}3\\1\end{pmatrix}$; basis is $\begin{pmatrix}3\\1\end{pmatrix}$

(c) $\begin{vmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{vmatrix} = \lambda^2-5\lambda-2=0,\ \lambda=(5\pm\sqrt{25+8})/2=(5\pm\sqrt{33})/2$

$\lambda_1=(5+\sqrt{33})/2$: $(\underset{\sim}{A}-\lambda_1\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives

$$\begin{pmatrix}(1-\frac{5+\sqrt{33}}{2}) & 2 & 0\\ 3 & (4-\frac{5+\sqrt{33}}{2}) & 0\end{pmatrix} \to \begin{matrix}x_2=\alpha\\ x_1=\frac{-3+\sqrt{33}}{6}\alpha\end{matrix} \quad \text{so } \underset{\sim}{e}_1=\alpha\begin{pmatrix}\frac{-3+\sqrt{33}}{6}\\ 1\end{pmatrix}$$

$\lambda_2=(5-\sqrt{33})/2$: Surely, we merely need to change $+\sqrt{33}$ to $-\sqrt{33}$ in the result, namely, $\underset{\sim}{e}_2=\beta\begin{pmatrix}\frac{-3-\sqrt{33}}{6}\\ 1\end{pmatrix}$

NOTE: If we use $\lambda_1\approx 5.37228$, say, in place of the exact value $(5+\sqrt{33})/2$, then when we do Gauss elimination to find the eigenvector the bottom row will not quite be 0 0 0, and the result will be that $x_1=x_2=0$ which is merely the trivial solution, not the desired eigenvector. That result makes sense because if λ is not exactly an eigenvalue then we get only the trivial solution. Thus, if we do wish to use $\lambda_1\approx 5.37228$ rather than the messy expression $(5+\sqrt{33})/2$, then we need to throw out the bottom row since it should consist of 3 zeros.

(e) $\begin{vmatrix}-\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & -\lambda\end{vmatrix} = -\lambda^3=0$ gives $\lambda=0$.

$\lambda=0$: $\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \to x_3=\alpha, x_2=\beta, x_1=\gamma$, so $\underset{\sim}{e}=\begin{pmatrix}\gamma\\ \beta\\ \alpha\end{pmatrix}=\alpha\begin{pmatrix}0\\0\\1\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\0\\0\end{pmatrix}$

That is, every nonzero vector in $\mathbb{R}^3$ is an eigenvector. A basis for that eigenspace is $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ or, indeed, any other basis for $\mathbb{R}^3$.

(g) $\begin{vmatrix}2-\lambda & 0 & 0\\ 0 & -5-\lambda & 0\\ 0 & 0 & 4-\lambda\end{vmatrix} = (2-\lambda)(-5-\lambda)(4-\lambda)=0$ gives $\lambda=2,-5,4$.

$\lambda_1=2$: $(\underset{\sim}{A}-2\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}0&0&0&0\\0&-7&0&0\\0&0&2&0\end{pmatrix} \to x_3=0, x_2=0, x_1=\alpha$, $\underset{\sim}{e}_1=\begin{pmatrix}\alpha\\0\\0\end{pmatrix}=\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}$,

a basis for which eigenspace is $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$\lambda_2=-5$: $(\underset{\sim}{A}+5\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}7&0&0&0\\0&0&0&0\\0&0&9&0\end{pmatrix} \to x_3=0, x_2=\beta, x_1=0$, $\underset{\sim}{e}_2=\beta\begin{pmatrix}0\\1\\0\end{pmatrix}$; a basis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\lambda_3=4$: $(\underset{\sim}{A}-4\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}-2&0&0&0\\0&-9&0&0\\0&0&0&0\end{pmatrix} \to x_3=\gamma, x_2=0, x_1=0$, $\underset{\sim}{e}_1=\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$; a basis is $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

(h) $\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & -5-\lambda & 3 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda)(4-\lambda) = 0$ gives $\lambda = 2, -5, 4$.

$\lambda_1 = 2$: $(\underset{\sim}{A} - 2\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} 0 & 1 & 6 & 0 \\ 0 & -7 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow x_3 = 0, x_2 = 0, x_1 = \alpha$

so $\underset{\sim}{e}_1 = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$\lambda_2 = -5$: $(\underset{\sim}{A} + 5\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} 7 & 1 & 6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 11 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow x_3 = 0, x_2 = \beta,$ $x_1 = -\beta/7$

so $\underset{\sim}{e}_2 = \begin{pmatrix} -\beta/7 \\ \beta \\ 0 \end{pmatrix} = \beta \begin{pmatrix} -1/7 \\ 1 \\ 0 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} -1 \\ 7 \\ 0 \end{pmatrix}$, say.

$\lambda_3 = 4$: $(\underset{\sim}{A} - 4\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} -2 & 1 & 6 & 0 \\ 0 & -9 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow x_3 = \gamma, x_2 = \gamma/3, x_1 = \frac{19}{6}\gamma$

so $\underset{\sim}{e}_3 = \begin{pmatrix} 19\gamma/6 \\ \gamma/3 \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} 19/6 \\ 1/3 \\ 1 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} 19 \\ 2 \\ 6 \end{pmatrix}$, say.

(j) $\begin{vmatrix} 4-\lambda & 4 & 4 \\ 4 & 4-\lambda & 4 \\ 4 & 4 & 4-\lambda \end{vmatrix} = \text{etc.} = \lambda^2(12-\lambda)$ gives $\lambda = 0, 0, 12$.

$\lambda_1 = 12$: $(\underset{\sim}{A} - 12\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} -8 & 4 & 4 & 0 \\ 4 & -8 & 4 & 0 \\ 4 & 4 & -8 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_3 = \alpha \\ x_2 = \alpha \\ x_1 = \alpha \end{matrix}$

so $\underset{\sim}{e}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda_2 = \lambda_3 = 0$: $(\underset{\sim}{A} - 0\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} 4 & 4 & 4 & 0 \\ 4 & 4 & 4 & 0 \\ 4 & 4 & 4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow x_3 = \beta, x_2 = \gamma, x_1 = -\beta - \gamma$

so $\underset{\sim}{e} = \begin{pmatrix} -\beta-\gamma \\ \gamma \\ \beta \end{pmatrix} = \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, a basis for which eigenspace is $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(k) $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -\lambda & 2 \\ 1 & 0 & 2-\lambda \end{vmatrix} = \lambda^2(3-\lambda) = 0$ gives $\lambda = 0, 0, 3$.

$\lambda_1 = 3$: $(\underset{\sim}{A} - 3\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} -2 & 0 & 2 & 0 \\ 1 & -3 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & -3 & 2 & 0 \\ -2 & 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_3 = \alpha, x_2 = \alpha, \\ x_1 = \alpha, \end{matrix}$

so $\underset{\sim}{e}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda_2 = \lambda_3 = 0$: $\rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow x_3 = \beta, x_2 = \gamma, x_1 = -2\beta$ so $\underset{\sim}{e} = \begin{pmatrix} -2\beta \\ \gamma \\ \beta \end{pmatrix} = \beta \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

a basis for which eigenspace is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(l) $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = -\lambda(\lambda+1)(\lambda-2) = 0$ gives $\lambda = 0, -1, 2$

$\lambda_1 = 2$: $(\underset{\sim}{A} - 2\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_3 = \overset{\substack{\text{2 for convenience} \\ \downarrow}}{2}\alpha, \\ x_2 = \alpha, \\ x_1 = \alpha \end{matrix}$

so $\underset{\sim}{e}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

$\lambda_2=-1$: $(\underset{\sim}{A}+\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}1&0&1&0\\0&1&1&0\\1&1&2&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&1&0\\0&1&1&0\\0&1&1&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}\rightarrow \underset{\sim}{x_3}=\beta,\ x_2=-\beta,\ x_1=-\beta$

so $\underset{\sim}{e}_2=\beta\begin{pmatrix}-1\\-1\\1\end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$.

$\lambda_3=0$: $(\underset{\sim}{A}+0\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{pmatrix}0&0&1&0\\0&0&1&0\\1&1&1&0\end{pmatrix}$ so $x_3=0,\ x_2=\gamma,\ x_1=-\gamma$

so $\underset{\sim}{e}_3=\gamma\begin{pmatrix}-1\\1\\0\end{pmatrix}$, a basis for which eigenspace is $\begin{pmatrix}-1\\1\\0\end{pmatrix}$.

4\. (q) The Maple commands

```
with(linalg):
A:=array([[2,0,0,0],[0,0,0,1],[0,0,0,1],[0,1,1,1]]);
eigenvects(A);
```

give this output:

λ → multiplicity → basis for the eigenspace

```
[2, 2, {[1000],[0112]}],
[0, 1, {[0,-1,1,0]}], [-1,1,{[0,-1,-1,1]}]
```

That is,

$\lambda_1=\lambda_2=2,\quad \underset{\sim}{e}=\alpha\,[1,0,0,0]^T+\beta\,[0,1,1,2]^T$

$\lambda_3=0,\quad \underset{\sim}{e}=\gamma\,[0,-1,1,0]^T$

$\lambda_4=-1,\quad \underset{\sim}{e}=\delta\,[0,-1,-1,1]^T$

(r) The Maple commands

```
with(linalg):
A:= array([[1,1,1,1],[2,2,2,2],[3,3,3,3],[4,4,4,4]]);
eigenvects(A);
```

give

```
[10,1,{[1 2 3 4]}], [0, 3, {[-1100],[-1010],[-1001]}]
```

5\. (c) The simplest test is this: $\underset{\sim}{x}$ is an eigenvector of $\underset{\sim}{A}$ iff $\underset{\sim}{A}\underset{\sim}{x}$ is a scalar multiple of $\underset{\sim}{x}$. (If so, the scalar multiple is λ.)

$$\begin{bmatrix}1&8&5&3\\2&16&10&6\\5&-14&-11&-3\\-1&-8&-5&-3\end{bmatrix}\begin{bmatrix}1\\2\\1\\1\end{bmatrix}=\begin{bmatrix}25\\50\\-37\\ \ \end{bmatrix}\leftarrow$$

Can stop right here: $\underset{\sim}{A}\underset{\sim}{x}$ is not a scalar multiple of $\underset{\sim}{x}$, so $\underset{\sim}{x}$ is not an eigenvector of $\underset{\sim}{A}$.

(d)
$$\begin{bmatrix} & & & \\ & '' & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}1\\0\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Yes, $\underset{\sim}{A}\underset{\sim}{x}=0\underset{\sim}{x}$, so $\underset{\sim}{x}=(1,0,1,-2)^T$ <u>is</u> an eigenvector of $\underset{\sim}{A}$ (and the corresponding eigenvalue is $\lambda=0$).

6\. (b) $(\underset{\sim}{A}-2\underset{\sim}{I})\underset{\sim}{x}=\underset{\sim}{0}$ gives $\begin{bmatrix}1&1&2&1&0\\-1&1&1&2&0\\0&2&3&3&0\\1&3&5&4&0\end{bmatrix}\rightarrow\begin{bmatrix}1&1&2&1&0\\0&2&3&3&0\\0&2&3&3&0\\0&2&3&3&0\end{bmatrix}\rightarrow\begin{bmatrix}1&1&2&1&0\\0&2&3&3&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\rightarrow x_4=\alpha,\ x_3=\beta,$

$x_2=(-3\alpha-3\beta)/2,$

$x_1=\frac{1}{2}\alpha-\frac{1}{2}\beta$

$$\text{so } \underset{\sim}{e} = \begin{bmatrix} \frac{1}{2}\alpha-\frac{1}{2}\beta \\ -\frac{3}{2}\alpha-\frac{3}{2}\beta \\ \beta \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 1/2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1/2 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}$$

7. (a) If $n=1$ then $\underset{\sim}{A}=(b)$. $bx=\lambda x$ has nontrivial solution only if $\lambda=b$. In (7.1), $\lambda_1 = b+2\sqrt{ac}\,\cancelto{0}{\cos\frac{\pi}{2}} = b.$ ✓ (Actually, we should restrict (7.1) to hold only if $n\geq 2$ because if $n=1$ then a and c in (7.1) are not defined.)
If $n=2$ then $\underset{\sim}{A}=\begin{pmatrix} b & c \\ a & b\end{pmatrix}$ and $\begin{vmatrix} b-\lambda & c \\ a & b-\lambda\end{vmatrix} = \lambda^2-2b\lambda+(b^2-ac)=0$ gives $\lambda=b\pm\sqrt{ac}$.
Meanwhile, (7.1) gives $\lambda_1 = b+2\sqrt{ac}\cos\pi/3 = b+\sqrt{ac}$ ✓
$\lambda_2 = b+2\sqrt{ac}\cos 2\pi/3 = b-\sqrt{ac}$ ✓

(c) $\underset{\sim}{A}=\begin{bmatrix} 2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{bmatrix}$. (7.1) gives
$\lambda_1 = 2+2\sqrt{1}\cos\frac{\pi}{5} = 3.6180$
$\lambda_2 = 2+2\sqrt{1}\cos\frac{2\pi}{5} = 2.6180$
$\lambda_3 = 2+2\sqrt{1}\cos\frac{3\pi}{5} = 1.3820$
$\lambda_4 = 2+2\sqrt{1}\cos\frac{4\pi}{5} = 0.3820$

Maple gives $(5\pm\sqrt{5})/2$ and $(3\pm\sqrt{5})/2$ which values agree with those given by (7.1).

8. The eigenvalues are the roots of an nth degree polynomial equation. Such equations always have at least one (and at most n) real or complex roots. Thus, no it is not possible.

9. No, for if $Ax=\lambda_1 x$
and $Ax=\lambda_2 x$
then subtraction gives $\underset{\sim}{0}=(\lambda_1-\lambda_2)\underset{\sim}{x}$. Since $\underset{\sim}{x}=\underset{\sim}{0}$ is not acceptable, we must have $\lambda_2=\lambda_1$.

10. $\underset{\sim}{A}\underset{\sim}{x}=\lambda\underset{\sim}{x}$ means that Ax is collinear with $\underset{\sim}{x}$, the scale factor then being λ. Surely, the figure in (a) is the generic case; it takes an exceptional choice of $\underset{\sim}{x}$ for $\underset{\sim}{A}\underset{\sim}{x}$ to be collinear with $\underset{\sim}{x}$.
(a) No, not collinear (b) Yes, $\lambda\approx 2$
(c) Yes, $\lambda=0$ (d) Yes, $\lambda\approx -1.5$

NOTE: This problem gives a nice graphical feeling for eigenvectors and eigenvalues, and you might wish to discuss it in class at the beginning of the discussion of the eigenvalue problem. Even if the space is more than 3-dimensional, the figures hold in a schematic sense.

11. Let $\lambda, \underset{\sim}{e}$ be an "eigenpair" of $\underset{\sim}{A}$: $\underset{\sim}{A}\underset{\sim}{e}=\lambda\underset{\sim}{e}$. Then it follows by scalar multiplication that $(k\underset{\sim}{A})\underset{\sim}{e}=(k\lambda)\underset{\sim}{e}$, so $k\underset{\sim}{A}$ has the eigenpair $k\lambda$ and $\underset{\sim}{e}$ (the same $\underset{\sim}{e}$ as for $\underset{\sim}{A}$).

12. The characteristic equation for $\underset{\sim}{A}^T$ is $\det(\underset{\sim}{A}^T-\lambda\underset{\sim}{I})=0$. But $\det(\underset{\sim}{A}^T-\lambda\underset{\sim}{I}) = \det(\underset{\sim}{A}-\lambda\underset{\sim}{I})^T = \det(\underset{\sim}{A}-\lambda\underset{\sim}{I})$ by property D10 of determinants in Section 10.4. Thus, $\underset{\sim}{A}$ and $\underset{\sim}{A}^T$ have the same characteristic equations and therefore the same λ's. But the eigenspaces corresponding to a given λ are, in general, different. For example, $\underset{\sim}{A}=\begin{pmatrix}1&1\\0&2\end{pmatrix}$ has $\lambda=1,2$ and $\lambda=1$ has the eigenspace $\underset{\sim}{e}=\alpha[1,0]^T$, whereas $\underset{\sim}{A}=\begin{pmatrix}1&0\\1&2\end{pmatrix}$ has $\lambda=1,2$ and $\lambda=1$ has the

different eigenspace $\mathbf{e} = \beta[1,-1]^T$.

13. $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$. Dot with $\mathbf{e}$: $\mathbf{e}\cdot(\mathbf{A}\mathbf{e}) = \lambda\mathbf{e}\cdot\mathbf{e}$, or, $\mathbf{e}^T\mathbf{A}\mathbf{e} = \lambda\,\mathbf{e}^T\mathbf{e}$, $\lambda = \mathbf{e}^T\mathbf{A}\mathbf{e}/\mathbf{e}^T\mathbf{e}$. The point is that if we wish to solve $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$ for λ we cannot merely say $\lambda = (\mathbf{A}\mathbf{e})/\mathbf{e}$ because vector division is not defined. In contrast, $\mathbf{e}\cdot(\mathbf{A}\mathbf{e}) = \lambda\mathbf{e}\cdot\mathbf{e}$ is a <u>scalar</u> equation.

14. Easy:
$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = (a_{11}-\lambda)\cdots(a_{nn}-\lambda) = 0, \text{ so } \lambda = a_{11},\dots,a_{nn}.$$

15. $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$, $\mathbf{A}\mathbf{A}\mathbf{e} = \lambda\mathbf{A}\mathbf{e} = \lambda^2\mathbf{e}$, so $\mathbf{A}^2\mathbf{e} = \lambda^2\mathbf{e}$. Then, $\mathbf{A}\mathbf{A}^2\mathbf{e} = \lambda^2\mathbf{A}\mathbf{e} = \lambda^3\mathbf{e}$, so $\mathbf{A}^3\mathbf{e} = \lambda^3\mathbf{e}$, and so on. Thus, if $\tilde{A}$ has an eigenpair $\lambda, \mathbf{e}$, then $\mathbf{A}^n$ has an eigenpair $\lambda^n, \mathbf{e}$.

16. (c)
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ has } \lambda_1 = 1,\ \mathbf{e}_1 = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix};\ \lambda_2 = 0,\ \mathbf{e}_2 = \beta\begin{pmatrix}0\\1\\0\end{pmatrix};\ \lambda_3 = -2,\ \mathbf{e}_3 = \gamma\begin{pmatrix}-1\\0\\3\end{pmatrix}.$$
$$\mathbf{A}^{10} = \begin{pmatrix} 1 & 0 & -341 \\ 0 & 0 & 0 \\ 0 & 0 & 1024 \end{pmatrix} \text{ has } \lambda_1 = \underset{\text{i.e., } 1^{10}\checkmark}{1},\ \mathbf{e}_1 = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix};\ \lambda_2 = \underset{0^{10}\checkmark}{0},\ \mathbf{e}_2 = \beta\begin{pmatrix}0\\1\\0\end{pmatrix};\ \lambda_3 = \underset{(-2)^{10}\checkmark}{1024},\ \mathbf{e} = \gamma\begin{pmatrix}-1\\0\\3\end{pmatrix}$$

17. (a) For $\mathbf{A}$, the maple commands

```
with(linalg):
A:= array([[2,2,0],[0,2,0],[2,0,2]]);
eigenvects(A);
```

give

```
[2, 3, {[0,0,1]}]
```

i.e., $\lambda = 2,2,2$, $\mathbf{e} = \alpha\begin{pmatrix}0\\0\\1\end{pmatrix}$.

For $\mathbf{A}^5$, continue with

```
B:= evalm(A^5);
eigenvects(B);
```

and obtain

```
[32, 3, {[0,0,1]}]
```

$\uparrow$ i.e., $2^5\checkmark$

18. (b)
$$\det(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}-\lambda\mathbf{I}) = \det(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}-\lambda\mathbf{Q}^{-1}\mathbf{I}\mathbf{Q}) = \det[\mathbf{Q}^{-1}(\mathbf{A}-\lambda\mathbf{I})\mathbf{Q}]$$
$$= \det\mathbf{Q}^{-1}\det(\mathbf{A}-\lambda\mathbf{I})\det\mathbf{Q} = 0$$
We can either cancel $\det\mathbf{Q}^{-1}$ and $\det\mathbf{Q}$ from the latter equation (because each is nonzero) or we can note that $\det\mathbf{Q}^{-1}\det\mathbf{Q} = \det(\mathbf{Q}^{-1}\mathbf{Q}) = \det\mathbf{I} = 1$. Thus, we obtain $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$, which is the same characteristic equation as for $\mathbf{A}$. Thus, $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ and $\mathbf{A}$ have the same λ's.

20. $\left.\begin{aligned}\mathbf{A}\mathbf{e}_j &= \lambda_j\mathbf{e}_j\\ \mathbf{B}\mathbf{e}_j &= \lambda_j\mathbf{e}_j\end{aligned}\right\}$ give $(\mathbf{A}-\mathbf{B})\mathbf{e}_j = \mathbf{0}$. Let $\mathbf{r}_i$ denote the ith row vector in $\mathbf{A}-\mathbf{B}$. Then $(\mathbf{A}-\mathbf{B})\mathbf{e}_j = \mathbf{0}$ requires that $\mathbf{r}_i\cdot\mathbf{e}_j = 0$ for each $j = 1,\dots,n$. If $\mathbf{r}_i$ is nonzero then $\{\mathbf{e}_1,\dots,\mathbf{e}_n,\mathbf{r}_i\}$ is LI. But we can't have $n+1$ LI vectors in n-space, so it

must be that $\underset{\sim}{r}_i = \underset{\sim}{0}$. Since that is true for each $i = 1, \dots, n$, it follows that $\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{O}$, or, $\underset{\sim}{A} = \underset{\sim}{B}$.

21. No, we know (from Theorem 9.9.3) that $\dim \mathbb{R}^n = n$ and we know (from Definition 9.9.2) that the dimension is the <u>greatest</u> <u>possible</u> number of LI vectors in the space; thus, we cannot have more than n LI vectors in $\mathbb{R}^n$.

22. (b)
$$\underset{\sim}{A} = \begin{pmatrix} 8 & 10 & 12 \\ 9 & 10 & 11 \\ 10 & 10 & 10 \end{pmatrix} = 30 \begin{pmatrix} 8/30 & 10/30 & 12/30 \\ 9/30 & 10/30 & 11/30 \\ 10/30 & 10/30 & 10/30 \end{pmatrix}$$
← This is a Markov matrix so it has $\lambda = 1$ among its eigenvalues. Then, by Exercise 11, $\underset{\sim}{A}$ has $\lambda = 30(1) = 30$ among its eigenvalues.

(c)
$$\underset{\sim}{A} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} = 5 \begin{pmatrix} .6 & .2 & .4 \\ .4 & .4 & .2 \\ 0 & .4 & .4 \end{pmatrix}$$
← This is a Markov matrix (the sum of each column is 1 and each $a_{ij} \geq 0$) so it has $\lambda = 1$ among its eigenvalues. Then, by Exercise 11, $\underset{\sim}{A}$ has $\lambda = 5(1) = 5$ among its eigenvalues.

(d)
$$\underset{\sim}{A} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = 20 \begin{pmatrix} .05 & \cdots & .05 \\ \vdots & & \vdots \\ .05 & \cdots & .05 \end{pmatrix}$$
← This is a Markov matrix, so it has $\lambda = 1$ among its eigenvalues. Thus, $\underset{\sim}{A}$ has $\lambda_1 = 20(1) = 20$ among its eigenvalues. To find its eigenspace use Gauss elimination: $(\underset{\sim}{A} - 20\underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ gives

$$\left[\begin{array}{cccccc} -19 & 1 & 1 & \cdots & 1 & 0 \\ 1 & -19 & 1 & \cdots & 1 & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & 1 & \cdots & 19 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccccc} 1 & -19 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & -19 & 1 & \cdots & 1 & 0 \\ \vdots & & & & & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 19 & 0 \\ -19 & 1 & 1 & 1 & \cdots & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss}} \left[\begin{array}{ccccccccc} 1 & -19 & 1 & 1 & 1 & \cdots & 1 & & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & & 0 \\ & \vdots & & & & \ddots & & & \vdots \\ & & & & & & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & & & 0 \end{array}\right]$$

gives $x_{20} = \alpha$, then $x_{19} = \alpha, \cdots, x_1 = \alpha$, so $\underset{\sim}{e}_1 = \alpha(1, 1, \dots, 1)^T$.

Also, it is obvious by inspection that 0 is an eigenvalue of A, since $\det \underset{\sim}{A}$ clearly $= 0$. To find its eigenspace use Gauss elimination: $(\underset{\sim}{A} - 0\underset{\sim}{I})\underset{\sim}{x} = 0$ gives

$$\left[\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_{20} = \beta_1,\ x_{19} = \beta_2, \dots, x_2 = \beta_{19}, \\ x_1 = -\beta_1 - \beta_2 - \cdots - \beta_{19} \end{array}$$

$$\text{so } \underset{\sim}{e} = \begin{bmatrix} -\beta_1 - \cdots - \beta_{19} \\ \beta_{19} \\ \beta_{18} \\ \vdots \\ \beta_1 \end{bmatrix} = \beta_1 \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} + \cdots + \beta_{19} \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for } \lambda = 0$$

$$\text{and } \underset{\sim}{e} = \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \text{ for } \lambda = 20.$$

Observe that the union of these two eigenspaces is all of $\mathbb{R}^{20}$, so every vector in $\mathbb{R}^{20}$ is in one of these eigenspaces. Thus, our search is completed; there are <u>only</u> the eigenvalues $\lambda = 0$ and $\lambda = 20$, and their eigenspaces are as given above. The eigenspace of $\lambda = 20$ is 1-dimensional and that of $\lambda = 0$ is 19-dim.

(e) Same idea as in (d). The result will be $\lambda = 60$ with the eigenspace $\underset{\sim}{e} = \alpha \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

and $\lambda = 0$ with the eigenspace $\underset{\sim}{e} = \beta_1 \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} + \cdots + \beta_{19} \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The eigenspace of $\lambda = 60$ is 1-dimensional and that of $\lambda = 0$ is 29-dimensional.

23. (b) $x(t) = q_1 e^{rt}$ and $y(t) = q_2 e^{rt}$ give $\begin{aligned} r q_1 \cancel{e^{rt}} &= q_1 \cancel{e^{rt}} + q_2 \cancel{e^{rt}} \\ r q_2 \cancel{e^{rt}} &= q_1 \cancel{e^{rt}} + q_2 \cancel{e^{rt}} \end{aligned}$

so $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = r \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, which is an eigenvalue problem with $\lambda = r$.

$\lambda_1 = 2, \ \underset{\sim}{e}_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = 0, \ \underset{\sim}{e}_2 = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thus, $\underset{\sim}{x}(t) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ and $\underset{\sim}{x}(t) = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{0t}$. Using linearity and superposition,

$\underset{\sim}{x}(t) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or, in scalar form, $\begin{aligned} x(t) &= \alpha e^{2t} + \beta \\ y(t) &= \alpha e^{2t} - \beta \end{aligned}$

(d) $x(t) = q_1 e^{rt}$ and $y(t) = q_2 e^{rt}$ give $\begin{aligned} r^2 q_1 &= q_1 + q_2 \\ r^2 q_2 &= q_1 + q_2 \end{aligned}$

so $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = r^2 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

$\lambda_1 = r_1^2 = 2, \ \underset{\sim}{e}_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = r_1^2 = 0, \ \underset{\sim}{e}_2 = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so $r_1 = \pm\sqrt{2}$

Thus, $\underset{\sim}{x}(t) = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\sqrt{2}t}, \ \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\sqrt{2}t}, \ \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{0t}$

so $\underset{\sim}{x}(t) = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\sqrt{2}t} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\sqrt{2}t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or, in scalar form,

$$\begin{aligned} x(t) &= \alpha_1 e^{+\sqrt{2}t} + \alpha_2 e^{-\sqrt{2}t} + \beta \\ y(t) &= \alpha_1 e^{+\sqrt{2}t} + \alpha_2 e^{-\sqrt{2}t} - \beta . \end{aligned}$$

The latter falls short of being the general solution because it contains only 3 arbitrary constants rather than 4. Evidently, the missing solution must not be of the assumed exponential form. Actually, the full solution is

$$\begin{aligned} x(t) &= \alpha_1 e^{+\sqrt{2}t} + \alpha_2 e^{-\sqrt{2}t} + \beta_1 + \beta_2 t \\ y(t) &= \alpha_1 e^{+\sqrt{2}t} + \alpha_2 e^{-\sqrt{2}t} - \beta_1 - \beta_2 t . \end{aligned} \qquad ♀$$

Sure enough, the missing terms, $\pm\beta_2 t$, are not of the assumed exponential form. To explore this point further, suppose we solve using the method of elimination instead (Sec. 3.9.3). Then we obtain the uncoupled ODE's

$$\begin{aligned} x'''' - 2x'' &= 0 \\ y'''' - 2y'' &= 0 \end{aligned} \qquad ★$$

Just as we miss the $\pm\beta_2 t$ terms in ♀ when we seek $x(t) = q_1 e^{rt}, y(t) = q_2 e^{rt}$, likewise we miss them when we try to solve ★ by seeking $\underset{\sim}{x}(t) = e^{rt}$

and $y(t) = e^{rt}$, for we get $r^4 - 2r^2 = 0$, so $r = 0, 0, +\sqrt{2}, -\sqrt{2}$. The repeated root 0 gives us only one solution so we obtain $x(t) = Ae^{0t} + Be^{\sqrt{2}t} + Ce^{-\sqrt{2}t}$ and similarly for $y(t)$. We can indeed find the missing solution — by other means, because it is not of the form e^{rt} — and find that $x(t) = (A+Dt)e^{0t} + Be^{\sqrt{2}t} + Ce^{-\sqrt{2}t}$ and $y(t) =$ etc.

(f) We obtain
$$\begin{pmatrix} 0 & 1 & -1 \\ -5 & 4 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = r\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$
and find these eigenpairs (where $\lambda = r$)
$$\lambda_1 = 0,\ \underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};\quad \lambda_2 = 1,\ \underset{\sim}{e}_2 = \beta\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix};\quad \lambda_3 = 2,\ \underset{\sim}{e}_3 = \gamma\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$$
so
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{0t} + \beta\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}e^{t} + \gamma\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}e^{2t},$$
which is the general solution.

(g) We obtain
$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = r\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$
and find these eigenpairs: $\lambda = r = 1, 1, 1,\ \underset{\sim}{e} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

so
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{t}$$

which is not the general solution, the shortfall occurring because $\lambda = 1$ is a repeated root.

(h) We obtain
$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = r^2\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$
and find these eigenpairs: $\lambda = r^2 = \overset{\swarrow r = \pm i}{-1}, -1, -1,\ \underset{\sim}{e} = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

so
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \alpha_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{it} + \alpha_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{-it}$$
or, equivalently,
$$x(t) = A\cos t + B\sin t$$
$$y(t) = 0$$
$$z(t) = A\cos t + B\sin t$$
which is not the general solution.

24.(a) $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$, so we want to verify that
$$\underset{\sim}{A}^2 - (a+d)\underset{\sim}{A} + (ad-bc)\underset{\sim}{I} = \underset{\sim}{0}.$$
Well,
$$\underset{\sim}{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix}$$
so
$$\underset{\sim}{A}^2 - (a+d)\underset{\sim}{A} + (ad-bc)\underset{\sim}{I} = \begin{pmatrix} a^2+bc-(a+d)a+(ad-bc) & ab+bd-(a+d)b+0 \\ ac+dc-(a+d)c+0 & bc+d^2-(a+d)d+(ad-bc) \end{pmatrix} \text{ does} = \underset{\sim}{0}$$

(b) Since $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$, it follows from the Cayley-Hamilton theorem that $\underset{\sim}{A}^2 - 4\underset{\sim}{A} + 3\underset{\sim}{I} = \underset{\sim}{0}$. If we multiply the latter equation by $\underset{\sim}{A}^{-1}$ we obtain

$$\underset{\sim}{A} - 4\underset{\sim}{I} + 3\underset{\sim}{A}^{-1} = \underset{\sim}{0}$$

and if we solve for $\underset{\sim}{A}^{-1}$ we obtain $\underset{\sim}{A}^{-1} = \frac{4}{3}\underset{\sim}{I} - \frac{1}{3}\underset{\sim}{A} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$.

25. (b) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, or, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, so $\begin{pmatrix} 1-\lambda & 0 \\ -\lambda & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then, $\begin{vmatrix} 1-\lambda & 0 \\ -\lambda & 1-\lambda \end{vmatrix} = 0$ gives $\lambda = 1$, and $(\underset{\sim}{A} - \overset{1}{\lambda}\underset{\sim}{B})\underset{\sim}{x} = \underset{\sim}{0}$ gives $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \to x_1 = 0, x_2 = \alpha$ so $\lambda = 1$ and $\underset{\sim}{e} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) $|\underset{\sim}{A} - \lambda\underset{\sim}{B}| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-2\lambda \end{vmatrix} = 2\lambda^2 - 6\lambda + 3 = 0$ gives $\lambda = (3 \pm \sqrt{3})/2$. Putting this into $(\underset{\sim}{A} - \lambda\underset{\sim}{B})\underset{\sim}{x} = \underset{\sim}{0}$ gives: $\lambda_1 = (3+\sqrt{3})/2$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix}$

$\lambda_2 = (3-\sqrt{3})/2$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} -1+\sqrt{3} \\ 1 \end{pmatrix}$

(d) We obtain

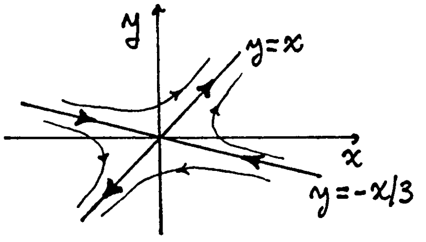
$\lambda_1 = \frac{1}{4}$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_2 = \lambda_3 = 1$, $\underset{\sim}{e} = \beta\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

26. Unfortunately, no, as some simple examples will show.

27. (a) $x' = x + y$
$y' = 3x - y$, $\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$, $\lambda_1 = +2$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -2$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Thus, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + \beta\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^{-2t}$

or, $x(t) = \alpha e^{2t} + 3\beta e^{-2t}$
$y(t) = \alpha e^{2t} - \beta e^{-2t}$.



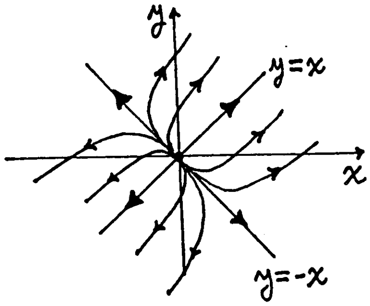
If the initial conditions are such that $\alpha = 0$, then we have the straight line solution $y = -\frac{1}{3}x$, with points on that line approaching the origin as $t \to +\infty$. If the initial conditions are such that $\beta = 0$, then we have the straight line solution $y = x$, with points on that line approaching the origin as $t \to -\infty$. Thus, the phase portrait is as sketched above, and is a saddle.

(b) $x' = -x - 3y$
$y' = x + y$, $\underset{\sim}{A} = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$, $\lambda = \pm\sqrt{2}i$, hence, a center

(c) $\underset{\sim}{A} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, $\lambda_1 = 3$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\lambda_2 = 5$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{3t} + \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{5t}$

or, $x(t) = \alpha e^{3t} + \beta e^{5t}$
$y(t) = -\alpha e^{3t} + \beta e^{5t}$.



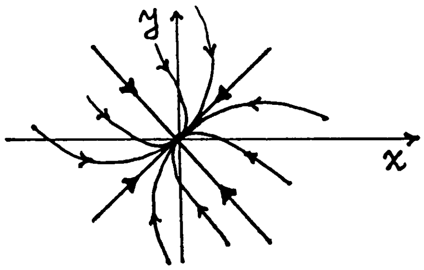
If the initial conditions are such that $\beta=0$, then we have the straight line solution $y=-x$, with points on that line approaching the origin as $t\to-\infty$. If the initial conditions are such that $\alpha=0$, then we have the straight line solution $y=+x$, with points on that line approaching the origin as $t\to-\infty$. For points not on those straight lines we see from the solution that $x(t)\sim\beta e^{5t}$ and $y(t)\sim\beta e^{5t}$ as $t\to+\infty$.

Thus, the phase portrait is as sketched above, and represents an unstable improper node.

(d) $x' = -3x+y$
$y' = x-3y$, $\underset{\sim}{A}=\begin{pmatrix}-3 & 1\\ 1 & -3\end{pmatrix}$, $\lambda_1=-2$, $\underset{\sim}{e}_1=\alpha\begin{pmatrix}1\\1\end{pmatrix}$; $\lambda_2=-4$, $\underset{\sim}{e}_2=\beta\begin{pmatrix}1\\-1\end{pmatrix}$.

Thus, $\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\alpha\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+\beta\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}$

or, $x(t)=\alpha e^{-2t}+\beta e^{-4t}$
$y(t)=\alpha e^{-2t}-\beta e^{-4t}$



If the initial conditions are such that $\beta=0$, then we have the straight line solution $y=x$, with points on that line approaching the origin as $t\to+\infty$. If the initial conditions are such that $\alpha=0$, then we have the straight line solution $y=-x$, with points on that line approaching the origin as $t\to+\infty$. For points not on those straight lines we see from the solution that $x(t)\sim\beta e^{-4t}$ and $y(t)\sim-\beta e^{-4t}$ as $t\to-\infty$, so $y\sim-x$ as $t\to-\infty$. Thus, the phase portrait is as sketched above, and represents a stable improper node.

(e) $x'=3x+y$
$y'=-x+3y$, $\underset{\sim}{A}=\begin{pmatrix}3 & 1\\ -1 & 3\end{pmatrix}$, $\lambda=3\pm i$, so unstable focus

(f) $x'=-2x-2y$
$y'=2x-2y$, $\underset{\sim}{A}=\begin{pmatrix}-2 & -2\\ 2 & -2\end{pmatrix}$, $\lambda=-2\pm 2i$, so stable focus

We don't need to go further, to answer the question, but let us complete the solution. $\lambda_1=-2+2i$ gives $\underset{\sim}{e}_1=\alpha\begin{pmatrix}-1\\i\end{pmatrix}$ and $\lambda_2=-2-2i$ gives $\underset{\sim}{e}_2=\beta\begin{pmatrix}-1\\-i\end{pmatrix}\equiv\gamma\begin{pmatrix}1\\i\end{pmatrix}$

so $\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\alpha\begin{pmatrix}-1\\i\end{pmatrix}e^{(-2+2i)t}+\gamma\begin{pmatrix}1\\i\end{pmatrix}e^{(-2-2i)t}$

or, $x(t)=e^{-2t}[-\alpha(\cos 2t+i\sin 2t)+\gamma(\cos 2t-i\sin 2t)]$
$y(t)=e^{-2t}[i\alpha(\quad\text{"}\quad)+i\gamma(\quad\text{"}\quad)]$

or, $x(t)=e^{-2t}(C_1\cos 2t-C_2\sin 2t)$ where $C_1=\gamma-\alpha$
$y(t)=e^{-2t}(C_2\cos 2t+C_1\sin 2t)$ and $C_2=i(\gamma+\alpha)$

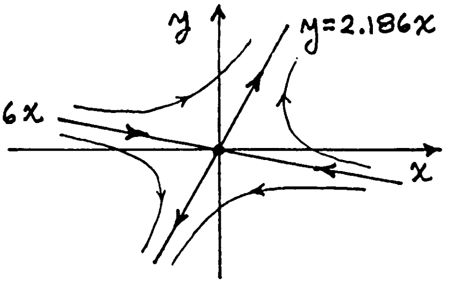
(g) $x'=x+2y$
$y'=3x+4y$, $\underset{\sim}{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $\lambda_1=\frac{5+\sqrt{33}}{2}\approx 5.37$, $\underset{\sim}{e}_1\approx\alpha\begin{pmatrix}1\\2.186\end{pmatrix}$;
$\lambda_2=\frac{5-\sqrt{33}}{2}\approx -0.37$, $\underset{\sim}{e}_2\approx\beta\begin{pmatrix}1\\-0.686\end{pmatrix}$.

Thus,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2.186 \end{pmatrix} e^{5.37t} + \beta \begin{pmatrix} 1 \\ -0.686 \end{pmatrix} e^{-0.37t}$$

or, $x(t) = \alpha e^{5.37t} + \beta e^{-0.37t}$

$y(t) = 2.186 \alpha e^{5.37t} - 0.686 \beta e^{-0.37t}$.



If the initial conditions are such that $\beta=0$, then we have the straight line solution $y=2.186x$, with points on that line approaching the origin as $t\to-\infty$. If the initial conditions are such that $\alpha=0$, then we have the straight line solution $y=-0.686x$, with points on that line approaching the origin as $t\to+\infty$. Thus, the phase portrait is as sketched above, and represents a <u>saddle</u>.

(h) $x' = 5x + y$

$y' = -8x + y$, $\underset{\sim}{A} = \begin{pmatrix} 5 & 1 \\ -8 & 1 \end{pmatrix}$, $\lambda = 3 \pm 2i$ so <u>unstable focus</u>.

NOTE: Section 11.2.2 is extremely brief, so you may wish to discuss the foregoing exercise in class.

Section 11.3

1. (b) $|\underset{\sim}{A} - \lambda \underset{\sim}{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 1 = 0$, $\lambda_1 = 1+\sqrt{2}$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$; $\lambda_2 = 1-\sqrt{2}$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$

An orthogonal basis of $\mathbb{R}^2$ is given by $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$. With $\alpha=\beta=1$, say, we have the orthog. basis $\left\{ \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix} \right\}$

(c) $|\underset{\sim}{A} - \lambda \underset{\sim}{I}| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2-1)$ so $\lambda_1 = 0$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_2 = 1$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$;

$\lambda_3 = -1$, $\underset{\sim}{e}_3 = \gamma\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ so an orthog. basis of $\mathbb{R}^3$ is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

(e) $|\underset{\sim}{A} - \lambda \underset{\sim}{I}| = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = -(\lambda-8)(\lambda-2)^2 = 0$

$\lambda_1 = 8$: $\begin{pmatrix} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ so $x_3 = \alpha$, $x_2 = \alpha$, $x_1 = \alpha$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 2$: $\begin{pmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$ $x_3 = \beta$, $x_2 = \gamma$, $x_1 = -\beta-\gamma$, $\underset{\sim}{e} = \begin{pmatrix} -\beta-\gamma \\ \gamma \\ \beta \end{pmatrix} = \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

To find two orthogonal vectors with the latter eigenspace, let $\underset{\sim}{e}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and let $\underset{\sim}{e}_3 = \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ such that $\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 2\beta + \gamma = 0$. Let $\beta=1$ and $\gamma=-2$, say. Then $\underset{\sim}{e}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Thus, an orthog. basis for $\mathbb{R}^3$ (from among the eigenvectors of $\underset{\sim}{A}$) is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

For (f), (h), (i), (j) I'll use Maple:

(f) $\lambda_1 = -9$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$; $\lambda_2 = \lambda_3 = 9$, $\underset{\sim}{e} = \beta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Let $\underset{\sim}{e}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{e}_3 = \beta\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ such that $\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 5\beta - 4\gamma = 0$. Let $\beta=4$ and $\gamma=5$, say, so $\underset{\sim}{e}_3 = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$. Thus, an orthog. basis is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$.

(h) $\lambda_1 = 6,\ \underset{\sim}{e}_1 = \alpha\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$; $\lambda_2=\lambda_3=\lambda_4=-3,\ \underset{\sim}{e} = \beta\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}+\delta\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

In the latter eigenspace the last vector is orthog. to the first two so it suffices to form two orthog. vectors from the first two. Let $\underset{\sim}{e}_2 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ and $\underset{\sim}{e}_3 = \beta\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$ such that $\underset{\sim}{e}_2\cdot\underset{\sim}{e}_3 = 2\beta+\gamma=0$. Letting $\beta=1$ and $\gamma=-2$, say, we have $\underset{\sim}{e}_3 = \begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$, so, with $\underset{\sim}{e}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, the desired orthog. basis is $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

(i) $\lambda_1=\lambda_2=0,\ \underset{\sim}{e} = \alpha\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$. As above, in this eigenspace we find the orthog. vectors $\underset{\sim}{e}_1 = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$ and $\underset{\sim}{e}_2 = \begin{pmatrix}0\\1\\1\\-2\end{pmatrix}$

$\lambda_3=\lambda_4=6,\ \underset{\sim}{e} = \gamma\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\delta\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$, in which eigenspace $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$ are orthog.

Thus, the desired orthog. basis is $\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$

(j) $\lambda_1=\lambda_2=\lambda_3=0,\ \underset{\sim}{e} = \alpha\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $\lambda_4=2,\ \underset{\sim}{e}_4=\delta\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

so the desired orthog. basis is $\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$.

2. (b) $\lambda_1=4,\ \underset{\sim}{e}_1=\alpha\begin{pmatrix}3\\1\end{pmatrix}$; $\lambda_2=-2,\ \underset{\sim}{e}_2=\beta\begin{pmatrix}-3\\1\end{pmatrix}$.

No, the eigenvectors of $\underset{\sim}{A}$ do not provide an orthog. basis for $\mathbb{R}^2$.

(c) $\lambda_1=\lambda_2=0,\ \underset{\sim}{e}=\alpha\begin{pmatrix}0\\1\end{pmatrix}$

3. (a) $\begin{vmatrix}a-\lambda & b\\ b & d-\lambda\end{vmatrix} = \lambda^2-(a+d)\lambda+(ad-b^2)=0$, $\lambda = \dfrac{a+d\pm\sqrt{(a+d)^2-4(ad-b^2)}}{2}$
$= \dfrac{a+d\pm\sqrt{(a-d)^2+4b^2}}{2}$, which are real because $(a-d)^2+4b^2\geq 0$.

(b) Taking the complex conjugate of (3.1), $\overline{\underset{\sim}{A}\underset{\sim}{e}} = \overline{\lambda\underset{\sim}{e}}$. Since $\underset{\sim}{A}$ is real, surely $\overline{\underset{\sim}{A}\underset{\sim}{e}} = \underset{\sim}{A}\overline{\underset{\sim}{e}}$. Also, $\overline{\lambda\underset{\sim}{e}} = \overline{\lambda}\,\overline{\underset{\sim}{e}}$ because $\overline{z_1z_2}=\overline{z}_1\overline{z}_2$ for any complex numbers z_1 and z_2. Thus, (3.1)$\Rightarrow$(3.2). Next, (3.3)$\Rightarrow$(3.4) by virtue of the properties $\underset{\sim}{x}\cdot\underset{\sim}{y} = \underset{\sim}{x}^T\underset{\sim}{y}$ and $(\underset{\sim}{A}\underset{\sim}{B})^T = \underset{\sim}{B}^T\underset{\sim}{A}^T$. Next, subtract the two equations in (3.4) and recall that $\underset{\sim}{A}^T=\underset{\sim}{A}$ by assumption, to obtain (3.5). Finally, denote $\underset{\sim}{e}=\begin{pmatrix}e_1\\ \vdots\\ e_n\end{pmatrix}$. Then $\underset{\sim}{e}^T\overline{\underset{\sim}{e}} = e_1\overline{e}_1+\cdots+e_n\overline{e}_n = |e_1|^2+\cdots+|e_n|^2 \neq 0$ because $\underset{\sim}{e}\neq\underset{\sim}{0}$. Thus, (3.5)$\Rightarrow$(3.6).

4. From the solution to Exercise 3(a) we see that λ will be of multiplicity 2 iff $a=d$ and $b=0$, in which case $\underset{\sim}{A}=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, with $\lambda_1=\lambda_2=a$ and $\underset{\sim}{e}=\begin{pmatrix} \alpha \\ \beta \end{pmatrix}=\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which is 2-dimensional.

5. Let us do only the left-hand equations over, in (2).

$$(\underset{\sim}{A}\underset{\sim}{e}_j)\cdot\underset{\sim}{e}_k=(\lambda_j\underset{\sim}{e}_j)\cdot\underset{\sim}{e}_k$$
$$(\underset{\sim}{A}\underset{\sim}{e}_j)^T\underset{\sim}{e}_k=(\lambda_j\underset{\sim}{e}_j)^T\underset{\sim}{e}_k$$
$$e_j^TA^Te_k=\lambda_j\,e_j^Te_k$$

But $\underset{\sim}{x}\cdot\underset{\sim}{y}=\underset{\sim}{x}^T\underset{\sim}{y}=\underset{\sim}{y}^T\underset{\sim}{x}$, so we can write the latter as

$$(A^Te_k)^Te_j=\lambda_j\,e_k^Te_j$$

or,

$$e_k^TA^{TT}e_j=\lambda_j\,e_k^Te_j$$
$$e_k^TAe_j=\lambda_j\,e_k^Te_j,$$

as in (2).

6. Re-express $\quad x_1''+2x_1-x_2=0\ ;\quad x_1(0)=x_{1_0},\ x_1'(0)=x_{1_0}'$
$\qquad\qquad\qquad x_2''-x_1+2x_2=0\ ;\quad x_2(0)=x_{2_0},\ x_2'(0)=x_{2_0}'$

as the first-order system

$$x_1'=u\ ;\quad x_1(0)=x_{1_0}$$
$$u'=-2x_1+x_2\ ;\quad u(0)=x_{1_0}'$$
$$x_2'=v\ ;\quad x_2(0)=x_{2_0}$$
$$v'=x_1-2x_2\ ;\quad v_1(0)=x_{2_0}'$$

Then it follows from Theorem 3.9.1 that the latter (and hence the former, to which it is equivalent) admits a unique solution on the interval $-\infty<t<\infty$.

7. (b) $\quad x_1(t)=\alpha\sin(t+\phi_1)+\beta\sin(\sqrt{3}\,t+\phi_2)$
$\qquad\quad x_2(t)=\alpha\sin(t+\phi_1)-\beta\sin(\sqrt{3}\,t+\phi_2)$

$$x_1(0)=1=\alpha\sin\phi_1+\beta\sin\phi_2 \qquad ①$$
$$x_2(0)=0=\alpha\sin\phi_1-\beta\sin\phi_2 \qquad ②$$
$$x_1'(0)=0=\alpha\cos\phi_1+\sqrt{3}\,\beta\cos\phi_2 \qquad ③$$
$$x_2'(0)=-3=\alpha\cos\phi_1-\sqrt{3}\,\beta\cos\phi_2 \qquad ④$$

①+② $\Rightarrow$ $1=2\alpha\sin\phi_1$ ⑤

③+④ $\Rightarrow$ $-3=2\alpha\cos\phi_1$ ⑥

①−② $\Rightarrow$ $1=2\beta\sin\phi_2$ ⑦

③−④ $\Rightarrow$ $3=2\sqrt{3}\,\beta\cos\phi_2$ ⑧

Then, $⑤^2+⑥^2 \Rightarrow 4\alpha^2=10,\ \alpha=\sqrt{10}/2\approx1.581$ *

$⑤ \Rightarrow \phi_1=\sin^{-1}(1/2\alpha)=\sin^{-1}0.3162=0.3217$ rad

$⑦^2+⑧^2 \Rightarrow 4\beta^2=10,\ \beta=\sqrt{10}/2\approx1.581$ *

$⑦ \Rightarrow \phi_2=\sin^{-1}(1/2\beta)=\sin^{-1}0.3162\approx0.3217$ rad

---

* The negative square roots will not yield anything new.

Thus, $x_1(t) = 1.581\sin(t+0.3217) + 1.581\sin(\sqrt{3}t+0.3217)$
$x_2(t) =$ " − " .

(c) As in (b),
$$\begin{aligned} x_1(0) = 0 &= \alpha\sin\phi_1 + \beta\sin\phi_2 \\ x_2(0) = 0 &= \alpha\sin\phi_1 - \beta\sin\phi_2 \\ x_1'(0) = 0 &= \alpha\cos\phi_1 + \sqrt{3}\beta\cos\phi_2 \\ x_2'(0) = 5 &= \alpha\cos\phi_1 - \sqrt{3}\beta\cos\phi_2 \end{aligned}$$

which give
$$\left.\begin{aligned} 2\alpha\sin\phi_1 &= 0 \\ 2\alpha\cos\phi_1 &= 5 \end{aligned}\right> \phi_1 = 0,\ \alpha = 5/2$$
$$\left.\begin{aligned} 2\beta\sin\phi_2 &= 0 \\ 2\sqrt{3}\beta\cos\phi_2 &= -5 \end{aligned}\right> \phi_2 = 0,\ \beta = -5/2\sqrt{3}$$

so $x_1(t) = \frac{5}{2}\sin t + \frac{5}{2\sqrt{3}}\sin\sqrt{3}t$
$x_2(t) =$ " − "

8. (a) For definiteness, suppose that $x_1 > x_2 > x_3$. Then the eqns. of motion are
$$\begin{aligned} x_1'' &= -x_1 - (x_1 - x_2) &&= -2x_1 + x_2 \\ x_2'' &= +(x_1 - x_2) - (x_2 - x_3) &&= x_1 - 2x_2 + x_3 \\ x_3'' &= (x_2 - x_3) - x_3 &&= x_2 - 2x_3 \end{aligned}$$

(b) Seeking $x_j = q_j\sin(\omega t + \phi)$, obtain
$$\begin{pmatrix} +2 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \overset{\lambda}{\overset{\downarrow}{\boxed{\omega^2}}}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Maple gives the eigens
$\lambda_1 = 2 - \sqrt{2} = 0.586$, $\underset{\sim}{e}_1 = \alpha(1, \sqrt{2}, 1)^T$, $\omega_1 = \sqrt{\lambda_1} = 0.765$
$\lambda_2 = 2$, $\underset{\sim}{e}_2 = \beta(1, 0, -1)^T$, $\omega_2 = \sqrt{\lambda_2} = 1.414$
$\lambda_3 = 2 + \sqrt{2} = 3.414$, $\underset{\sim}{e}_3 = \gamma(1, -\sqrt{2}, 1)^T$, $\omega_3 = \sqrt{\lambda_3} = 1.848$

so $x_1(t) = \alpha\sin(0.765t + \phi_1) + \beta\sin(1.414t + \phi_2) + \gamma\sin(1.848t + \phi_3)$
$x_2(t) = \sqrt{2}\alpha\sin($ " $) + 0\beta\sin($ " $) - \sqrt{2}\gamma\sin($ " $)$
$x_3(t) = \alpha\sin($ " $) - \beta\sin($ " $) + \gamma\sin($ " $)$,
the natural frequencies being $\omega_1, \omega_2, \omega_3$ and their mode shapes being defined by $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$.

(d)
$$\begin{aligned} x_1(0) = 1 &= \alpha\sin\phi_1 + \beta\sin\phi_2 + \gamma\sin\phi_3 \\ x_2(0) = 0 &= \sqrt{2}\alpha\sin\phi_1 - \sqrt{2}\gamma\sin\phi_3 \\ x_3(0) = 0 &= \alpha\sin\phi_1 - \beta\sin\phi_2 + \gamma\sin\phi_3 \\ x_1'(0) = 0 &= 0.765\alpha\cos\phi_1 + 1.414\beta\cos\phi_2 + 1.848\gamma\cos\phi_3 \\ x_2'(0) = 0 &= 0.765\sqrt{2}\alpha\cos\phi_1 - 1.848\sqrt{2}\gamma\cos\phi_3 \\ x_3'(0) = 0 &= 0.765\alpha\cos\phi_1 - 1.414\beta\cos\phi_2 + 1.848\gamma\cos\phi_3 \end{aligned}$$

The are <u>nonlinear</u> equations in $\alpha, \beta, \gamma, \phi_1, \phi_2, \phi_3$, but they are <u>linear</u> equations in the groupings $\alpha\sin\phi_1$, $\beta\sin\phi_2$, $\gamma\sin\phi_3$, $\alpha\cos\phi_1$, $\beta\cos\phi_2$, $\gamma\cos\phi_3$. Using the Maple linsolve command, the first 3 eqns. give
$$\begin{aligned} \alpha\sin\phi_1 &= 1/4 \\ \beta\sin\phi_2 &= 1/2 \\ \gamma\sin\phi_3 &= 1/4 \end{aligned}$$
and the last 3 eqns. give
$$\begin{aligned} \alpha\cos\phi_1 &= 0 \\ \beta\cos\phi_2 &= 0 \\ \gamma\cos\phi_3 &= 0 \end{aligned}$$

Thus, $\phi_1=\phi_2=\phi_3=\pi/2$, $\alpha=1/4$, $\beta=1/2$, $\gamma=1/4$ so

$$\begin{aligned} x_1(t) &= \tfrac14\sin(0.765t+\pi/2)+\tfrac12\sin(1.414t+\pi/2)+\tfrac14\sin(1.848t+\pi/2)\\ &= 0.25\cos 0.765t+0.5\cos 1.414t+0.25\cos 1.848t\\ x_2(t) &= \tfrac{\sqrt2}{4}\sin(0.765t+\pi/2)-\tfrac{\sqrt2}{4}\sin(1.848t+\pi/2)\\ &= 0.354\cos 0.765t-0.354\cos 0.765t\\ x_3(t) &= 0.25\cos 0.765t-0.5\cos 1.414t+0.25\cos 1.848t \end{aligned}$$

10. $\underset{\sim}{A}=\begin{pmatrix}21&-1\\-1&21\end{pmatrix}$, $\begin{vmatrix}21-\lambda&-1\\-1&21-\lambda\end{vmatrix}=\lambda^2-42\lambda+440=0$

$$\lambda_1=22\ (\omega_1=\sqrt{22}),\quad \underset{\sim}{e}_1=\alpha\begin{pmatrix}1\\-1\end{pmatrix};$$
$$\lambda_2=20\ (\omega_2=\sqrt{20}),\quad \underset{\sim}{e}_2=\beta\begin{pmatrix}1\\1\end{pmatrix},$$

so
$$\underset{\sim}{x}(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\alpha\begin{pmatrix}\sin(\sqrt{22}t+\phi_1)\\-\sin(\quad''\quad)\end{pmatrix}+\beta\begin{pmatrix}\sin(\sqrt{20}t+\phi_2)\\\sin(\sqrt{20}t+\phi_2)\end{pmatrix}.$$

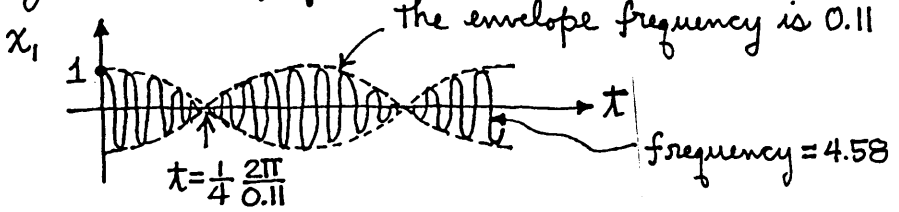
$$\left.\begin{array}{l}\left.\begin{array}{l}x_1(0)=1=\alpha\sin\phi_1+\beta\sin\phi_2\\ x_2(0)=0=-\alpha\sin\phi_1+\beta\sin\phi_2\end{array}\right\}\rightarrow\begin{array}{l}\alpha\sin\phi_1=1/2\\ \beta\sin\phi_2=1/2\end{array}\\ \left.\begin{array}{l}x_1'(0)=0=\sqrt{22}\,\alpha\cos\phi_1+\sqrt{20}\,\beta\cos\phi_2\\ x_2'(0)=0=-\sqrt{22}\,\alpha\cos\phi_1+\sqrt{20}\,\beta\cos\phi_2\end{array}\right\}\rightarrow\begin{array}{l}\alpha\cos\phi_1=0\\ \beta\cos\phi_2=0\end{array}\end{array}\right\}\rightarrow\begin{array}{l}\phi_1=\phi_2=\pi/2\\ \alpha=\beta=1/2\end{array}$$

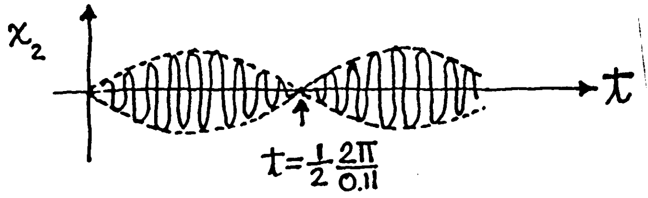
so
$$x_1(t)=\tfrac12(\cos\sqrt{20}t+\cos\sqrt{22}t)=\tfrac12 2\cos\tfrac{\sqrt{20}+\sqrt{22}}{2}t\cos\tfrac{\sqrt{20}-\sqrt{22}}{2}t\approx\cos 4.58t\cos .11t$$
$$x_2(t)=\tfrac12(\quad''\quad-\quad''\quad)=-\tfrac12 2\sin(\quad''\quad)\sin(\quad''\quad)\approx\sin 4.58t\sin .11t$$

We can plot these using maple as follows,

```
with(plots):
implicitplot({x=cos(4.58*t)*cos(.11*t), x=sin(4.58*t)sin(.11*t)},
t=0..60, x=-1.5..1.5, numpoints=1000);
```

but the plots are very "spiky" and inaccurate. Increasing numpoints to 20000, say, gives much improvement but it's still not good enough. Let's merely sketch the graphs (as, indeed, is called for in the question):





Observe the slow energy transfer between the two masses: initially x_1 is vibrating with unit amplitude and x_2 is "quiet"; then the x_2 amplitude increases and x_1 diminishes, and so on.

11. (a) $\mathbf{x} = \sum_1^n a_j \mathbf{e}_j$

$$\mathbf{A}\mathbf{x} = \sum_1^n a_j \mathbf{A}\mathbf{e}_j = \sum_1^n a_j \lambda_j \mathbf{e}_j$$

$$\mathbf{A}\mathbf{x}\cdot\mathbf{x} = \left(\sum_1^n a_j\lambda_j\mathbf{e}_j\right)\cdot\left(\sum_1^n a_k\mathbf{e}_k\right) = \sum_{k=1}^n\sum_{j=1}^n a_j a_k \overbrace{\mathbf{e}_j\cdot\mathbf{e}_k}^{\leftarrow 0 \text{ if } j\neq k}\lambda_j$$

$$= \sum_{j=1}^n a_j^2 \|\mathbf{e}_j\|^2 \lambda_j$$

so

$$|\mathbf{A}\mathbf{x}\cdot\mathbf{x}| = \left|\sum_{j=1}^n \lambda_j a_j^2 \|\mathbf{e}_j\|^2\right| \leq \sum_1^n |\lambda_j| a_j^2 \|\mathbf{e}_j\|^2 \leq |\lambda_1| \sum_1^n a_j^2 \|\mathbf{e}_j\|^2 = |\lambda_1|\,\mathbf{x}\cdot\mathbf{x}$$

so

$$|\lambda_1| \geq \left|\frac{\mathbf{A}\mathbf{x}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}\right| \quad (\text{for all } \mathbf{x}\neq\mathbf{0})$$

(b) $\mathbf{A} = \begin{pmatrix}2&0&0\\0&-1&-2\\0&-2&-1\end{pmatrix}$, λ's $= 1, -3$, so $\lambda_1 = -3$ and $|R(\mathbf{x})|$ should be $\leq |-3| = 3$ for all $\mathbf{x}$.

For $\mathbf{x} = (1,0,0)^T$,

$$|R(\mathbf{x})| = \frac{\left|(1\,0\,0)\begin{pmatrix}2&0&0\\0&-1&-2\\0&-2&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right|}{(1\,0\,0)\begin{pmatrix}1\\0\\0\end{pmatrix}} = 2 \text{ is } \leq 3\ ✓$$

For $\mathbf{x} = (0,1,0)^T$, $|R(\mathbf{x})| = 1$ is ≤ 3 ✓

For $\mathbf{x} = (0,0,1)^T$, $|R(\mathbf{x})| = 1$ is ≤ 3 ✓

For $\mathbf{x} = (1,2,3)^T$, $|R(\mathbf{x})| = 35/14 = 2.5$ is ≤ 3 ✓

For $\mathbf{x} = (3,-1,4)^T$, $|R(\mathbf{x})| = 17/26 = 0.654$ is ≤ 3 ✓

12. (a) $\mathbf{x}^{(0)} = \sum_1^n a_j \mathbf{e}_j$, $\quad \mathbf{x}^{(1)} = \mathbf{A}\mathbf{x}^{(0)} = \sum_1^n a_j \mathbf{A}\mathbf{e}_j = \sum_1^n a_j\lambda_j\mathbf{e}_j$

$$\mathbf{x}^{(2)} = \mathbf{A}\mathbf{x}^{(1)} = \sum_1^n a_j\lambda_j\mathbf{A}\mathbf{e}_j = \sum_1^n a_j\lambda_j^2\mathbf{e}_j, \text{ and so on.}$$

(b) Using Maple,

```
with(linalg):
A:= array([[0,0,1],[0,0,1],[1,1,1]]);
eigenvects(A);
```

gives $\lambda_1 = 2$, $\mathbf{e}_1 = \alpha(1,1,2)^T$; $\lambda_2 = -1$, $\mathbf{e}_2 = \beta(-1,-1,1)^T$; $\lambda_3 = 0$, $\mathbf{e}_3 = \gamma(1,-1,0)^T$

$$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}\overset{\mathbf{x}^{(0)}}{\begin{pmatrix}1\\0\\0\end{pmatrix}}\overset{\mathbf{x}^{(1)}}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\overset{\mathbf{x}^{(2)}}{\begin{pmatrix}1\\1\\1\end{pmatrix}}\overset{\mathbf{x}^{(3)}}{\begin{pmatrix}1\\1\\3\end{pmatrix}}\overset{\mathbf{x}^{(4)}}{\begin{pmatrix}3\\3\\5\end{pmatrix}}\overset{\mathbf{x}^{(5)}}{\begin{pmatrix}5\\5\\11\end{pmatrix}} \text{ so } \mathbf{e}\approx\begin{pmatrix}5\\5\\11\end{pmatrix},\ \lambda\approx\frac{\mathbf{x}^{(4)T}\mathbf{A}\mathbf{x}^{(4)}}{\mathbf{x}^{(4)T}\mathbf{x}^{(4)}} = \frac{\mathbf{x}^{(4)}\cdot\mathbf{x}^{(5)}}{\mathbf{x}^{(4)}\cdot\mathbf{x}^{(4)}} = 1.98$$

$$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}\overset{\mathbf{x}^{(0)}}{\begin{pmatrix}0\\1\\0\end{pmatrix}}\overset{\mathbf{x}^{(1)}}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\overset{\mathbf{x}^{(2)}}{\begin{pmatrix}1\\1\\1\end{pmatrix}}$$ is proceeding as above, giving $\mathbf{e}_1 \approx \begin{pmatrix}5\\5\\11\end{pmatrix}$, $\lambda \approx 1.98$ again

$$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}\overset{\mathbf{x}^{(0)}}{\begin{pmatrix}0\\0\\1\end{pmatrix}}\overset{\mathbf{x}^{(1)}}{\begin{pmatrix}1\\1\\1\end{pmatrix}}$$ is proceeding as above, once again, so we conclude that $\mathbf{e}_1 \approx \begin{pmatrix}5\\5\\11\end{pmatrix}$ (times any nonzero scale factor) and $\lambda_1 \approx 1.98$.

Of course, the <u>exact</u> values are $\mathbf{e}_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\lambda_1 = 2$.

13. (b) Maple gives $\lambda_1 = 4$, $\mathbf{e}_1 = \alpha(1,1,1)^T$; $\lambda_2 = \lambda_3 = 1$, $\mathbf{e} = \beta(1,-1,0)^T + \gamma(1,0,-1)^T$

$$\begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix} \overset{x^{(0)}}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \overset{x^{(1)}}{\begin{pmatrix}2\\1\\1\end{pmatrix}} \overset{x^{(2)}}{\begin{pmatrix}6\\5\\5\end{pmatrix}} \overset{x^{(3)}}{\begin{pmatrix}22\\21\\21\end{pmatrix}} \overset{x^{(4)}}{\begin{pmatrix}86\\85\\85\end{pmatrix}} \overset{x^{(5)}}{\begin{pmatrix}342\\341\\341\end{pmatrix}} \text{ so } \mathbf{e} \approx \begin{pmatrix}342\\341\\341\end{pmatrix}, \lambda \approx \frac{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(5)}}{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(4)}} = \frac{87382}{21846} \approx 3.99991$$

$$\begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix} \overset{x^{(0)}}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \overset{x^{(1)}}{\begin{pmatrix}1\\2\\1\end{pmatrix}} \overset{x^{(2)}}{\begin{pmatrix}5\\6\\5\end{pmatrix}} \overset{x^{(3)}}{\begin{pmatrix}21\\22\\21\end{pmatrix}} \overset{x^{(4)}}{\begin{pmatrix}85\\86\\85\end{pmatrix}} \overset{x^{(5)}}{\begin{pmatrix}341\\342\\341\end{pmatrix}} \text{ so } \mathbf{e} \approx \begin{pmatrix}341\\342\\341\end{pmatrix}, \lambda \approx \frac{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(5)}}{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(4)}} = \frac{87382}{21846} = 3.99991$$

$$\begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix} \overset{x^{(0)}}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \overset{x^{(1)}}{\begin{pmatrix}1\\1\\2\end{pmatrix}} \overset{x^{(2)}}{\begin{pmatrix}5\\5\\6\end{pmatrix}} \overset{x^{(3)}}{\begin{pmatrix}21\\21\\22\end{pmatrix}} \overset{x^{(4)}}{\begin{pmatrix}85\\85\\86\end{pmatrix}} \overset{x^{(5)}}{\begin{pmatrix}341\\341\\342\end{pmatrix}} \text{ so } \mathbf{e} \approx \begin{pmatrix}341\\341\\342\end{pmatrix}, \lambda \approx \text{ " } = \text{ " } = 3.99991$$

We have no reason to favor one $\mathbf{e}$ above the others, so they are equally acceptable approximations to $\mathbf{e}_1$; e.g., $\mathbf{e}_1 \approx (342, 341, 341)^T$, $\lambda_1 \approx 3.99991$

(e) Maple gives $\lambda_1 = 7$, $\mathbf{e}_1 = (0,1,1)^T$; $\lambda_2 = -2$, $\mathbf{e}_2 = (1,-1,1)^T$; $\lambda_3 = 1$, $\mathbf{e}_3 = (2,1,-1)^T$

$$\begin{pmatrix} 0&1&-1\\1&3&4\\-1&4&3 \end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\-1\end{pmatrix} \begin{pmatrix}2\\-1\\1\end{pmatrix} \begin{pmatrix}-2\\3\\-3\end{pmatrix} \begin{pmatrix}6\\-5\\5\end{pmatrix} \begin{pmatrix}-10\\11\\-11\end{pmatrix} \text{ so } \mathbf{e} \approx \begin{pmatrix}-10\\11\\-11\end{pmatrix}, \lambda \approx \frac{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(5)}}{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(4)}} = \frac{-170}{86} = -1.98$$

$$\begin{pmatrix} 0&1&-1\\1&3&4\\-1&4&3 \end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}1\\3\\4\end{pmatrix} \begin{pmatrix}-1\\26\\23\end{pmatrix} \begin{pmatrix}3\\169\\174\end{pmatrix} \begin{pmatrix}-5\\1206\\1195\end{pmatrix} \begin{pmatrix}11\\8393\\8414\end{pmatrix} \text{ so } \mathbf{e} \approx \begin{pmatrix}11\\8393\\8414\end{pmatrix}, \lambda \approx \frac{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(5)}}{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(4)}} = \frac{20176633}{2882486} = 6.9997$$

$$\begin{pmatrix} 0&1&-1\\1&3&4\\-1&4&3 \end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}-1\\4\\3\end{pmatrix} \begin{pmatrix}1\\23\\26\end{pmatrix} \begin{pmatrix}-3\\174\\169\end{pmatrix} \begin{pmatrix}5\\1195\\1206\end{pmatrix} \begin{pmatrix}-11\\8414\\8393\end{pmatrix} \text{ so } \mathbf{e} \approx \begin{pmatrix}-11\\8414\\8393\end{pmatrix}, \lambda \approx \frac{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(5)}}{\mathbf{x}^{(4)} \cdot \mathbf{x}^{(4)}} = \frac{20176633}{2882486} = 6.9997$$

Comparing the three results we conclude that $\lambda_1 \approx 6.9997$ (exact $= 7$) and $\mathbf{e}_1 \approx (11, 8393, 8414)^T$ (exact $= (0,1,1)^T$). The first element in $\mathbf{e}$ does not tend to zero <u>absolutely</u>, but relative to the other two elements it <u>does</u>. Note that the first of the three calculations converged to the <u>next</u> largest λ (namely, $\lambda = -2$) because the initial vector $\mathbf{x}^{(0)} = (1,0,0)^T$ <u>happened</u> to be orthogonal to $\mathbf{e}_1 = (0,1,1)^T$. That is, in the expansion

$$\mathbf{x}^{(0)} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3$$

a_1 is 0, so

$$\begin{aligned} \mathbf{x}^{(0)} &= a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \\ \mathbf{x}^{(1)} &= a_2 \mathbf{A} \mathbf{e}_2 + a_3 \mathbf{A} \mathbf{e}_3 = a_2 \lambda_2 \mathbf{e}_2 + a_3 \lambda_3 \mathbf{e}_3 \\ \mathbf{x}^{(2)} &= \cdots = a_2 \lambda_2^2 \mathbf{e}_2 + a_3 \lambda_3^2 \mathbf{e}_3 \\ &\vdots \\ \mathbf{x}^{(k)} &= a_2 (-2)^k \mathbf{e}_2 + a_3 (1)^k \mathbf{e}_3 \sim a_2 (-2)^k \mathbf{e}_2 \end{aligned}$$

as $k \to \infty$ because $(-2)^k$ grows faster than 1^k. That is, the vector sequence converges to $\mathbf{e}_2$ rather than $\mathbf{e}_1$. (The $a_2(-2)^k$ scale factor is irrelevant because an eigenvector can be scaled arbitrarily anyhow.)

NOTE: It's true that if we start with $\mathbf{x}^{(0)} = (1,0,0)^T$ then we converge to

$\underset{\sim}{e}_2$ and λ_2 — if we use a perfect computer. But a real computer incurs roundoff errors so that, effectively, the initial vector $\underset{\sim}{x}^{(0)}$ is slightly different from $(1,0,0)^T$. Thus, in the equation

$$\underset{\sim}{x}^{(k)} = a_1(7)^k \underset{\sim}{e}_1 + a_2(-2)^k \underset{\sim}{e}_2 + a_3(1)^k \underset{\sim}{e}_3$$

imagine a_1 being very small but not zero. Then $\underset{\sim}{x}^{(k)}$ will approach $a_2(-2)^k\underset{\sim}{e}_2$ and get close to it, but eventually the $(7)^k$ factor will cause $a_1(7)^k$ to dominate, and $\underset{\sim}{x}^{(k)}$ will be attracted to, and converge to, $\underset{\sim}{e}_1$.

(g) NOTE: This $\underset{\sim}{A}$ matrix is not symmetric. Nonetheless, the power method will still work because $\underset{\sim}{A}$ has 4 LI eigenvectors; thus, its $\underset{\sim}{e}_j$'s comprise a basis (though not an orthogonal one) so (12.7) still holds. In fact, Maple gives the eigens of

$$\underset{\sim}{A} = \begin{pmatrix} 2&1&1&2\\0&1&1&0\\0&1&1&0\\2&1&1&2 \end{pmatrix} \text{ as } \lambda_1=4, \underset{\sim}{e}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix};\ \lambda_2=2, \underset{\sim}{e}_2=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix};\ \lambda_3=\lambda_4=0,\ \underset{\sim}{e}=\alpha\underset{\uparrow\ \underset{\sim}{e}_3\text{, say}}{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}+\beta\underset{\uparrow\ \underset{\sim}{e}_4\text{, say}}{\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}}$$

Note that the $\underset{\sim}{e}_j$'s are LI but not orthog. since $\underset{\sim}{e}_1\cdot\underset{\sim}{e}_2 \neq 0$. The power method will give convergence to $\underset{\sim}{e}_1$.

15. NOTE: In Exercise 13(g), above, we noted that the power method works if the eigenvectors of **A** are LI (thus, the method can work even if **A** is not symmetric), so that we can expand the initial vector as in (12.7). Likewise, for the eigenvector expansion method to work we merely need the eigenvectors of **A** to be LI so that they provide a basis; then we can indeed expand **x** and **c** in terms of them, as we've done in Section 11.3.2. The only difference is that the c_j's cannot be calculated conveniently by the formula $c_j = (c\cdot e_j)/(e_j\cdot e_j)$ since that formula is for orthogonal bases; we need to use Gauss elimination or other such method. We will need to do that in Exercises 15(d)-(i).

(d) $\underset{\sim}{A}$ looks symmetric, at first glance, but is not. Maple gives $\lambda_1=4$, $\underset{\sim}{e}_1=(1,0,0,1)^T$; $\lambda_2=2$, $\underset{\sim}{e}_2=(1,-1,-1,1)^T$; $\lambda_3=\lambda_4=0$, $\underset{\sim}{e}=\alpha(1,0,0,-1)^T+\beta(0,1,-1,0)^T$ so we can take $\underset{\sim}{e}_3=(1,0,0,-1)^T$ and $\underset{\sim}{e}_4=(0,1,-1,0)^T$. Note that the $\underset{\sim}{e}$'s are LI but not orthog. since $\underset{\sim}{e}_1\cdot\underset{\sim}{e}_2\neq 0$. Thus, to expand

$$\underset{\sim}{c}=(3,-1,1,0)^T = c_1\underset{\sim}{e}_1 + c_2\underset{\sim}{e}_2 + c_3\underset{\sim}{e}_3 + c_4\underset{\sim}{e}_4$$

write
$$\left.\begin{array}{rl} c_1+c_2+c_3 & =3\\ -c_2 \qquad +c_4 & =-1\\ -c_2 \qquad -c_4 & =1\\ c_1+c_2-c_3 & =0 \end{array}\right\}\quad \begin{array}{l} c_1=3/2,\ c_2=0,\\ c_3=3/2,\ c_4=-1\end{array}$$

$\Lambda=1$ is distinct from the λ_j's, so (33) applies and gives

$$\underset{\sim}{x} = \frac{3/2}{4-1}\underset{\sim}{e}_1 + 0\underset{\sim}{e}_2 + \frac{3/2}{0-1}\underset{\sim}{e}_3 - \frac{1}{0-1}\underset{\sim}{e}_4 = \frac{1}{2}\underset{\sim}{e}_1 - \frac{3}{2}\underset{\sim}{e}_3 + \underset{\sim}{e}_4,\ \text{which} = \begin{pmatrix}-1\\1\\-1\\2\end{pmatrix}$$

(e) The λ's and $\underset{\sim}{e}$'s of $\underset{\sim}{A}$ are given in part (d). In this case $\Lambda=4$ coincides with $\lambda_1=4$ so, according to (ii) on page 562 we need to see if $c_1=0$.

$$\underset{\sim}{c}=(1,2,0,3)=c_1\underset{\sim}{e}_1+c_2\underset{\sim}{e}_2+c_3\underset{\sim}{e}_3+c_4\underset{\sim}{e}_4$$

gives

$$\left.\begin{aligned} c_1+c_2+c_3 \qquad &=1\\ -c_2 \qquad +c_4 &=2\\ -c_2 \qquad -c_4 &=0\\ c_1+c_2-c_3 \qquad &=3 \end{aligned}\right\}\quad \begin{aligned}&c_1=3,\ c_2=-1,\\ &c_3=-1,\ c_4=1.\end{aligned}$$

Since $c_1\neq 0$ there is, according to (ii), no solution.

(f) Same as (e), but this time

$$\left.\begin{aligned} c_1+c_2+c_3 \qquad &=2\\ -c_2 \qquad +c_4 &=0\\ -c_2 \qquad -c_4 &=1\\ c_1+c_2-c_3 \qquad &=-2 \end{aligned}\right\}\quad \begin{aligned}&c_1=1/2,\ c_2=-1/2,\\ &c_3=2,\ c_4=-1/2.\end{aligned}$$

Since $c_1\neq 0$ there is, according to (ii), no solution.

(g) This time $\Lambda=0$ coincides with $\lambda_3=\lambda_4=0$ so, according to (iii), there will be no solution if c_3 and c_4 are not both zero, and a 2-parameter family of solutions if they are zero. Thus, expand

$$\underset{\sim}{c}=(1,3,3,1)^T=c_1\underset{\sim}{e}_1+\cdots+c_4\underset{\sim}{e}_4$$

or,

$$\left.\begin{aligned} c_1+c_2+c_3 \qquad &=1\\ -c_2 \qquad +c_4 &=3\\ -c_2 \qquad -c_4 &=3\\ c_1+c_2-c_3 \qquad &=1 \end{aligned}\right\}\quad \begin{aligned}&c_1=4,\ c_2=-3,\\ &c_3=0,\ c_4=0\end{aligned}$$

so, according to (35), we have the 2-parameter family of solutions

$$\underset{\sim}{x}=\frac{4}{4-0}\underset{\sim}{e}_1-\frac{3}{2-0}\underset{\sim}{e}_2+\alpha\underset{\sim}{e}_3+\beta\underset{\sim}{e}_4=\underset{\sim}{e}_1-\frac{3}{2}\underset{\sim}{e}_2+\alpha\underset{\sim}{e}_3+\beta\underset{\sim}{e}_4\quad(\alpha,\beta\text{ arbitrary})$$

$$=\begin{pmatrix}-1/2\\3/2\\3/2\\-1/2\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+\beta\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

(h) Similar to (g), but we obtain $c_1=5$, $c_2=-5/2$, $c_3=-3/2$, $c_4=-1/2$. Since c_3 and c_4 are not both zero there is no solution.

(i) Similar to (g) but this time $c_1=9/2$, $c_2=-3$, $c_3=-1/2$, $c_4=0$. It's true that $c_4=0$, but c_3 is not zero, so there is no solution. Actually, we could have seen that result by inspection in this case since the original system is

$$\begin{bmatrix}2&1&1&2\\0&1&1&0\\0&1&1&0\\2&1&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\3\\3\\2\end{bmatrix}$$

in which the first and fourth equations (and hence the whole system) are clearly inconsistent.

16. NOTE: The point made in this exercise is an important one, and one which is not confined to this single example, namely, that it is generally true that the basis that is most convenient, in a given application, is the one that is generated by the matrix (or, in other cases, differential) operator that is contained within the given problem. Other examples of this "principle," within this text, are the following: investigation of the stability of the equilibrium population vector in Example 4 of Section 11.2; proof of convergence of the power method in Exercise 12; the eigenfunction expansion of u(x,t) and F(x,t) in Exercise 17 of Section 18.3; and the derivation of the stability condition $r < 1/2$ for the explicit finite-difference solution of the diffusion equation in Exercise 13 of Section 18.6. You may wish to emphasize this point if any of these examples are discussed in class.

Let us compare the two approaches. If, as in the text, we use as our basis the (presumed LI) eigenvectors of $\underset{\sim}{A}$ then we obtain the uncoupled equations (32),

$$(\lambda_1-\Lambda)a_1=c_1,$$
$$(\lambda_2-\Lambda)a_2=c_2,$$

and so on. If, instead, the base vectors chosen $(\underset{\sim}{e}_1,\dots,\underset{\sim}{e}_n)$ are not eigenvectors of $\underset{\sim}{A}$ then (29) gives

$$\sum_1^n a_j\underset{\sim}{A}\underset{\sim}{e}_j=\Lambda\sum_1^n a_j\underset{\sim}{e}_j+\sum_1^n c_j\underset{\sim}{e}_j .$$

Now, instead of $\underset{\sim}{A}\underset{\sim}{e}_j$ equalling $\lambda_j\underset{\sim}{e}_j$, it is simply "some vector" and must be expanded in terms of all of the $\underset{\sim}{e}_j$'s: $\underset{\sim}{A}\underset{\sim}{e}_j=\sum_1^n p_{kj}\underset{\sim}{e}_k$, say, where the p_{kj}'s are computable. Thus,

$$\sum_{k=1}^n\Big(\sum_{j=1}^n p_{kj}a_j\Big)\underset{\sim}{e}_k=\sum_{k=1}^n(\Lambda a_k+c_k)\underset{\sim}{e}_k ,$$

where we have changed the dummy index on the right-hand side from j to k for convenience. The $\underset{\sim}{e}_k$'s linear independence implies that

$$\sum_{j=1}^n p_{kj}a_j=\Lambda a_k+c_k \qquad (k=1,2,\dots,n)$$

or, in matrix form, $\underset{\sim}{P}\underset{\sim}{a}=\Lambda\underset{\sim}{a}+\underset{\sim}{c}$, which is a system of coupled equations for $a_1,\dots,a_n$. In fact, the latter is of the same form as the original coupled system $\underset{\sim}{A}\underset{\sim}{x}=\Lambda\underset{\sim}{x}+\underset{\sim}{c}$.

17. (a)

$$\begin{array}{l|l}
\underset{\sim}{K}\underset{\sim}{e}_j=\lambda_j\underset{\sim}{M}\underset{\sim}{e}_j & \underset{\sim}{K}\underset{\sim}{e}_k=\lambda_k\underset{\sim}{M}\underset{\sim}{e}_k \\
\underset{\sim}{e}_k\cdot\underset{\sim}{K}\underset{\sim}{e}_j=\lambda_j\,\underset{\sim}{e}_k\cdot\underset{\sim}{M}\underset{\sim}{e}_j & \underset{\sim}{K}\underset{\sim}{e}_k\cdot\underset{\sim}{e}_j=\lambda_k\,\underset{\sim}{M}\underset{\sim}{e}_k\cdot\underset{\sim}{e}_j \\
\underset{\sim}{e}_k^T\underset{\sim}{K}\underset{\sim}{e}_j=\lambda_j\,\underset{\sim}{e}_k^T\underset{\sim}{M}\underset{\sim}{e}_j & (\underset{\sim}{K}\underset{\sim}{e}_k)^T\underset{\sim}{e}_j=\lambda_k(\underset{\sim}{M}\underset{\sim}{e}_k)^T\underset{\sim}{e}_j \\
 & \underset{\sim}{e}_k^T\underset{\sim}{K}^T\underset{\sim}{e}_j=\lambda_k\,\underset{\sim}{e}_k^T\underset{\sim}{M}^T\underset{\sim}{e}_j
\end{array}$$

If we subtract these two equations, and recall that $K^T=K$ and $M^T=M$ by assumption, then we obtain $0=(\lambda_j-\lambda_k)\underset{\sim}{e}_k^T\underset{\sim}{M}\underset{\sim}{e}_j$. Since λ_j and λ_k are distinct, then $e_k^T M e_j=\underset{\sim}{e}_k\cdot(\underset{\sim}{M}\underset{\sim}{e}_j)=0$ or, equivalently, $\underset{\sim}{e}_j\cdot(\underset{\sim}{M}\underset{\sim}{e}_k)=0$.

(b) $|\underset{\sim}{K}-\lambda\underset{\sim}{M}| = \begin{vmatrix} 1-3\lambda & 1 \\ 1 & 1-2\lambda \end{vmatrix} = 6\lambda^2-5\lambda = 0$ so $\lambda = 0, 5/6.$

$\lambda_1 = 0$: $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\lambda_2 = 5/6$: $(\underset{\sim}{K}-\lambda\underset{\sim}{M})\underset{\sim}{q} = \begin{pmatrix} -3/2 & 1 \\ 1 & -2/3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \underset{\sim}{0}$ gives $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

Then,

$$\underset{\sim}{e}_1\cdot(\underset{\sim}{M}\underset{\sim}{e}_2) = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 6\beta \\ 6\beta \end{pmatrix} = 0 \checkmark$$

and

$$\underset{\sim}{e}_2\cdot(\underset{\sim}{M}\underset{\sim}{e}_1) = \beta\begin{pmatrix} 2 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 3\alpha \\ -2\alpha \end{pmatrix} = 0 \checkmark$$

Section 11.4

1. (b) I will use Maple for the various calculations.
$\underset{\sim}{A} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$ gives $\lambda = 0, 0$, $\underset{\sim}{e} = (-2,1)^T$. $\underset{\sim}{A}$ does not have 2 LI eigenvectors so it is not diagonalizable.

(c) The Maple commands:

```
with(linalg):
A:= array([[1,0],[1,0]]);
eigenvects(A);
```

gives $\lambda_1 = 0$, $\underset{\sim}{e}_1 = (0,1)^T$; $\lambda_2 = 1$, $\underset{\sim}{e}_2 = (1,1)^T$ so chose $\underset{\sim}{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

```
Q: = array([[0,1],[1,1]]);
R: = inverse(Q);
```

gives

$$R = Q^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we should have $\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Indeed, the command

```
evalm(R&*A&*Q);
```

does give $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(e) $\lambda = 0, 0, 0$, $\underset{\sim}{e} = (1,0,0)^T$. Since $\underset{\sim}{A}$ does not have 3 LI eigenvectors it is not diagonalizable.

(f) $\lambda_1 = 0$, $\underset{\sim}{e}_1 = (0,1,-1)^T$; $\lambda_2 = \lambda_3 = 2$, $\underset{\sim}{e} = \beta(1,0,0)^T + \gamma(0,1,1)^T$ so we can take $\underset{\sim}{e}_2 = (1,0,0)^T$ and $\underset{\sim}{e}_3 = (0,1,1)^T$ and $\underset{\sim}{Q} \overset{*}{=} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$. Then $\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$
$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Check: $\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.\checkmark$

(g) $\lambda_1 = 3$, $\underset{\sim}{e}_1 = (2,1,-1)^T$; $\lambda_2 = 7$, $\underset{\sim}{e}_2 = (0,1,1)^T$; $\lambda_3 = 0$, $\underset{\sim}{e}_3 = (1,-1,1)^T$ so $\underset{\sim}{Q} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.
Then

$$\underset{\sim}{Q}^{-1} = \begin{pmatrix} 1/3 & 1/6 & -1/6 \\ 0 & 1/2 & 1/2 \\ 1/3 & -1/3 & 1/3 \end{pmatrix} \quad\text{and}\quad \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(h) $\lambda_1 = 1$, $\underset{\sim}{e}_1 = (1,1,0)^T$; $\lambda_2 = -1$, $\underset{\sim}{e}_2 = (1,-1,0)^T$; $\lambda_3 = 0$, $\underset{\sim}{e}_3 = (0,0,1)^T$ so $\underset{\sim}{Q} \overset{*}{=} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then $\underset{\sim}{Q}^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

---

* In this example $\underset{\sim}{A}$ is symmetric and our $\underset{\sim}{e}_j$'s are orthogonal. If we want $\underset{\sim}{Q}^{-1}$ to be given simply by $\underset{\sim}{Q}^T$ then we should normalize the $\underset{\sim}{e}_j$'s. I chose not to do that.

(i) $\lambda_1 = 0$, $\underset{\sim}{e}_1 = (0,1,-1,0)^T$; $\lambda_2 = 4$, $\underset{\sim}{e}_2 = (1,0,0,0)^T$; $\lambda_3 = \lambda_4 = 2$, $\underset{\sim}{e} = \alpha(0,0,0,1)^T + \beta(0,1,1,0)^T$ so let $e_3 = (0,0,0,1)^T$ and $\underset{\sim}{e}_4 = (0,1,1,0)^T$. Not bothering to normalize the $\underset{\sim}{e}_j$'s, which step would permit us to use the formula $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$ (because I'm merely using Maple to compute $\underset{\sim}{Q}^{-1}$ anyway), we have

$$\underset{\sim}{Q} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(j) $\lambda_1 = 4$, $\underset{\sim}{e}_1 = (1,1,1,1)^T$; $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\underset{\sim}{e} = \alpha(-1,1,0,0)^T + \beta(-1,0,1,0)^T + \gamma(-1,0,0,1)^T$ so choose $\underset{\sim}{e}_2 = (-1,1,0,0)^T$, $\underset{\sim}{e}_3 = (-1,0,1,0)^T$, $\underset{\sim}{e}_4 = (-1,0,0,1)^T$. (We could obtain an ON set of $\underset{\sim}{e}$'s, but let us accept this LI set.) Then

$$\underset{\sim}{Q} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & & & \bigcirc \\ & \lambda_2 & & \\ & & \lambda_3 & \\ \bigcirc & & & \lambda_4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{where } \underset{\sim}{Q}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

2. (b) $\underset{\sim}{x}' = \underset{\sim}{A}\underset{\sim}{x}$ where $\underset{\sim}{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ has $\lambda_1 = -1$, $\underset{\sim}{e}_1 = (-2,1)^T$; $\lambda_2 = 3$, $\underset{\sim}{e}_2 = (2,1)^T$ so set $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ where

$$\underset{\sim}{Q} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then $\underset{\sim}{Q}\underset{\sim}{\tilde{x}}' = \underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ or $\underset{\sim}{\tilde{x}}' = \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ so $\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$, which gives

$$\tilde{x}(t) = C_1 e^{-t}$$
$$\tilde{y}(t) = C_2 e^{3t}.$$

Then

$$\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 e^{-t} \\ C_2 e^{3t} \end{pmatrix} \text{ so } \begin{aligned} x(t) &= -2C_1 e^{-t} + 2C_2 e^{3t} \\ y(t) &= C_1 e^{-t} + C_2 e^{3t} \end{aligned}$$

or, in vector form,

$$\underset{\sim}{x}(t) = C_1 \overset{\underset{\sim}{e}_1 \searrow}{\begin{pmatrix} -2 \\ 1 \end{pmatrix}} e^{\overset{\lambda_1 = -1}{\downarrow} -t} + C_2 \overset{\underset{\sim}{e}_2 \searrow}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} e^{\overset{\lambda_2 \searrow}{3t}}.$$

(c) $\underset{\sim}{x}'' = \underset{\sim}{A}\underset{\sim}{x}$ where $\underset{\sim}{A} = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$ has $\lambda_1 = -2$, $\underset{\sim}{e}_1 = (1,-1)^T$; $\lambda_2 = 3$, $\underset{\sim}{e}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}^T$ so set $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ where

$$\underset{\sim}{Q} = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}.$$

Then $\underset{\sim}{Q}\underset{\sim}{\tilde{x}}'' = \underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ or $\underset{\sim}{\tilde{x}}'' = \underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ so $\begin{pmatrix} \tilde{x}'' \\ \tilde{y}'' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$, which gives

$$\tilde{x}(t) = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t$$
$$\tilde{y}(t) = C_3 \cosh\sqrt{3}t + C_4 \sinh\sqrt{3}t.$$

Then

$$\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}} = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t \\ C_3 \cosh\sqrt{3}t + C_4 \sinh\sqrt{3}t \end{pmatrix}$$

so

$$x(t) = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t + 4C_3 \cosh\sqrt{3}t + 4C_4 \sinh\sqrt{3}t$$
$$y(t) = -C_1 \cos\sqrt{2}t - C_2 \sin\sqrt{2}t + C_3 \cosh\sqrt{3}t + C_4 \sinh\sqrt{3}t$$

(d) $\utilde{x}'+\utilde{A}\utilde{x}=\utilde{0}$ where $\utilde{A}=\begin{pmatrix}2&1&0\\1&2&1\\0&1&2\end{pmatrix}$ has $\lambda_1=2+\sqrt{2}$, $\utilde{e}_1=(1,\sqrt{2},1)^T$; $\lambda_2=2-\sqrt{2}$, $\utilde{e}_2=(1,-\sqrt{2},1)^T$; $\lambda_3=2$, $\utilde{e}_3=(-1,0,1)^T$ so set $\utilde{x}=\utilde{Q}\utilde{\tilde{x}}$ where $\utilde{Q}=\begin{pmatrix}1&1&-1\\\sqrt{2}&-\sqrt{2}&0\\1&1&1\end{pmatrix}$.

Then $\utilde{Q}\utilde{\tilde{x}}'+\utilde{A}\utilde{Q}\utilde{\tilde{x}}=\utilde{0}$ or $\utilde{\tilde{x}}'+\utilde{Q}^{-1}\utilde{A}\utilde{Q}\utilde{\tilde{x}}=\utilde{0}$ so $\begin{pmatrix}\tilde{x}'\\\tilde{y}'\\\tilde{z}'\end{pmatrix}+\begin{pmatrix}2+\sqrt{2}&0&0\\0&2-\sqrt{2}&0\\0&0&2\end{pmatrix}\begin{pmatrix}\tilde{x}\\\tilde{y}\\\tilde{z}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

Thus, $\tilde{x}(t)=C_1e^{-(2+\sqrt{2})t}$, $\tilde{y}(t)=C_2e^{-(2-\sqrt{2})t}$, $\tilde{z}(t)=C_3e^{-2t}$

so

$$\utilde{x}(t)=\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix}=\utilde{Q}\utilde{\tilde{x}}=\begin{pmatrix}1&1&-1\\\sqrt{2}&-\sqrt{2}&0\\1&1&1\end{pmatrix}\begin{pmatrix}C_1e^{-(2+\sqrt{2})t}\\C_2e^{-(2-\sqrt{2})t}\\C_3e^{-2t}\end{pmatrix}$$

and

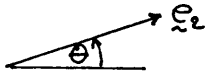
$$\begin{aligned}x(t)&=C_1e^{-(2+\sqrt{2})t}+C_2e^{-(2-\sqrt{2})t}+C_3e^{-2t}\\y(t)&=\sqrt{2}C_1e^{-(2+\sqrt{2})t}-\sqrt{2}C_2e^{-(2-\sqrt{2})t}\\z(t)&=C_1e^{-(2+\sqrt{2})t}+C_2e^{-(2-\sqrt{2})t}+C_3e^{-2t}\end{aligned}$$

3. If $\utilde{A}$ is singular that merely means that 0 is among its eigenvalues – which has nothing to do with whether or not $\utilde{A}$ has n LI eigenvectors. For example, the matrix in Exercise 1(b) has $\lambda=0,0$ and is *not* diagonalizable, whereas the matrix in Exercise 1(f) has $\lambda=0,2,2$ and *is* diagonalizable.

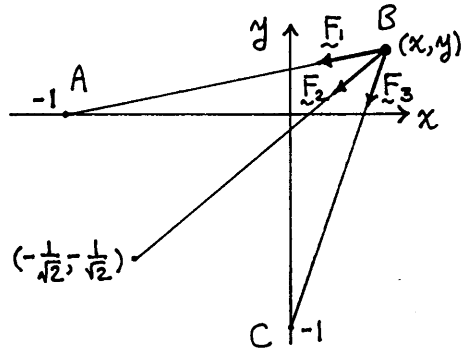
4. We need $\utilde{e}_2$. We find that $\lambda_2=(k_1+k_2+\sqrt{k_1^2+k_2^2})/(2m)$ and that

$$\utilde{e}_2=\begin{pmatrix}k_1+\sqrt{k_1^2+k_2^2}\\k_2\end{pmatrix}$$

so $\tan\theta=\dfrac{k_2}{k_1+\sqrt{k_1^2+k_2^2}}$. Let $k_2=\alpha k_1$,



where $0<\alpha<\infty$. Then $\tan\theta=\dfrac{\alpha}{1+\sqrt{1+\alpha^2}}\equiv f(\alpha)$. By inspection we see that f increases from 0 to 1 as α increases from 0 to ∞. Thus, $0<\tan\theta<1$, so $0<\theta<\pi/4$. ✓



5. (b) This one is tedious and may be difficult for the non-advanced student.

$$\utilde{F}_1=k_1\underbrace{\left(\sqrt{(1+x)^2+y^2}-1\right)}_{\text{the stretch in spring \#1}}\underbrace{\frac{-(1+x)\hat{\utilde{i}}-y\hat{\utilde{j}}}{\sqrt{(1+x)^2+y^2}}}_{\text{Unit vector from B toward A}}$$

$$\sim k_1\left(\sqrt{1+2x}-1\right)\frac{-(1+x)\hat{\utilde{i}}-y\hat{\utilde{j}}}{\sqrt{1+2x}}=k_1\left[1-(1+2x)^{-1/2}\right]\left[-(1+x)\hat{\utilde{i}}-y\hat{\utilde{j}}\right]$$

$$\sim k_1\left[1-(1-x+\cdots)\right]\left[-(1+x)\hat{\utilde{i}}-y\hat{\utilde{j}}\right]\sim k_1x\left[-(1+x)\hat{\utilde{i}}-y\hat{\utilde{j}}\right]\sim -k_1x\hat{\utilde{i}}$$

through first-order terms.

$$\underset{\sim}{F}_2 = k_2\left(\sqrt{(\tfrac{1}{\sqrt2}+x)^2+(\tfrac{1}{\sqrt2}+y)^2}-1\right)\frac{-(\frac{1}{\sqrt2}+x)\underset{\sim}{\hat{\imath}}-(\frac{1}{\sqrt2}+y)\underset{\sim}{\hat{\jmath}}}{\sqrt{(\frac{1}{\sqrt2}+x)^2+(\frac{1}{\sqrt2}+y)^2}}$$

$$\sim k_2\left(\sqrt{\tfrac{1}{2}+\tfrac{2}{\sqrt2}x+\tfrac{1}{2}+\tfrac{2}{\sqrt2}y}-1\right)\frac{-(1/\sqrt2+x)\hat{\imath}-(1/\sqrt2+y)\hat{\jmath}}{\sqrt{\frac{1}{2}+\frac{2}{\sqrt2}x+\frac{1}{2}+\frac{2}{\sqrt2}y}}$$

$$= k_2\left(1-(1+\sqrt2(x+y))]^{-1/2}\right)\left[-(\tfrac{1}{\sqrt2}+x)\underset{\sim}{\hat{\imath}}-(\tfrac{1}{\sqrt2}+y)\underset{\sim}{\hat{\jmath}}\right]$$

$$\sim k_2\left(1-\left[1-\tfrac{\sqrt2}{2}(x+y)\right]\right)\left[\qquad\text{''}\qquad\right]\sim k_2\tfrac{1}{\sqrt2}(x+y)\left(-\tfrac{1}{\sqrt2}\hat{\imath}-\tfrac{1}{\sqrt2}\hat{\jmath}\right)$$

$$= -k_2\,\frac{x+y}{2}(\underset{\sim}{\hat{\imath}}+\underset{\sim}{\hat{\jmath}})$$

$$\underset{\sim}{F}_3 = k_3\left(\sqrt{x^2+(1+y)^2}-1\right)\frac{-x\hat{\imath}-(1+y)\hat{\jmath}}{\sqrt{x^2+(1+y)^2}}$$

$$\sim k_3\left(1-[1+2y]^{-1/2}\right)\left(-x\hat{\imath}-(1+y)\hat{\jmath}\right)\ \sim\ k_3\left(1-(1-y)\right)\left(-x\underset{\sim}{\hat{\imath}}-(1+y)\underset{\sim}{\hat{\jmath}}\right)\sim -k_3y\underset{\sim}{\hat{\jmath}}$$

so $m(x''\underset{\sim}{\hat{\imath}}+y''\hat{\jmath}) = \underset{\sim}{F}_1+\underset{\sim}{F}_2+\underset{\sim}{F}_3$
gives, through first order (i.e., the linearized equation),

$$m(x''\hat{\imath}+y''\hat{\jmath}) = -k_1x\hat{\imath}-k_2\frac{x+y}{2}(\hat{\imath}+\hat{\jmath})-k_3y\hat{\jmath}.$$

Now, $m=4$, $k_1=4$, $k_2=3$, $k_3=1$, so

$\underset{\sim}{\hat{\imath}}$: $4x'' = -4x-\frac{3}{2}(x+y)$

$\underset{\sim}{\hat{\jmath}}$: $4y'' = -\frac{3}{2}(x+y)-y$

or,

$$x'' = -\tfrac{11}{8}x-\tfrac{3}{8}y$$
$$y'' = -\tfrac{3}{8}x-\tfrac{5}{8}y \quad\text{or}\quad \underset{\sim}{x}''=\underset{\sim}{A}\underset{\sim}{x} \text{ where } \underset{\sim}{A}=\begin{pmatrix}-11/8 & -3/8\\ -3/8 & -5/8\end{pmatrix} \text{ has } \lambda_1\approx -0.470,\ \underset{\sim}{e}_1=\begin{pmatrix}-0.414\\1\end{pmatrix},$$

and $\lambda_2\approx -1.53$, $\underset{\sim}{e}_2\approx\begin{pmatrix}2.42\\1\end{pmatrix}$ so set $\underset{\sim}{x}=\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ where $\underset{\sim}{Q}=\begin{pmatrix}-.414 & 2.42\\ 1 & 1\end{pmatrix}$.

Then $\underset{\sim}{Q}\underset{\sim}{\tilde{x}}''=\underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ or $\underset{\sim}{\tilde{x}}''=\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{\tilde{x}}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$, so

bars merely for convenience

$$\tilde{x}''=\lambda_1\tilde{x} \quad\text{or}\quad \tilde{x}''+0.470\tilde{x}=0 \quad\text{so}\quad \tilde{x}(t)=\bar{C}_1\cos 0.686t+\bar{C}_2\sin 0.686t$$
$$\tilde{y}''=\lambda_2\tilde{y} \quad\text{or}\quad \tilde{y}''+1.53\tilde{y}=0 \quad\text{so}\quad \tilde{y}(t)=\bar{C}_3\cos 1.24t+\bar{C}_4\sin 1.24t$$

Finally,
$$\underset{\sim}{x}(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\underset{\sim}{Q}\underset{\sim}{\tilde{x}}=\begin{pmatrix}-.414 & 2.42\\ 1 & 1\end{pmatrix}\begin{pmatrix}\bar{C}_1\cos 0.686t+\bar{C}_2\sin 0.686t\\ \bar{C}_3\cos 1.24t+\bar{C}_4\sin 1.24t\end{pmatrix}$$

gives

$$x(t)\approx -0.414\bar{C}_1\cos 0.686t-0.414\bar{C}_2\sin 0.686t+2.42\bar{C}_3\cos 1.24t+2.42\bar{C}_4\sin 1.24t$$
$$y(t)\approx \bar{C}_1\cos 0.686t+\bar{C}_2\sin 0.686t+\bar{C}_3\cos 1.24t+\bar{C}_4\sin 1.24t.$$

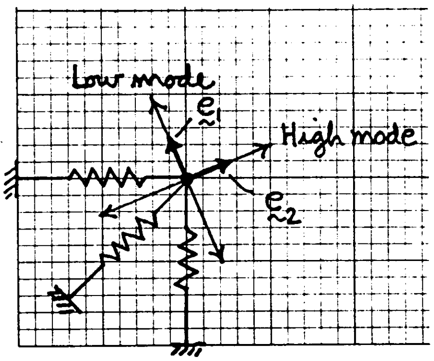
To understand the latter in terms of eigenmodes let us re-express it as

$$\underset{\sim}{x}(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}-0.414\\ 1\end{pmatrix}(\bar{C}_1\cos 0.686t+\bar{C}_2\sin 0.686t)+\begin{pmatrix}2.42\\ 1\end{pmatrix}(\bar{C}_3\cos 1.24t+\bar{C}_4\sin 1.24t).$$

Or, in an equivalent and clearer form,

$$\underset{\sim}{x}(t) = \begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \underbrace{C_5 \underset{\text{mode shape}}{\begin{pmatrix}-0.414\\ 1\end{pmatrix}} \sin(\underset{\underset{\text{frequency}}{\uparrow}}{0.686}t+\phi_1)}_{\text{Low mode}} + \underbrace{C_2 \underset{\text{mode shape}}{\begin{pmatrix}2.42\\ 1\end{pmatrix}} \sin(\underset{\underset{\text{frequency}}{\uparrow}}{1.24}t+\phi_2)}_{\text{High mode}}$$

Drawing to scale, the low and high modes are along the lines shown.



6. If we partition Q into columns as $\underset{\sim}{Q} = (\underset{\sim}{q}_1, \dots, \underset{\sim}{q}_n)$ then, by the definition of matrix multiplication, it follows that $\underset{\sim}{A}\underset{\sim}{Q} = \underset{\sim}{A}(\underset{\sim}{q}_1, \dots, \underset{\sim}{q}_2) = (\underset{\sim}{A}\underset{\sim}{q}_1, \dots, \underset{\sim}{A}\underset{\sim}{q}_n)$

7. Easy: If $Q^{-1}AQ = D$, then

$$\underset{\sim}{Q}\underset{\sim}{Q}^{-1}\underset{\sim}{A}\underset{\sim}{Q} = \underset{\sim}{Q}\underset{\sim}{D},$$
$$\underset{\sim}{A}\underset{\sim}{Q} = \underset{\sim}{Q}\underset{\sim}{D},$$
$$\underset{\sim}{A}\underset{\sim}{Q}\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{-1},$$
$$\text{so} \quad \underset{\sim}{A} = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{-1}.$$

Then

$$\underset{\sim}{A}^2 = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{-1}\underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{I}\underset{\sim}{D}\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}\underset{\sim}{D}^2\underset{\sim}{Q}^{-1}$$
$$\underset{\sim}{A}^3 = \underset{\sim}{A}\underset{\sim}{A}^2 = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{-1}\underset{\sim}{Q}\underset{\sim}{D}^2\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{I}\underset{\sim}{D}^2\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}\underset{\sim}{D}^3\underset{\sim}{Q}^{-1}, \text{ and so on.}$$

8. (b) Let's use Maple.

```
with(linalg):
A:= array([[2,2,1],[1,3,1],[1,2,2]]);
eigenvects(A);
```

gives $\lambda_1 = \lambda_2 = 1$, $\underset{\sim}{e} = \alpha(-2,1,0)^T + \beta(-1,0,1)^T$; $\lambda_3 = 5$, $\underset{\sim}{e}_3 = (1,1,1)^T$, so choose $\underset{\sim}{e}_1 = (-2,1,0)^T$ and $\underset{\sim}{e}_2 = (-1,0,1)^T$, and $\underset{\sim}{Q} = \begin{pmatrix} -2 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$

Then

```
Q:= array([[-2,-1,1],[1,0,1],[0,1,1]]);
P:= inverse(Q);
```

gives

$$Q^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 2 & -1\\ -1 & -2 & 3\\ 1 & 2 & 1\end{pmatrix}$$

Also, $D^{1000} = \begin{pmatrix} 1^{1000} & 0 & 0\\ 0 & 1^{1000} & 0\\ 0 & 0 & 5^{1000}\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & a\end{pmatrix}$, where it will be convenient to use "a" as shorthand for 5^{1000}. Finally,

```
R:= array([[1,0,0],[0,1,0],[0,0,a]]);
evalm(Q&*R&*P);
```

gives, after factoring out 1/4, $\underset{\sim}{A}^{1000} = \frac{1}{4}\begin{pmatrix} 3+a & -2+2a & -1+a \\ -1+a & 2+2a & -1+a \\ -1+a & -2+2a & 3+a \end{pmatrix}$ $(a = 5^{1000})$

Of course, the latter $\approx \frac{5^{1000}}{4}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

9. (a) $\mathcal{I}_{xx} = \int_0^1\int_0^3 (y^2 + 0)\sigma\,dx\,dy = 3\sigma/3 = \sigma$

$\mathcal{I}_{xy} = \int_0^1\int_0^3 xy\sigma\,dx\,dy = 9\sigma/4 = \mathcal{I}_{yx}$

$\mathcal{I}_{yy} = \int_0^1\int_0^3 (x^2+0)\sigma\,dx\,dy = 9\sigma$

$\mathcal{I}_{xz} = \mathcal{I}_{zx} = 0$, $\mathcal{I}_{zz} = \int_0^1\int_0^3 (x^2+y^2)\sigma\,dx\,dy = 10\sigma$

so $\underset{\sim}{\mathcal{I}} = \sigma\begin{pmatrix} 1 & -9/4 & 0 \\ -9/4 & 9 & 0 \\ 0 & 0 & 10 \end{pmatrix}$

Maple gives $\lambda_1 \approx 9.5890\sigma$, $\underset{\sim}{e}_1 \approx (+.2534, -.9674, 0)^T$; ← I've scaled Maple's $\underset{\sim}{e}_1$ by -1 so $\det \underset{\sim}{Q} = +1$, for right-handed $\tilde{x}, \tilde{y}, \tilde{z}$ system.

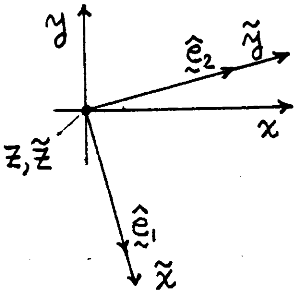
$\lambda_2 \approx 0.411\sigma$, $\underset{\sim}{e}_2 \approx (.9674, .2534, 0)^T$;

$\lambda_3 = 10\sigma$, $\underset{\sim}{e}_3 = (0,0,1)^T$.

Thus, with $\underset{\sim}{Q} = \begin{pmatrix} .2534 & .9674 & 0 \\ -.9674 & .2534 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $\tilde{\mathcal{I}} = \underset{\sim}{Q}^{-1}\underset{\sim}{\mathcal{I}}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

$= \begin{pmatrix} 9.5890\sigma & 0 & 0 \\ 0 & 0.411\sigma & 0 \\ 0 & 0 & 10\sigma \end{pmatrix}$

so $\mathcal{I}_{\tilde{x}\tilde{x}} = 9.5890\sigma$, $\mathcal{I}_{\tilde{y}\tilde{y}} \approx 0.411\sigma$, $\mathcal{I}_{\tilde{z}\tilde{z}} = 10\sigma$.



(d) $\mathcal{I}_{xx} = \int_0^2\int_{-2}^2 (y^2+0)\sigma\,dx\,dy - \int_0^1\int_0^1 (y^2+0)\sigma\,dx\,dy = 32\sigma/3 - \sigma/3 = 31\sigma/3$

$\mathcal{I}_{xy} = \mathcal{I}_{yx} = \int_0^2\int_{-2}^2 xy\sigma\,dx\,dy - \int_0^1\int_0^1 xy\sigma\,dx\,dy = -\sigma/4$

$\mathcal{I}_{yy} = \int_0^2\int_{-2}^2 (x^2+0)\sigma\,dx\,dy - \int_0^1\int_0^1 (x^2+0)\sigma\,dx\,dy = 31\sigma/3$

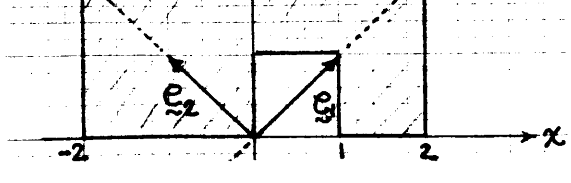
$\mathcal{I}_{xz} = \mathcal{I}_{zx} = 0$, $\mathcal{I}_{zz} = \iint (x^2+y^2)\sigma\,dx\,dy = \mathcal{I}_{xx} + \mathcal{I}_{yy}$
$= 62\sigma/3$

so $\underset{\sim}{\mathcal{I}} = \sigma\begin{pmatrix} 31/3 & +1/4 & 0 \\ +1/4 & 31/3 & 0 \\ 0 & 0 & 62/3 \end{pmatrix}$. Maple gives $\lambda_1 = 127\sigma/12$, $\underset{\sim}{e}_1 = (1, 1, 0)^T$;

$\lambda_2 = 121\sigma/12$, $\underset{\sim}{e}_2 = (-1, 1, 0)^T$;

$\lambda_3 = 62\sigma/3$, $\underset{\sim}{e}_3 = (0,0,1)^T$. Thus, with

$\underset{\sim}{Q} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $\tilde{\mathcal{I}} = \underset{\sim}{Q}^{-1}\underset{\sim}{\mathcal{I}}\underset{\sim}{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 127\sigma/12 & 0 & 0 \\ 0 & 121\sigma/12 & 0 \\ 0 & 0 & 62\sigma/3 \end{pmatrix}$,



so $\mathcal{I}_{\tilde{x}\tilde{x}} = 127\sigma/12$,

$\mathcal{I}_{\tilde{y}\tilde{y}} = 121\sigma/12$,

$\mathcal{I}_{\tilde{z}\tilde{z}} = 62\sigma/3$.

10. (b) With $\lambda_1, \underset{\sim}{e}_1$ given by (10.2) and $\underset{\sim}{A}$ given by (10.1), (10.4) becomes

$$\begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ or, by Gauss elimination,

$$\begin{pmatrix} 0 & -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_4 = 2/3,\ x_3 = 1, \\ x_2 = 1,\ x_1 = \alpha \end{matrix}$$

so $\underset{\sim}{e}_2 = (\alpha, 1, 1, 2/3)^T$, as in (10.6). Putting this into (10.5), (10.5) becomes

$$\begin{pmatrix} 0 & -1 & 2 & 0 & \alpha \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -3 & 3 & 2/3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 0 & \alpha \\ 0 & 0 & 1 & 0 & \alpha+1 \\ 0 & 0 & 1 & 0 & \alpha+1 \\ 0 & 0 & -1 & 3 & \alpha+2/3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 2 & 0 & \alpha \\ 0 & 0 & 1 & 0 & \alpha+1 \\ 0 & 0 & 0 & 3 & 2\alpha+5/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_4 = \frac{2}{3}\alpha + \frac{5}{9},\ x_3 = \alpha+1, \\ x_2 = \alpha+2,\ x_1 = \beta \end{matrix}$$

so $\underset{\sim}{e}_3 = (\beta, \alpha+2, \alpha+1, \frac{2}{3}\alpha + \frac{5}{9})^T$, where α, β are arbitrary. Take $\alpha = \beta = 0$, then, for simplicity. Then $\underset{\sim}{P} = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4)$ is as given in (10.8) and if we work out $\underset{\sim}{P}^{-1}\underset{\sim}{A}\underset{\sim}{P}$ (using maple, for example) then we obtain the result shown in (10.9). This wasn't asked for, but here are suitable maple commands: with(linalg):

```
A:=array([[2,-1,2,0],[0,3,-1,0],[0,1,1,0],[0,1,-3,5]]);
P:=array([[1,0,0,0],[0,1,2,0],[0,1,1,0],[0,2/3,5/9,1]]);
B:=inverse(P);
evalm(B&*A&*P);
```

which give the result shown in (10.9).

(c)
$$\alpha_1 \underset{\sim}{e}_1 + \alpha_2 \underset{\sim}{e}_2 + \alpha_3 \underset{\sim}{e}_3 + \alpha_4 \underset{\sim}{e}_4 = \underset{\sim}{0} \qquad ①$$

$$\alpha_1(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_1 + \alpha_2(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_2 + \alpha_3(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_3 + \alpha_4(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_4 = \underset{\sim}{0}$$

or, $$\alpha_1 \underset{\sim}{0} + \alpha_2 \underset{\sim}{e}_1 + \alpha_3 \underset{\sim}{e}_2 + \alpha_4(\lambda_4 - \lambda_1)\underset{\sim}{e}_4 = \underset{\sim}{0} \qquad ②$$

Then $$\alpha_2(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_1 + \alpha_3(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_2 + \alpha_4(\lambda_4 - \lambda_1)(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_4 = \underset{\sim}{0}$$

or $$\alpha_2 \underset{\sim}{0} + \alpha_3 \underset{\sim}{e}_1 + \alpha_4(\lambda_4 - \lambda_1)^2 \underset{\sim}{e}_4 = \underset{\sim}{0} \qquad ③$$

Then $$\alpha_3(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_1 + \alpha_4(\lambda_4 - \lambda_1)^2(\underset{\sim}{A} - \lambda_1 \underset{\sim}{I})\underset{\sim}{e}_4 = \underset{\sim}{0}$$

or $$\alpha_3 \underset{\sim}{0} + \alpha_4(\lambda_4 - \lambda_1)^3 \underset{\sim}{e}_4 = \underset{\sim}{0} \qquad ④$$

Since $\underset{\sim}{e}_4 \neq \underset{\sim}{0}$ and $\lambda_4 \neq \lambda_1$, ④ implies that $\alpha_4 = 0$. Then ③ gives $\alpha_3 = 0$, ② gives $\alpha_2 = 0$, and ① gives $\alpha_1 = 0$. Since, for ① to be true, we need $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, it follows that $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4$ are LI.

(d) In this part we show how to apply the foregoing results (namely, the result that $\underset{\sim}{P}^{-1}\underset{\sim}{A}\underset{\sim}{P} = \underset{\sim}{J}$ is in Jordan canonical form) to a system of coupled ODE's, namely,

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \qquad (10.12)$$

As seen in (10.2) $\underset{\sim}{A}$ does not have 4 LI eigenvectors and is therefore not diagonalizable. Nevertheless, the transformation $\underset{\sim}{x} = P\underset{\sim}{\tilde{x}}$ reduces (10.12) to the triangular (more specifically, Jordan) form

$$\begin{pmatrix} \tilde{x}_1'(t) \\ \tilde{x}_2'(t) \\ \tilde{x}_3'(t) \\ \tilde{x}_4'(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix}$$

Then $\tilde{x}_4' = 5\tilde{x}_4$ gives $\tilde{x}_4 = C_1 e^{5t}$, $\tilde{x}_3' = 2\tilde{x}_3$ gives $\tilde{x}_3 = C_2 e^{2t}$, $\tilde{x}_2' = 2\tilde{x}_2 + C_2 e^{2t}$ gives $\tilde{x}_2 = C_3 e^{2t} + C_2 t e^{2t}$, and $\tilde{x}_1' = 2\tilde{x}_1 + C_3 e^{2t} + C_2 t e^{2t}$ gives $\tilde{x}_1 = C_4 e^{2t} + \frac{1}{2} C_2 t^2 e^{2t} + C_3 t e^{2t}$. Finally,

$$\underset{\sim}{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \underset{\sim}{P}\,\underset{\sim}{\tilde{x}}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2/3 & 5/9 & 1 \end{pmatrix} \begin{pmatrix} C_4 e^{2t} + \frac{1}{2} C_2 t^2 e^{2t} + C_3 t e^{2t} \\ C_3 e^{2t} + C_2 t e^{2t} \\ C_2 e^{2t} \\ C_1 e^{5t} \end{pmatrix}$$

gives

$$x_1(t) = (C_4 + \tfrac{1}{2} C_2 t^2 + C_3 t) e^{2t}$$
$$x_2(t) = (C_3 + C_2 t + 2C_2) e^{2t}$$
$$x_3(t) = (C_3 + C_2 t + C_2) e^{2t}$$
$$x_4(t) = (\tfrac{5}{9} C_2 + \tfrac{2}{3} C_3 + \tfrac{2}{3} C_2 t) e^{2t} + C_1 e^{5t}$$

Section 11.5

1. (12) is $\underset{\sim}{x}(t) = \underset{\sim}{Q} e^{\underset{\sim}{D}t} \underset{\sim}{Q}^{-1} \underset{\sim}{c} + \int_0^t \underset{\sim}{Q} e^{\underset{\sim}{D}(t-\tau)} \underset{\sim}{Q}^{-1} \underset{\sim}{f}(\tau)\, d\tau$

NOTE: all t_0's in Sec 11.5 should be 0's. This is fixed in the 3rd printing, 10/99.

(b) $x' = 2x + 4y + 1$; $x(0) = 0$
$y' = x - y$; $y(0) = 0$ so $\underset{\sim}{A} = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underset{\sim}{A}$ gives $\lambda_1 = -2$, $\underset{\sim}{e}_1 = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $\lambda_2 = 3$, $\underset{\sim}{e}_2 = \beta\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. These $\underset{\sim}{e}$'s are LI so $\underset{\sim}{A}$ is diagonalizable and (12) applies. With

$$\underset{\sim}{D} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \underset{\sim}{Q} = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}, \quad \underset{\sim}{Q}^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}$$

(12) gives $\underset{\sim}{x}(t) = \underset{\sim}{0} + \int_0^t \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2(t-\tau)} & 0 \\ 0 & e^{3(t-\tau)} \end{pmatrix} \frac{1}{5} \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau$

$$= \frac{1}{5}\int_0^t \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -e^{2(\tau-t)} \\ e^{3(t-\tau)} \end{pmatrix} d\tau = \frac{1}{5}\int_0^t \begin{pmatrix} e^{2(\tau-t)} + 4e^{3(t-\tau)} \\ -e^{2(\tau-t)} + e^{3(t-\tau)} \end{pmatrix} dt$$

so $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} \frac{1}{2} e^{2(\tau-t)} - \frac{4}{3} e^{3(t-\tau)} \\ -\frac{1}{2} e^{2(\tau-t)} - \frac{1}{3} e^{3(t-\tau)} \end{pmatrix}\Bigg|_0^t = \begin{pmatrix} -\frac{1}{6} - \frac{1}{10} e^{-2t} + \frac{4}{15} e^{3t} \\ -\frac{1}{6} + \frac{1}{10} e^{-2t} + \frac{1}{15} e^{3t} \end{pmatrix}$

(c) $x' = 2x - y - 3t + 1;\ x(0) = 1$
$y' = -x + 2y + 3\ ;\ y(0) = 0$ so $\underset{\sim}{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} -3t+1 \\ 3 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $t_0 = 0$

$\underset{\sim}{A}$ gives $\lambda_1 = 1$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 3$, $\underset{\sim}{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. These $\underset{\sim}{e}$'s are LI (indeed, they are orthogonal) so $\underset{\sim}{A}$ is diagonalizable and (12) applies. With

$$\underset{\sim}{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},\ \underset{\sim}{Q} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\ \underset{\sim}{Q}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(12) gives
$$\underset{\sim}{x}(t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{(t-\tau)} & 0 \\ 0 & e^{3(t-\tau)} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3\tau+1 \\ 3 \end{pmatrix} d\tau$$
$$= \frac{1}{2}\begin{pmatrix} e^t + e^{-3t} \\ e^t - e^{3t} \end{pmatrix} + \frac{1}{2}\int_0^t \begin{pmatrix} (4-3\tau)e^{t-\tau} - (3\tau+2)e^{3(t-\tau)} \\ (4-3\tau)e^{t-\tau} + (3\tau+2)e^{3(t-\tau)} \end{pmatrix} d\tau$$
$$\text{so } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \quad '' \quad + \frac{1}{2}\begin{pmatrix} 4t + e^t - e^{3t} \\ -2 + 2t + e^t + e^{3t} \end{pmatrix} = \begin{pmatrix} 2t + e^t \\ t - 1 + e^t \end{pmatrix}$$

(d) $x' = x + 2y - t - 1\ ;\ x(0) = 0$
$y' = 4x + 8y - 4t - 8\ ;\ y(0) = 3$ so $\underset{\sim}{A} = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} -t-1 \\ -4t-8 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $t_0 = 0$

$\underset{\sim}{A}$ gives $\lambda_1 = 0$, $\underset{\sim}{e}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$; $\lambda_2 = 9$, $\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. These $\underset{\sim}{e}$'s are LI so $\underset{\sim}{A}$ is diagonalizable and (12) applies. With

$$\underset{\sim}{D} = \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix},\ \underset{\sim}{Q} = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix},\ \underset{\sim}{Q}^{-1} = \frac{1}{9}\begin{pmatrix} -4 & 1 \\ 1 & 2 \end{pmatrix}$$

(12) gives
$$\underset{\sim}{x}(t) = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{9t} \end{pmatrix}\frac{1}{9}\begin{pmatrix} -4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \int_0^t \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{9(t-\tau)} \end{pmatrix}\frac{1}{9}\begin{pmatrix} -4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -\tau-1 \\ -4\tau-8 \end{pmatrix} d\tau$$
$$= \frac{1}{9}\begin{pmatrix} -6 + 6e^{9t} \\ 3 + 24e^{9t} \end{pmatrix} + \frac{1}{9}\int_0^t \begin{pmatrix} 8 - (9\tau+17)e^{9(t-\tau)} \\ -4 - (36\tau+68)e^{9(t-\tau)} \end{pmatrix} d\tau$$
$$\text{so } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -6 + 6e^{9t} \\ 3 + 24e^{9t} \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 9t + 2 - 2e^{9t} \\ 8 - 8e^{9t} \end{pmatrix} = \begin{pmatrix} t - \frac{4}{9} + \frac{4}{9}e^{9t} \\ \frac{11}{9} + \frac{16}{9}e^{9t} \end{pmatrix}$$

(e) $x' = -y + \sin t;\ x(0) = 0$
$y' = -9x + 4\ ;\ y(0) = 1$ so $\underset{\sim}{A} = \begin{pmatrix} 0 & -1 \\ -9 & 0 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} \sin t \\ 4 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $t_0 = 0$

$\underset{\sim}{A}$ gives $\lambda_1 = 3$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$; $\lambda_2 = -3$, $\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. These $\underset{\sim}{e}$'s are LI so $\underset{\sim}{A}$ is diagonalizable and (12) applies. With $\underset{\sim}{D} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$, $\underset{\sim}{Q} = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$, $\underset{\sim}{Q}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$, (12) gives
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} e^{3(t-\tau)} & 0 \\ 0 & e^{-3(t-\tau)} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \sin\tau \\ 4 \end{pmatrix} d\tau$$
$$= \frac{1}{6}\begin{pmatrix} -e^{3t} + e^{-3t} \\ 3e^{3t} + 3e^{-3t} \end{pmatrix} + \frac{1}{6}\int_0^t \begin{pmatrix} (3\sin\tau - 4)e^{3(t-\tau)} + (3\sin\tau + 4)e^{-3(t-\tau)} \\ (-9\sin\tau + 12)e^{3(t-\tau)} + (9\sin\tau + 12)e^{-3(t-\tau)} \end{pmatrix} d\tau$$

$$x(t) = \frac{4}{9} - \frac{1}{10}\cos t - \frac{61}{180}e^{3t} - \frac{1}{180}e^{-3t}$$
$$y(t) = \frac{9}{10}\sin t + \frac{61}{60}e^{3t} - \frac{1}{60}e^{-3t}$$

(j) $x' = y \quad ; \ x(0) = 2$
$y' = -x - 2y \ ; \ y(0) = 9$ so $\underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$, $t_0 = 0$

$\underset{\sim}{A}$ gives $\lambda_1 = \lambda_2 = -1$, $\underset{\sim}{e} = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Since we don't have 2 LI eigenvectors, $\underset{\sim}{A}$ is not diagonalizable.

(l) $x' = x + y + z - 1 \ ; \ x(0) = 0$
$y' = x + y + z \ ; \ y(0) = 0$ so $\underset{\sim}{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\underset{\sim}{f}(t) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $t_0 = 0$
$z' = x + y + z \ ; \ z(0) = 0$

$\underset{\sim}{A}$ gives $\lambda_1 = 3$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_2 = \lambda_3 = 0$, $\underset{\sim}{e} = \alpha\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ so choose $\underset{\sim}{e}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\underset{\sim}{e}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. (We could obtain orthogonal $\underset{\sim}{e}_2, \underset{\sim}{e}_3$ vectors but we don't <u>need</u> to.) Since $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are LI, $\underset{\sim}{A}$ is diagonalizable and (12) applies. With

$$\underset{\sim}{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{Q} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \underset{\sim}{Q}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

(12) gives
$$\underset{\sim}{x}(t) = \underset{\sim}{0} + \int_0^t \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{3(t-\tau)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} d\tau = \frac{1}{3}\int_0^t \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{3(t-\tau)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} d\tau$$
$$= \frac{1}{3}\int_0^t \begin{pmatrix} -e^{3(t-\tau)} - 2 \\ -e^{3(t-\tau)} + 1 \\ -e^{3(t-\tau)} + 1 \end{pmatrix} d\tau \quad \text{so} \quad \begin{aligned} x(t) &= \frac{1}{9} - \frac{2}{3}t - \frac{1}{9}e^{3t} \\ y(t) &= \frac{1}{9} + \frac{1}{3}t - \frac{1}{9}e^{3t} \\ z(t) &= \frac{1}{9} + \frac{1}{3}t - \frac{1}{9}e^{3t} \end{aligned}$$

2. Just let the initial condition be $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, where the c_j's are arbitrary. For instance, consider part (c):

(c) From the solution to 1(c) we can begin with

$$\underset{\sim}{x}(t) = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{homogeneous soln.}} + \underbrace{\frac{1}{2}\begin{pmatrix} 4t + e^t - e^{3t} \\ -2 + 2t + e^t + e^{3t} \end{pmatrix}}_{\text{particular soln.}}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (c_1 + c_2)e^t \\ (-c_1 + c_2)e^{3t} \end{pmatrix} + \quad ''$$
$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_3 e^t \\ c_4 e^{3t} \end{pmatrix} \quad + \quad '' \quad \text{where we've set } (c_1 + c_2)/2 \equiv c_3 \text{ and } (-c_1 + c_2)/2 \equiv c_4$$

so
$$x(t) = c_3 e^t - c_4 e^{3t} + 2t + (e^t - e^{3t})/2$$
$$y(t) = c_3 e^t + c_4 e^{3t} - 1 + t + (e^t + e^{3t})/2$$

or, $$x(t) = (C_3+\tfrac{1}{2})e^{t} - (C_4+\tfrac{1}{2})e^{3t} + 2t \equiv C_5e^{t} + C_6e^{3t} + 2t$$
$$y(t) = (C_3+\tfrac{1}{2})e^{t} + (C_4+\tfrac{1}{2})e^{3t} + t - 1 \equiv C_5e^{t} - C_6e^{3t} + t - 1$$

3. (b) $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ gives $\lambda_1 = 4$, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 0$, $\mathbf{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

so $\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{Q}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $e^{\mathbf{A}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^4 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^4+1 & e^4-1 \\ e^4-1 & e^4+1 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix}$ gives $\lambda_1 = 6$, $\mathbf{e}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $\lambda_2 = -6$, $\mathbf{e}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$,

so $\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$, $\mathbf{Q}^{-1} = \frac{1}{24}\begin{pmatrix} 6 & 4 \\ 6 & -4 \end{pmatrix}$, $e^{\mathbf{A}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} = \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} e^6 & 0 \\ 0 & e^{-6} \end{pmatrix}\frac{1}{24}\begin{pmatrix} 6 & 4 \\ 6 & -4 \end{pmatrix}$

$$= \frac{1}{24}\begin{pmatrix} 12e^6+12e^{-6} & 8e^6-8e^{-6} \\ 18e^6-18e^{-6} & 12e^6+12e^{-6} \end{pmatrix} = \begin{pmatrix} \cosh 6 & \frac{2}{3}\sinh 6 \\ \frac{3}{2}\sinh 6 & \cosh 6 \end{pmatrix}$$

(f) $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$, $e^{\mathbf{A}} = \begin{pmatrix} e^5 & 0 \\ 0 & 1 \end{pmatrix}$

(g) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\lambda_1 = 0$, $\mathbf{e}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = -1$, $\mathbf{e}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; $\lambda_3 = 2$, $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

so $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\mathbf{Q}^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 3 & 0 \\ -2 & -2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$,

$$e^{\mathbf{A}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^2 \end{pmatrix}\frac{1}{6}\begin{pmatrix} -3 & 3 & 0 \\ -2 & -2 & 2 \\ 1 & 1 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3+2e^{-1}+e^2 & -3+2e^{-1}+e^2 & -2e^{-1}+2e^2 \\ -3+2e^{-1}+e^2 & 3+2e^{-1}+e^2 & -2e^{-1}+2e^2 \\ -2e^{-1}+2e^2 & -2e^{-1}+2e^2 & 2e^{-1}+4e^2 \end{pmatrix}$$

(i) $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, $\lambda_1 = 5$, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_2 = \lambda_3 = 1$, $\mathbf{e} = \alpha\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ so let $\mathbf{e}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$,

and $\mathbf{e}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. $\mathbf{D} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\mathbf{Q}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{pmatrix}$,

$$e^{\mathbf{A}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^5 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}\frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} e^5+3e & 2e^5-2e & e^5-e \\ e^5-e & 2e^5+2e & e^5-e \\ e^5-e & 2e^5-2e & e^5+3e \end{pmatrix}$$

(k) This $\mathbf{A}$ is not diagonalizable because $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\mathbf{e} = \alpha(1,0,0,0)^T$ gives only one LI eigenvector. However, $\mathbf{A}$ is nilpotent, so

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots = \mathbf{I} + \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathbf{0}\text{'s} = \begin{pmatrix} 1 & 1 & 4 & \frac{31}{2} \\ 0 & 1 & 4 & 17 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(l) Same idea as in (k):

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots = \mathbf{I} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 9 & 3 & 0 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{pmatrix} + \mathbf{0}\text{'s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 7 & 3 & 1 & 0 \\ \frac{15}{2} & \frac{7}{2} & 1 & 1 \end{pmatrix}$$

4. (a) The Maple commands

```
with(linalg):
A:=array([[1,2],[0,3]]);
exponential(A);
```

give $e^{\mathbf{A}} = \begin{pmatrix} e & e^3-e \\ 0 & e^3 \end{pmatrix}$

Section 11.6

1\. (b) $\underset{\sim}{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (c) $\underset{\sim}{A} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ (e) $\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ (f) $\underset{\sim}{A} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$

2\. (b) $\lambda_1 = -1,\ \underset{\sim}{e}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix};\ \lambda_2 = 2,\ \underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix};\ \lambda_3 = 0,\ \underset{\sim}{e}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$\underset{\sim}{Q} = (\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3) = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{pmatrix},\ f = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \lambda_3 \tilde{x}_3^2 = -\tilde{x}_1^2 + 2\tilde{x}_2^2 + 0\tilde{x}_3^2$$

f and $\underset{\sim}{A}$ are neither positive nor negative definite.

(c) $\lambda_1 = 1/2,\ \underset{\sim}{e}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix};\ \lambda_2 = 3/2,\ \underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \underset{\sim}{Q} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},\ f = \frac{1}{2}\tilde{x}_1^2 + \frac{3}{2}\tilde{x}_2^2,$ so f and $\underset{\sim}{A}$ are positive definite

(e) $\lambda_1 = 2,\ \underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix};\ \lambda_2 = \lambda_3 = \lambda_4 = 0,\ \underset{\sim}{e} = \alpha\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

↑ these are orthog., so let them be $\underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4$

$$\underset{\sim}{Q} = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix},\ f = 2\tilde{x}_1^2 + 0\tilde{x}_2^2 + 0\tilde{x}_3^2 + 0\tilde{x}_4^2.$$

f and $\underset{\sim}{A}$ are not quite positive definite (i.e., f "definitely positive"), but we say that it is positive <u>semi</u>-definite (since $f \geq 0$ for all $\underset{\sim}{x} \neq \underset{\sim}{0}$)

(f) $\lambda_1 = \lambda_2 = 2,\ \underset{\sim}{e} = \alpha\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\underset{\sim}{e}_1} + \beta\underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\underset{\sim}{e}_2};\ \lambda_3 = \lambda_4 = -2,\ \underset{\sim}{e} = \gamma\underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\underset{\sim}{e}_3} + \delta\underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{\underset{\sim}{e}_4}$

$$\underset{\sim}{Q} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{pmatrix},\ f = 2\tilde{x}_1^2 + 2\tilde{x}_2^2 - 2\tilde{x}_3^2 - 2\tilde{x}_4^2$$

so f and $\underset{\sim}{A}$ are neither positive definite nor negative definite.

3\. (a) $a_{11} = 4 > 0,\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15 > 0$, so f and $\underset{\sim}{A}$ are positive definite.

Check: $\lambda_1 = 3,\ \lambda_2 = 5$ and both are positive. ✓

(b) $a_{11} = 1 > 0$, but $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, so <u>not</u> positive definite.

Check: $\lambda_1 = 2,\ \lambda_2 = \underline{\underline{0}}$ ✓

(c) $a_{11} = 2 > 0,\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$ so f and $\underset{\sim}{A}$ <u>are</u> positive definite.

Check: $\lambda_1 = 1,\ \lambda_2 = 3$ ✓

(d) $a_{11} = 0 \not> 0$ so we can stop right there; f and $\underset{\sim}{A}$ are <u>not</u> positive definite.

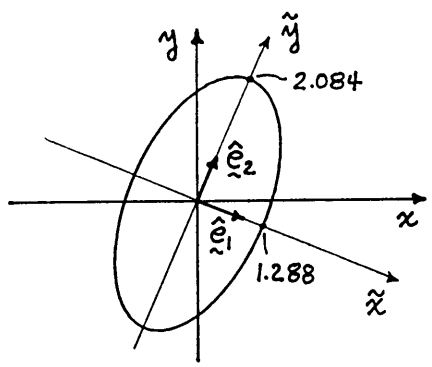
Check: λ's $= -1, -1, 2$. ✓

(e) $a_{11} = 6 > 0,\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0,\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 36 > 0$ so f and $\underset{\sim}{A}$ are positive definite. Check: λ's $= 6, 2, 3$. ✓

4. (a) $f = 3x^2+2y^2-2xy = 6$, $\underset{\sim}{A} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, $\lambda_1 = \frac{5+\sqrt{5}}{2}$, $\underset{\sim}{e}_1 = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$; $\lambda_2 = \frac{5-\sqrt{5}}{2}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}$

Then let $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$, where $\underset{\sim}{Q} = [\underset{\sim}{\hat{e}}_1, \underset{\sim}{\hat{e}}_2]$, gives $f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 = 6$ or, $\frac{5+\sqrt{5}}{2}\tilde{x}^2 + \frac{5-\sqrt{5}}{2}\tilde{y}^2 = 6$, which is an ellipse

$$\left(\frac{\tilde{x}}{1.288}\right)^2 + \left(\frac{\tilde{y}}{2.084}\right)^2 = 1$$



with the intercepts 1.288 and 2.084 on the $\tilde{x}$ and $\tilde{y}$ axes, respectively. But how are the $\tilde{x}, \tilde{y}$ axes oriented in the x,y plane? Recall that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[\left(\underset{\sim}{\hat{e}}_1\right)\left(\underset{\sim}{\hat{e}}_2\right) \right] \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

Thus, the point $\tilde{x}=1, \tilde{y}=0$ (which will locate for us the orientation of the positive $\tilde{x}$ axis) corresponds to $\begin{pmatrix} x \\ y \end{pmatrix} = \left[\left(\underset{\sim}{\hat{e}}_1\right)\left(\underset{\sim}{\hat{e}}_2\right) \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\underset{\sim}{\hat{e}}_1\right)$

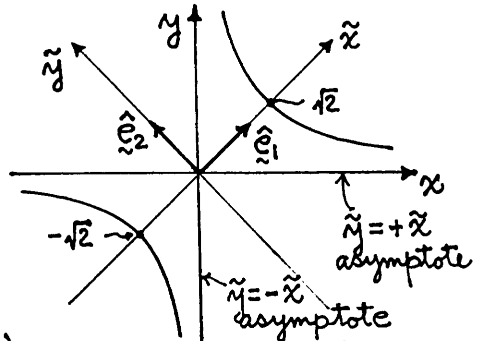
and, similarly, $\tilde{x}=0, \tilde{y}=1$ corresponds to $\begin{pmatrix} x \\ y \end{pmatrix} = \left[\left(\underset{\sim}{\hat{e}}_1\right)\left(\underset{\sim}{\hat{e}}_2\right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\underset{\sim}{\hat{e}}_2\right)$.

Thus, the picture is as sketched (not to scale) above. We see that the transformation $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ effects a pure rotation from x,y to $\tilde{x},\tilde{y}$. (Such transformations were discussed in the optional Sec. 10.7, but that section is not a prerequisite for this exercise.)

(b) Of course, it's easy to see that this is an hyperbola, but let us proceed in the manner specified. $A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, $\lambda_1 = \frac{1}{2}$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -\frac{1}{2}$, $\underset{\sim}{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then proceed as in (a): $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$, where $\underset{\sim}{Q} = [\underset{\sim}{\hat{e}}_1, \underset{\sim}{\hat{e}}_2]$, gives $f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 = 1$, or,

$$\frac{1}{2}\tilde{x}^2 - \frac{1}{2}\tilde{y}^2 = 1, \text{ or, } \left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 - \left(\frac{\tilde{y}}{\sqrt{2}}\right)^2 = 1$$

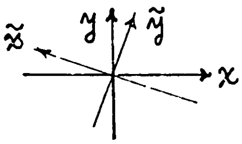
which is the hyperbola sketch (not to scale) at the right, the asymptotes being $\tilde{y} = \pm\tilde{x}$.



(c) $f = 4y^2+3xy = 1$, $\underset{\sim}{A} = \begin{pmatrix} 0 & 3/2 \\ 3/2 & 4 \end{pmatrix}$, $\lambda_1 = -\frac{1}{2}$, $\underset{\sim}{e}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$; $\lambda_2 = \frac{9}{2}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Then proceed as in (a): $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$, where $\underset{\sim}{Q} = (\underset{\sim}{\hat{e}}_1, \underset{\sim}{\hat{e}}_2)$ gives $f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 = 1$, or,

$$-\frac{1}{2}\tilde{x}^2 + \frac{9}{2}\tilde{y}^2 = 1, \text{ or, } -\left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 + \left(\frac{\tilde{y}}{\sqrt{2/3}}\right)^2 = 1,$$

which gives the hyperbola shown (not to scale) below.



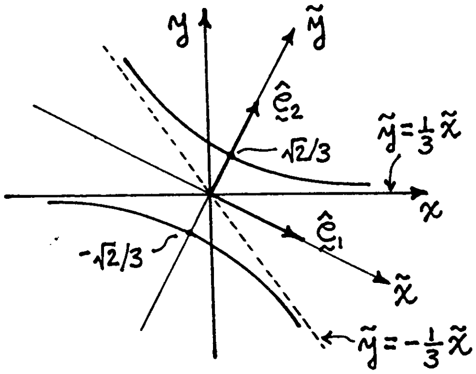
However, we see immediately that the transformation from x, y to $\tilde{x}, \tilde{y}$ is, this time, a rotation plus a reflection. (If you did study Section 10.7, in particular Example 1 therein, then you will realize that this outcome follows from the fact that $\det \underset{\sim}{Q} = -1$ rather than $+1$.) If we prefer a pure rotation we can achieve that by reversing the direction of one of the $\underset{\sim}{e}_j$'s. For example, let us take $\underset{\sim}{e}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\underset{\sim}{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Then we obtain

$$f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 = 1,$$

or,

$$-\left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 + \left(\frac{\tilde{y}}{\sqrt{2}/3}\right)^2 = 1$$

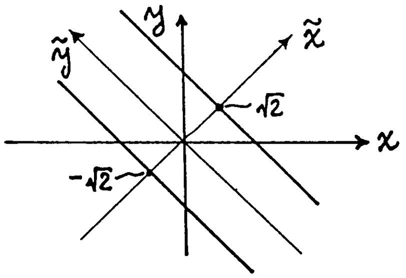
as before, but this time the picture is as shown (not to scale) at the right: an hyperbola with real intercepts $\tilde{y} = \pm\sqrt{2}/3$ and asymptotes $\tilde{y} = \pm\frac{1}{3}\tilde{x}$.



(e) $f = x^2 + y^2 + 2xy = 4$, $\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\lambda_1 = 2$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 0$, $\underset{\sim}{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then proceed as in (a): $\underset{\sim}{x} = \underset{\sim}{Q}\,\tilde{\underset{\sim}{x}}$, where $\underset{\sim}{Q} = (\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2)$ gives $f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 = 4$, or,

$$2\tilde{x}^2 + 0\tilde{y}^2 = 4,$$

or, $\left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 + 0\tilde{y}^2 = 1$ or $\tilde{x} = \pm\sqrt{2}$



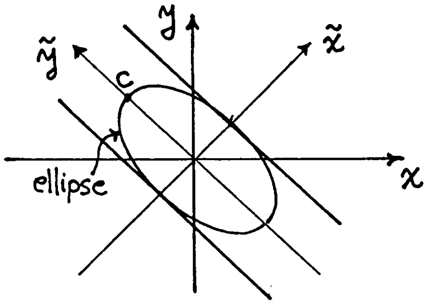
as shown at the right. We can think of this case as a borderline between an ellipse and an hyperbola for it is the limit of the ellipses

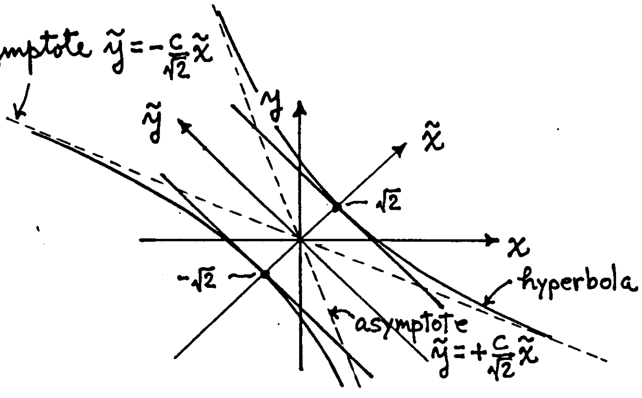
$$\left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 + \left(\frac{\tilde{y}}{c}\right)^2 = 1$$

as $c \to \infty$ (below left) and it is also the limit of the hyperbolas

$$\left(\frac{\tilde{x}}{\sqrt{2}}\right)^2 - \left(\frac{\tilde{y}}{c}\right)^2 = 1$$

as $c \to \infty$.





5. (a) $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{\tilde{x}}$ is $\begin{array}{l} x_1 = \tilde{x}_1 - \frac{1}{2}\tilde{x}_2 \\ x_2 = \tilde{x}_2 \end{array}$, or, $\underset{\sim}{x} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \underset{\sim}{\tilde{x}}$ so $\underset{\sim}{Q} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}$.

A modal matrix is necessarily composed of ON columns and this Q does not have ON column vectors, so it cannot possibly be a modal matrix.

(b)
$$\begin{aligned} f &= x_1^2 + x_2^2 + 4x_3^2 + 2x_1x_2 - x_1x_3 \\ &= (x_1^2 + 2x_1x_2 + x_2^2) + 4(x_3^2 - \tfrac{1}{4}x_1x_3 + \tfrac{1}{64}x_1^2) - \tfrac{1}{16}x_1^2 \\ &= \tilde{x}_1^2 + 4\tilde{x}_2^2 - \tfrac{1}{16}\tilde{x}_3^2 , \end{aligned}$$

where

$$\begin{aligned} \tilde{x}_1 &= x_1 + x_2 \\ \tilde{x}_2 &= x_3 - \tfrac{1}{8}x_1 \\ \tilde{x}_3 &= x_1 \end{aligned} \qquad \text{or,} \qquad \begin{aligned} x_1 &= \tilde{x}_3 \\ x_2 &= \tilde{x}_1 - \tilde{x}_3 \\ x_3 &= \tilde{x}_2 + \tfrac{1}{8}\tilde{x}_3 \end{aligned}$$

(c)
$$\begin{aligned} f &= x_1^2 + 4x_1x_2 + x_2x_3 \\ &= (x_1^2 + 4x_1x_2 + 4x_2^2) - 4x_2^2 + x_2x_3 \\ &= (x_1 + 2x_2)^2 - 4(x_2^2 - \tfrac{1}{4}x_2x_3 + \tfrac{1}{64}x_3^2) + \tfrac{1}{16}(x_2 - \tfrac{1}{8}x_3)^2 \\ &= \tilde{x}_1^2 - 4\tilde{x}_2^2 + \tfrac{1}{16}\tilde{x}_3^2 \end{aligned}$$

where

$$\tilde{x}_1 = x_1 + 2x_2 , \quad \tilde{x}_2 = x_2 - \tfrac{1}{8}x_3 , \quad \tilde{x}_3 = \tfrac{1}{4}x_3$$

(d)
$$\begin{aligned} f &= 4x_1^2 + 2x_1x_3 + x_2x_3 \\ &= 4(x_1^2 + \tfrac{1}{2}x_1x_3 + \tfrac{1}{16}x_3^2) - \tfrac{1}{4}x_3^2 + x_2x_3 \\ &= 4(x_1 + \tfrac{1}{4}x_3)^2 - \tfrac{1}{4}(x_3^2 - 4x_2x_3 + 4x_2^2) + x_2^2 \\ &= 4\tilde{x}_1^2 - \tfrac{1}{4}\tilde{x}_2^2 + \tilde{x}_3^2 \end{aligned}$$

where

$$\begin{aligned} \tilde{x}_1 &= x_1 + \tfrac{1}{4}x_3 \\ \tilde{x}_2 &= 2x_2 - x_3 \\ \tilde{x}_3 &= x_2 \end{aligned}$$

(e)
$$\begin{aligned} f = x_3^2 + 2x_1x_3 &= (x_3^2 + 2x_1x_3 + x_1^2) - x_1^2 \\ &= (x_1 + x_3)^2 - x_1^2 \\ &= \tilde{x}_1^2 - \tilde{x}_2^2 + 0\tilde{x}_3^2 \end{aligned}$$

where $\tilde{x}_1 = x_1 + x_3$, $\tilde{x}_2 = x_1$, and $\tilde{x}_3 = ax_1 + bx_2 + cx_3$ for any choice of constants a, b, c

(f) $f = 2x_1x_2 = (x_1^2 + 2x_1x_2 + x_2^2) - x_1^2 - x_2^2$. Evidently not.

6. Then $V = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 - Pz \sim \frac{1}{2}kx^2 + \frac{1}{2}ky^2 - P\left(\frac{x^2 - xy + y^2}{L}\right)$

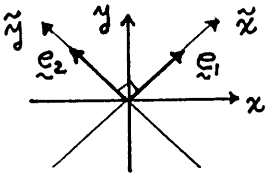
$f(x,y) = \left(\frac{k}{2} - \frac{P}{L}\right)x^2 + \left(\frac{k}{2} - \frac{P}{L}\right)y^2 + \frac{P}{L}xy ,$

so $\underset{\sim}{A} = \begin{pmatrix} \frac{k}{2} - \frac{P}{L} & \frac{P}{2L} \\ \frac{P}{2L} & \frac{k}{2} - \frac{P}{L} \end{pmatrix}$, with

$$\lambda_1 = \frac{k}{2}\left(1 - \frac{P}{kL}\right), \quad \underset{\sim}{e}_1 = \begin{pmatrix}1\\1\end{pmatrix}$$
$$\lambda_2 = \frac{k}{2}\left(1 - \frac{3P}{kL}\right), \quad \underset{\sim}{e}_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

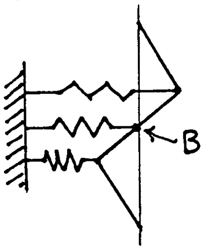
Thus, the modal matrix transformation $\underset{\sim}{x} = \underset{\sim}{Q}\,\underset{\sim}{\tilde{x}}$, or $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$, reduces f to the canonical form

$$f = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2.$$

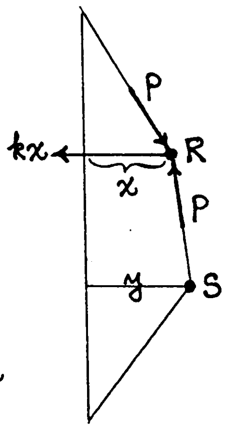


Imagine the f surface plotted normal to the plane of the paper. If P=0 then $\lambda_1 = k/2 > 0$ and $\lambda_2 = k/2 > 0$, so the f surface has a minimum at the origin. As P is increased λ_1 and λ_2 both decrease so the valley in the $\tilde{x}$ direction and the valley in the $\tilde{y}$ direction become shallower until, when P reaches the value kL/3, the valley in the $\tilde{y}$ direction becomes flat; as P increases further the f surface develops a hill at the origin along the $\tilde{y}$ axis, so the system becomes unstable because movement in the $\pm\tilde{y}$ direction gives a reduction in the potential V. (This saddle persists, as P is increased further, until P increases beyond kL and the saddle gives way to a maximum.)

Thus, the critical value of P, to initiate buckling, is $P_{cr} = kL/3$. Buckling will then occur along the $\tilde{y}$ axis, along which $y = -x$, so that the system will begin to collapse in the "s-shape" configuration shown at the right. (Of course, we can't predict if it will be ⟩ or ⟨ but only that $y \sim -x$ when it starts to buckle.) Observe that in this buckling configuration the middle spring is inactive because there is no horizontal displacement of the point B. Thus, whether that spring is kept (as in Example 4) or deleted (as in this exercise) is immaterial. That is why the buckling load and the buckling configuration are the same, whether that spring is included or not.



NOTE: We used a potential method in Example 4 because the method is convenient and because it focuses on the quadratic form for V. Alternatively, we could have used force balances (in the horizontal direction) on the pins at R and S. At R, for example, the horizontal force balance gives $-kx + P\frac{x}{L} - P\frac{y-x}{L} = 0$, which is the same as we obtain from (25) by setting $\partial V/\partial x = 0$. Similarly for the pin at S. The result would be the equilibrium equations



$$\begin{pmatrix} \frac{k}{2}-\frac{P}{L} & \frac{P}{2L} \\ \frac{P}{2L} & \frac{k}{2}-\frac{P}{L} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ or, } \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\overset{\lambda}{\boxed{\frac{kL}{P}}}\begin{pmatrix} x \\ y \end{pmatrix}$$

with $\lambda_1 = 1$, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 3$, $e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. That is, we have the unique trivial solution $x = y = 0$ unless $\lambda = 1$ or 3; $\lambda = 1$ means $P = kL$ and $\lambda = 3$ means $P = kL/3$. The latter occurs before the former so $P_{cr} = kL/3$, and the corresponding buckling configuration is the nontrivial solution $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, as obtained above. Incidentally, we have shown the compressive force in each link as P, in the diagram above. Of course those forces are not exactly P, but they $\sim P$ as $x \to 0$ and $y \to 0$.

NOTE: You might consider expanding on Example 4 in class. For example, the physical significance of the eigenvectors is not discussed in the text in Example 4, nor is the final note, above, that relates the potential approach to the force equilibrium approach.

CHAPTER 12

Section 12.2

1. (b) $\|\underset{\sim}{v}\| = (1+i)\overline{(1+i)} + (1-i)\overline{(1-i)} = (1+i)(1-i) + (1-i)(1+i) = 4$, so $\hat{\underset{\sim}{v}} = \frac{1}{4}(1+i, 1-i)$

   (c) $\|\underset{\sim}{v}\| = \sqrt{(1^2 + 3^2 + (-2)^2 + 0^2)} = \sqrt{14}$, so $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{14}}(1,3,-2,0)$

   (f) $\|\underset{\sim}{v}\| = \sqrt{(i\bar{i} + (-i)(\overline{-i}))} = \sqrt{2}$, so $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{2}}(i,0,0,-i)$

   (g) $\|\underset{\sim}{v}\| = \sqrt{(x^2+y^2+z^2+(ict)(\overline{ict}))}$, so $\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{x^2+y^2+z^2+c^2t^2}}(x,y,z,ict)$

2. (b) Vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ are a basis for $\mathbb{C}^3$ if any given vector $\underset{\sim}{u}$ can be expanded uniquely in terms of them: $a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + a_3\underset{\sim}{v}_3 = \underset{\sim}{u}$, or,

$$\begin{aligned} a_1v_{11} + a_2v_{21} + a_3v_{31} &= u_1 \\ a_1v_{12} + a_2v_{22} + a_3v_{32} &= u_2 \quad \text{or} \quad [(\underset{\sim}{v}_1)(\underset{\sim}{v}_2)(\underset{\sim}{v}_3)]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \underset{\sim}{u}, \\ a_1v_{13} + a_2v_{23} + a_3v_{33} &= u_3 \end{aligned}$$

which admits a unique solution for $\underset{\sim}{a}$ iff $r[\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3] = 3$. Well,

$$\det|\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3| = \det\begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & -2 & 4 \end{vmatrix} = 8 \neq 0$$ so $r = 3$. Yes, they are a basis for $\mathbb{C}^3$.

   (c) As in (b), above, $\det\begin{vmatrix} 4 & 3+i & -2-2i \\ 1-2i & i & 1-4i \\ 0 & -2i & 4i \end{vmatrix} = 0$, so the set is not a basis for $\mathbb{C}^3$.

NOTE: In maple, i is denoted as I. To evaluate the preceding determinant using maple, use these commands:

```
with(linalg):
A:=array([[4,3+I,-2-2*I],[1-2*I,I,1-4*I],[0,-2*I,4*I]]);
det(A);
```

   (d) These vectors are orthogonal, so they are surely LI. Thus, they are a basis for $\mathbb{C}^3$.

   (e) No; the dimension of $\mathbb{C}^3$ is 3, and the dimension of a space is, by definition, the greatist number of LI vectors that can be found in the space. Thus, the 4 given vectors cannot possibly be LI, so they cannot be a basis for $\mathbb{C}^3$.

3. Need merely verify that $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = 0$, $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = 0$, $\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 0$ so the set is orthogonal. It follows, then, that it is a basis for $\mathbb{C}^3$.

4. We need to be careful since it follows from the definition that the order of the vectors matters, as found in (5a).

$$\underset{\sim}{u} = \alpha_1\underset{\sim}{e}_1 + \cdots + \alpha_k\underset{\sim}{e}_k \quad \star$$

Since $\underset{\sim}{u}\cdot\underset{\sim}{v} = \overline{\underset{\sim}{v}\cdot\underset{\sim}{u}}$ we need to be careful to pre-dot (or postdot) both sides of $\star$ with $\underset{\sim}{e}_j$. Postdotting, for example, gives

$$\begin{aligned} \underset{\sim}{u}\cdot\underset{\sim}{e}_j &= (\alpha_1\underset{\sim}{e}_1 + \cdots + \alpha_k\underset{\sim}{e}_k)\cdot\underset{\sim}{e}_j = (\alpha_1\underset{\sim}{e}_1)\cdot\underset{\sim}{e}_j + \cdots + (\alpha_k\underset{\sim}{e}_k)\cdot\underset{\sim}{e}_k \\ &= \alpha_1(\underset{\sim}{e}_1\cdot\underset{\sim}{e}_j) + \cdots + \alpha_k(\underset{\sim}{e}_k\cdot\underset{\sim}{e}_j) = 0 + \cdots + 0 + \alpha_j(\underset{\sim}{e}_j\cdot\underset{\sim}{e}_j) + 0 + \cdots + 0 \end{aligned}$$

so $\alpha_j = \underset{\sim}{u}\cdot\underset{\sim}{e}_j / \underset{\sim}{e}_j\cdot\underset{\sim}{e}_j$, as in (23) in Sec. 9.9. Note that the 3rd equality follows from definition $\underset{\sim}{u}\cdot\underset{\sim}{v} = \sum_1^k u_j\bar{v}_j$; a scale factor in $\underset{\sim}{u}$ comes out without a complex conjugate bar.

Consider the basis $\mathbf{e}_1 = (i,1,0)$, $\mathbf{e}_2 = (2,2i,1)$, $\mathbf{e}_3 = (1,i,-4)$.

(a) $$\mathbf{u} = (2+4i,-3i,2) = \frac{(2+4i)(-i)+(-3i)(1)+(2)(0)}{(i)(-i)+(1)(1)+(0)(0)}\mathbf{e}_1 + \frac{(2+4i)(2)+(-3i)(-2i)+(2)(1)}{(2)(2)+(2i)(-2i)+(1)(1)}\mathbf{e}_2$$
$$+ \frac{(2+4i)(1)+(-3i)(-i)+(2)(-4)}{(1)(1)+(i)(-i)+(-4)(-4)}\mathbf{e}_3$$
$$= \frac{4-5i}{2}\mathbf{e}_1 + \frac{8i}{9}\mathbf{e}_2 - \frac{9-4i}{18}\mathbf{e}_3$$

(b) $\mathbf{u} = (0,0,1) = 0\mathbf{e}_1 + \frac{1}{9}\mathbf{e}_2 - \frac{4}{18}\mathbf{e}_3$

(c) $\mathbf{u} = (1,1,1) = \frac{1-i}{2}\mathbf{e}_1 + \frac{3-2i}{9}\mathbf{e}_2 + \frac{3+i}{18}\mathbf{e}_3$

(d) $\mathbf{u} = (i,2i,3i) = \frac{1+2i}{2}\mathbf{e}_1 + \frac{4+5i}{9}\mathbf{e}_2 + \frac{2-11i}{18}\mathbf{e}_3$

(e) $\mathbf{u} = (0,i,0) = \frac{i}{2}\mathbf{e}_1 - \frac{2}{9}\mathbf{e}_2 + \frac{1}{18}\mathbf{e}_3$

(f) $\mathbf{u} = (1-i,0,0) = -\frac{1+i}{2}\mathbf{e}_1 + \frac{2-2i}{9}\mathbf{e}_2 + \frac{1-i}{18}\mathbf{e}_3$

6. $$\mathbf{u}\cdot\mathbf{u} = u_1\bar{u}_1 + \cdots + u_n\bar{u}_n = |u_1|^2 + \cdots + |u_n|^2 \begin{array}{l} > 0 \text{ if } u_1,\dots,u_n \text{ are not all } 0 \\ = 0 \text{ if } \quad '' \quad \text{are all } 0, \end{array}$$

which proves (5b). Next,
$$\begin{aligned}(\alpha\mathbf{u}+\beta\mathbf{v})\cdot\mathbf{w} &= [\alpha u_1+\beta v_1,\dots,\alpha u_n+\beta v_n]\cdot(w_1,\dots,w_n) \\ &= (\alpha u_1+\beta v_1)\bar{w}_1 + \cdots + (\alpha u_n+\beta v_n)\bar{w}_n \\ &= \alpha(u_1\bar{w}_1+\cdots+u_n\bar{w}_n) + \beta(v_1\bar{w}_1+\cdots+v_n\bar{w}_n) \\ &= \alpha(\mathbf{u}\cdot\mathbf{w}) + \beta(\mathbf{v}\cdot\mathbf{w})\end{aligned}$$

7. We will mimic the proof given in Chapter 9.
$$(\mathbf{u}+\alpha\mathbf{v})\cdot(\mathbf{u}+\alpha\mathbf{v}) \geq 0$$
$$\|\mathbf{u}\|^2 + \alpha\,\mathbf{v}\cdot\mathbf{u} + \bar{\alpha}\,\mathbf{u}\cdot\mathbf{v} + \alpha\bar{\alpha}\,\|\mathbf{v}\|^2 \geq 0$$
$$\|\mathbf{u}\|^2 + \alpha\,\overline{\mathbf{u}\cdot\mathbf{v}} + \bar{\alpha}\,\mathbf{u}\cdot\mathbf{v} + |\alpha|^2\|\mathbf{v}\|^2 \geq 0$$

Let $\alpha = a+ib$ and $\mathbf{u}\cdot\mathbf{v} = c+id$. Then
$$\|\mathbf{u}\|^2 + (a+ib)(c-id) + (a-ib)(c+id) + |\alpha|^2\|\mathbf{v}\|^2 \geq 0$$
$$\|\mathbf{u}\|^2 + 2ac + 2bd + (a^2+b^2)\|\mathbf{v}\|^2 \geq 0$$

Set $\partial/\partial a = 2c + 2a\|\mathbf{v}\|^2 = 0 \rightarrow$ optimal $a = -c/\|\mathbf{v}\|^2$

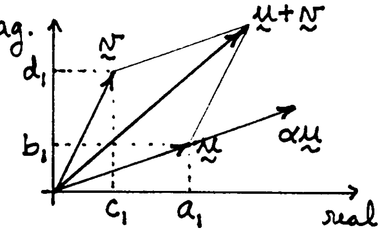
Set $\partial/\partial b = 2d + 2b\|\mathbf{v}\|^2 = 0 \rightarrow$ optimal $b = -d/\|\mathbf{v}\|^2$

so, with these choices,
$$\|\mathbf{u}\|^2 - \frac{2c^2}{\|\mathbf{v}\|^2} - \frac{2d^2}{\|\mathbf{v}\|^2} + \frac{c^2+d^2}{\|\mathbf{v}\|^4}\|\mathbf{v}\|^2 \geq 0,$$
$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - 2(c^2+d^2) + (c^2+d^2) \geq 0,$$
$$c^2+d^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2$$
$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\| \quad \checkmark$$

8. (7a): $\|\alpha \underset{\sim}{u}\| = \sqrt{\sum_1^n |\alpha u_j|^2} = |\alpha| \sqrt{\sum_1^n |u_j|^2} = |\alpha| \underset{\sim}{u}$.

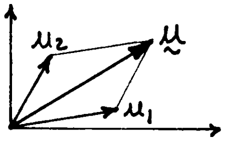
(7b): $\|\underset{\sim}{u}\| = \sqrt{\sum_1^n |u_j|^2}$ clearly >0 for all $\underset{\sim}{u} \neq \underset{\sim}{0}$ (i.e., for the u_j's not all zero) and $=0$ for $\underset{\sim}{u} = \underset{\sim}{0}$.

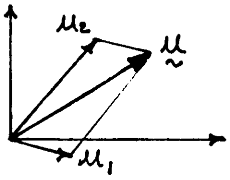
9. Yes for $\mathbb{C}^1$ since if $\underset{\sim}{u} = (u_1) = (a_1 + ib_1)$ and $\underset{\sim}{v} = (v_1) = (c_1 + id_1)$ are any two vectors in $\mathbb{C}^1$ then they, and $\alpha \underset{\sim}{u} = (\alpha a_1 + i\alpha b_1)$, and $\underset{\sim}{u} + \underset{\sim}{v} = (a_1 + c_1 + i(b_1 + d_1))$ can be displayed graphically as shown at the right.



No for $\mathbb{C}^2$ and $\mathbb{C}^3$ since we would need 4 dimensions for $\mathbb{C}^2$ and 6 for $\mathbb{C}^3$. Suppose we try to use a single complex plane for $\mathbb{C}^2$, say. Since u_1 and u_2 are complex, in $\underset{\sim}{u} = (u_1, u_2)$, we might try representing $\underset{\sim}{u}$ as shown at the right [in (a)]. However, although the components u_1, u_2 uniquely determine the arrow representation of $\underset{\sim}{u}$, the reverse is not true, as seen by comparing (a) and (b). Further, there does not then exist a unique zero vector because $\underset{\sim}{u} = (u_1, -u_1)$ will be displayed as a point (i.e., as $\underset{\sim}{0}$) for any complex number u_1.

(a)





Section 12.3

1. (b) $\underset{\sim}{A} = \begin{pmatrix} 1+i & 3i \\ 1 & 2-i \end{pmatrix}$. $\det \underset{\sim}{A} = 3+i$; $M_{11} = 2-i, M_{12} = 1, M_{21} = 3i, M_{22} = 1+i$

$A_{11} = 2-i, A_{12} = -1, A_{21} = -3i, A_{22} = 1+i$, $\underset{\sim}{A}^{-1} = \frac{1}{3-2i}\begin{pmatrix} 2-i & -3i \\ -1 & 1+i \end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix} 8+i & 6-9i \\ -3-2i & 1+5i \end{pmatrix}$

(c) $\underset{\sim}{A} = \begin{pmatrix} 3 & 1+i \\ 1-i & 2 \end{pmatrix}$. $\det \underset{\sim}{A} = 4$; $M_{11} = 2, M_{12} = 1-i, M_{21} = 1+i, M_{22} = 3$

$A_{11} = 2, A_{12} = -1+i, A_{21} = -1-i, A_{22} = 3$, $\underset{\sim}{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1-i \\ -1+i & 3 \end{pmatrix}$

2. (d) The Maple commands

```
with(linalg):
A:= array([[1,3,I],[0,1+I,1],[0,0,2]]);
inverse(A);
```

gives $\underset{\sim}{A}^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -6+6i & 3-5i \\ 0 & 2-2i & -1+i \\ 0 & 0 & 2 \end{pmatrix}$

(e) $\underset{\sim}{A}^{-1}$ does not exist because $\det \underset{\sim}{A} = 0$. (The 3rd row is a multiple of the 1st.)

3. (b) $|\underset{\sim}{A} - \lambda \underset{\sim}{I}| = \begin{vmatrix} 1+i-\lambda & 3i \\ 1 & 2-i-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + (3-2i) = 0$ so $\lambda = (3 \pm \sqrt{-3+8i})/2$

$\lambda_1 = (3+\sqrt{-3+8i})/2$: $\begin{pmatrix} 1+i-\frac{3+\sqrt{-3+8i}}{2} & 3i \\ 1 & 2-i-\frac{3+\sqrt{-3+8i}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

gives, by Gauss elim., $\begin{pmatrix} 1+i-\frac{3+\sqrt{-3+8i}}{2} & 3i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so

$$x_2 = \alpha,\ x_1 = \frac{-3i\alpha}{1+i-\frac{3+\sqrt{-3+8i}}{2}} = \frac{-6i\alpha}{-1+2i-\sqrt{-3+8i}}\,\frac{-1+2i+\sqrt{-3+8i}}{-1+2i+\sqrt{-3+8i}} = \frac{12-6i\sqrt{-3+8i}+6i}{\cancel{-4}-4i+\cancel{1}+\cancel{3}-8i}\alpha$$
$$= \frac{2i-1+\sqrt{-3+8i}}{2}\alpha$$

so $\underset{\sim}{e}_1 = \alpha\left(\frac{2i-1+\sqrt{-3+8i}}{2}, 1\right)^T$

Similarly, $\lambda_2 = (3-\sqrt{-3+8i})/2$ gives (by changing all $+\sqrt{\ }$'s to $-\sqrt{\ }$'s)

$$e_2 = \beta\left(\frac{2i-1-\sqrt{-3+8i}}{2}, 1\right)^T$$

The $\underset{\sim}{A}$ matrix was not Hermitian so the λ's need not be real (indeed, they are not) and the $\underset{\sim}{e}$'s need not be orthogonal (indeed, they are not since $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = \alpha\bar{\beta}(0.115 - 2.865i) \neq 0$). Finally the Maple command evalf(sqrt(-3+8*I)); gives $1.6649 + 2.4025i$, so

$\lambda_1 \approx 2.332 + 1.201i$, $\underset{\sim}{e}_1 \approx \alpha(0.332 + 2.201i, 1)^T$
$\lambda_2 \approx 0.668 - 1.201i$, $\underset{\sim}{e}_2 \approx \beta(-1.332 - 0.201i, 1)^T$

4. (c) The Maple commands

```
with(linalg):
A:=array([[3,1+I],[1-I,2]]);
eigenvects(A);
```

give $\lambda_1 = 4$, $\underset{\sim}{e}_1 = (1, \frac{1}{2} - \frac{1}{2}I)^T$
$\lambda_2 = 1$, $\underset{\sim}{e}_2 = (-\frac{1}{2} - \frac{1}{2}I, 1)^T$

In this case $\underset{\sim}{A}$ is Hermitian and, sure enough, the λ's are real and $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = (1)(-\frac{1}{2} + \frac{1}{2}i) + (\frac{1}{2} - \frac{1}{2}i)(1) = 0$ so the $\underset{\sim}{e}$'s are orthogonal.

(d) $\lambda_1 = 1$, $\underset{\sim}{e}_1 = (1,0,0)^T$
$\lambda_2 = 2$, $\underset{\sim}{e}_2 = (4+i, 1, 1-i)^T$
$\lambda_3 = 1+i$, $\underset{\sim}{e}_3 = (1, i/3, 0)^T$

$\underset{\sim}{A}$ is not Hermitian so the λ's need not be real and the $\underset{\sim}{e}$'s need not be orthogonal. In fact, not all the λ's are real, and the $\underset{\sim}{e}$'s are not orthogonal (e.g., $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = 1$).

(e) $\lambda_1 = 0$, $\underset{\sim}{e}_1 = (2-2i, 0, 1)^T$
$\lambda_2 = -3$, $\underset{\sim}{e}_2 = (-i/3, 1, -\frac{2}{3} + \frac{2}{3}i)^T$
$\lambda_3 = 3$, $\underset{\sim}{e}_3 = (i/3, 1, \frac{2}{3} - \frac{2}{3}i)^T$

$\underset{\sim}{A}$ is Hermitian and, sure enough, the λ's are real and the $\underset{\sim}{e}$'s are orthogonal: $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = (2-2i)(\frac{i}{3}) + 0 + (1)(-\frac{2}{3} - \frac{2}{3}i) = 0$ ✓

$\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = (2-2i)(-\frac{i}{3}) + 0 + (1)(\frac{2}{3} + \frac{2}{3}i) = 0$ ✓

$\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = (-i/3)(-i/3) + 1 + (-\frac{2}{3} + \frac{2}{3}i)(\frac{2}{3} + \frac{2}{3}i) = 0$ ✓

5. (a) $\lambda_1=-1,\ \mathbf{e}_1=(-1,1)^T$; $\lambda_2=2,\ \mathbf{e}_2=(2,1)^T$

(b) $\lambda_1=2+i,\ \mathbf{e}_1=(-i,1)^T$; $\lambda_2=2-i,\ \mathbf{e}_2=(i,1)^T$

(c) $\lambda_1=(5+\sqrt{33})/2,\ \mathbf{e}_1=(1,(3+\sqrt{33})/4)^T$; $\lambda_2=(5-\sqrt{33})/2,\ \mathbf{e}_2=(1,(3-\sqrt{33})/4)^T$

(d) $\lambda_1=(3+4i)/2,\ \mathbf{e}_1=(1,(5-4i)/10)^T$; $\lambda_2=(3-4i)/2,\ \mathbf{e}_2=(1,(5+4i)/10)^T$

7. (b) $\mathbf{A}=\begin{pmatrix}2 & -2i\\ 2i & 5\end{pmatrix}$ from (16). $\lambda_1=6,\ \mathbf{e}_1=(-\frac{i}{2},1)^T,\ \|\mathbf{e}_1\|^2=\mathbf{e}_1\cdot\mathbf{e}_1=\frac{5}{4}$ so $\hat{\mathbf{e}}_1=\frac{2}{\sqrt5}(-\frac{i}{2},1)^T$;

$\lambda_2=1,\ \mathbf{e}_2=(2i,1)^T,\ \|\mathbf{e}_2\|^2=\mathbf{e}_2\cdot\mathbf{e}_2=5$ so $\hat{\mathbf{e}}_2=\frac{1}{\sqrt5}(2i,1)^T$.

so $\mathbf{x}=\mathbf{U}\tilde{\mathbf{x}}$ gives $f=\lambda_1|\tilde{x}_1|^2+\lambda_2|\tilde{x}_2|^2=6|\tilde{x}_1|^2+1|\tilde{x}_2|^2$,

where
$$\mathbf{U}=(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)=\begin{pmatrix}-i/\sqrt5 & 2i/\sqrt5\\ 2/\sqrt5 & 1/\sqrt5\end{pmatrix}\text{ and }\mathbf{U}^{-1}=\mathbf{U}^*=\begin{pmatrix}i/\sqrt5 & 2/\sqrt5\\ -2i/\sqrt5 & 1/\sqrt5\end{pmatrix}$$

(c) $\mathbf{A}=\begin{pmatrix}0 & i & 0\\ -i & 0 & 2+2i\\ 0 & 2-2i & 0\end{pmatrix}$. $\lambda_1=0,\ \mathbf{e}_1=(2-2i,0,1)^T, \|\mathbf{e}_1\|^2=\mathbf{e}_1\cdot\mathbf{e}_1=9$ so $\hat{\mathbf{e}}_1=\frac{1}{3}(2-2i,0,1)^T$

$\lambda_2=-3,\ \mathbf{e}_2=(-i/3,1,-\frac{2}{3}+\frac{2}{3}i)^T,\ \|\mathbf{e}_2\|^2=\mathbf{e}_2\cdot\mathbf{e}_2=2$ so $\hat{\mathbf{e}}_2=\frac{1}{\sqrt2}(-\frac{i}{3},1,\frac{-2+2i}{3})^T$

$\lambda_3=3,\ \mathbf{e}_3=(\frac{i}{3},1,\frac{2}{3}-\frac{2}{3}i)^T,\ \|e_3\|^2=\mathbf{e}_3\cdot\mathbf{e}_3=2$ so $\hat{\mathbf{e}}_3=\frac{1}{\sqrt2}(\frac{i}{3},1,\frac{2}{3}-\frac{2i}{3})^T$

so $\mathbf{x}=\mathbf{U}\tilde{\mathbf{x}}$ gives $f=\lambda_1|\tilde{x}_1|^2+\lambda_2|\tilde{x}_2|^2+\lambda_3|\tilde{x}_3|^2$

$=0|\tilde{x}_1|^2-3|\tilde{x}_2|^2+3|\tilde{x}_3|^2$

where
$$\mathbf{U}=(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3)=\begin{pmatrix}\frac{2-2i}{3} & -\frac{i}{3\sqrt2} & \frac{i}{3\sqrt2}\\ 0 & 1/\sqrt2 & 1/\sqrt2\\ \frac{1}{3} & \frac{-2+2i}{3\sqrt2} & \frac{2-2i}{3\sqrt2}\end{pmatrix}\text{ and }\mathbf{U}^{-1}=\mathbf{U}^*=\text{etc.}$$

(d) $\mathbf{A}=\begin{pmatrix}4 & 0 & -i\\ 0 & 4 & 0\\ i & 0 & 4\end{pmatrix}$. $\lambda_1=4,\ \mathbf{e}_1=(0,1,0)^T=\hat{\mathbf{e}}_1$

$\lambda_2=3,\ \mathbf{e}_2=(i,0,1)^T,\ \|\mathbf{e}_2\|^2=\mathbf{e}_2\cdot\mathbf{e}_2=2$ so $\mathbf{e}_2=\frac{1}{\sqrt2}(i,0,1)^T$

$\lambda_3=5,\ \mathbf{e}_3=(-i,0,1)^T,\ \hat{\mathbf{e}}_3=\frac{1}{\sqrt2}(-i,0,1)^T$

so $\mathbf{x}=\mathbf{U}\tilde{\mathbf{x}}$ gives $f=\lambda_1|\tilde{x}_1|^2+\lambda_2|\tilde{x}_2|^2+\lambda_3|\tilde{x}_3|^2=4|\tilde{x}_1|^2+3|\tilde{x}_2|^2+5|\tilde{x}_3|^2$

where
$$\mathbf{U}=(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3)=\begin{pmatrix}0 & i/\sqrt2 & -i/\sqrt2\\ 1 & 0 & 0\\ 0 & 1/\sqrt2 & 1/\sqrt2\end{pmatrix}\text{ and }\mathbf{U}^{-1}=\mathbf{U}^*=\text{etc.}$$

8. If $\mathbf{Q}$ is a modal matrix of $\mathbf{A}$, then

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}=\mathbf{D},\quad \mathbf{A}\mathbf{Q}=\mathbf{Q}\mathbf{D},\quad \mathbf{A}=\mathbf{Q}\mathbf{D}\mathbf{Q}^{-1},$$
$$\mathbf{A}^2=\mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}=\mathbf{Q}\mathbf{D}^2\mathbf{Q}^{-1}$$
$$\vdots$$
$$\mathbf{A}^k=\mathbf{Q}\mathbf{D}^k\mathbf{Q}^{-1}=\mathbf{Q}\begin{pmatrix}\lambda_1^k & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \lambda_n^k\end{pmatrix}\mathbf{Q}^{-1}$$

In this exercise each $\mathbf{A}$ matrix is Hermitian. If we use normalized eigenvectors in $\mathbf{Q}$ and denote that normalized modal matrix as $\mathbf{U}$ then

$$\mathbf{A}^k=\mathbf{U}\mathbf{D}^k\mathbf{U}^*$$

since $\mathbf{U}^{-1}=\mathbf{U}^T$.

(a) $\mathbf{A}=\begin{pmatrix}2 & 1-i\\ 1+i & 3\end{pmatrix}$, $\lambda_1=4,\ \mathbf{e}_1=(1-i,2)^T,\ \|\mathbf{e}_1\|=\sqrt6$ so $\hat{\mathbf{e}}_1=\frac{1}{\sqrt6}(1-i,2)^T$;

$$\lambda_2 = 1,\ \mathbf{e}_2 = (-1+i, 1)^T,\ \|\mathbf{e}_2\| = \sqrt{3},\ \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{3}}(-1+i,1)^T$$

$$\text{so } \mathbf{A}^{1000} = \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{-1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4^{1000} & 0 \\ 0 & 1^{1000} \end{pmatrix} \begin{pmatrix} \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2+4^{1000} & (1-i)(4^{1000}-1) \\ (1+i)(4^{1000}-1) & (2)4^{1000}+1 \end{pmatrix}$$

NOTE: The maple evalm(A^1000) command is not helpful since the 4^{1000}'s will appear not in that form but in the "written out" form with a great many digits. As a check, we might note that the formula above holds with k in place of the 1000's, for any integer k. With $k=2$, say, we can compare the above result with the easily-worked-out A^2 matrix. They agree. ✓ Of course, $2+4^{1000} \approx 4^{1000}$ and so on, but we have left all terms intact.

(c) $\mathbf{A} = \begin{pmatrix} 0 & -i \\ i & 2 \end{pmatrix}$, $\lambda_1 = 1+\sqrt{2}$, $\mathbf{e}_1 = (1, (1+\sqrt{2})i)^T$, $\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{4+2\sqrt{2}}}(1, (1+\sqrt{2})i)^T$

$\lambda_2 = 1-\sqrt{2}$, $\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{4-2\sqrt{2}}}(1, (1-\sqrt{2})i)^T$

$$\text{so } \mathbf{A}^{1000} = \begin{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ \frac{(1+\sqrt{2})i}{\sqrt{4+2\sqrt{2}}} & \frac{(1-\sqrt{2})i}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} (1+\sqrt{2})^{1000} & 0 \\ 0 & (1-\sqrt{2})^{1000} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} & -\frac{(1+\sqrt{2})i}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & -\frac{(1-\sqrt{2})i}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} .3827 & .9239 \\ 6.3083i & -.3827i \end{pmatrix} \begin{pmatrix} (1+\sqrt{2})^{1000} & 0 \\ 0 & (1-\sqrt{2})^{1000} \end{pmatrix} \begin{pmatrix} .3827 & -6.3083i \\ .9239 & .3827i \end{pmatrix}$$

$$= \begin{pmatrix} .1465(1+\sqrt{2})^{1000} + .8536(1-\sqrt{2})^{1000}, & [-2.4142(1+\sqrt{2})^{1000} + .3536(1-\sqrt{2})^{1000}]i \\ [2.4142(1+\sqrt{2})^{1000} - .3536(1-\sqrt{2})^{1000}]i, & 39.7946(1+\sqrt{2})^{1000} + .1465(1-\sqrt{2})^{1000} \end{pmatrix}$$

Of course, $(1+\sqrt{2})^{1000} >> (1-\sqrt{2})^{1000}$ so

$$\mathbf{A}^{1000} \approx (1+\sqrt{2})^{1000} \begin{pmatrix} .1465 & -2.4142i \\ 2.4142i & 39.7946 \end{pmatrix}$$

9. $\bar{f} = \overline{\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}} = \overline{\bar{\mathbf{x}}^T}\ \overline{\mathbf{A}\mathbf{x}}$ because $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ for any complex numbers z_1 and z_2 [see (14b) in Sec 21.2]

$= \mathbf{x}^T \overline{\mathbf{A}\mathbf{x}}$ because $\bar{\bar{\mathbf{x}}} = \mathbf{x}$, of course

$= \overline{(\mathbf{A}\mathbf{x})^T}\mathbf{x}$ because $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$

$= \overline{\mathbf{x}^T \mathbf{A}^T}\mathbf{x}$ by (3d) in Sec. 10.3

$= \bar{\mathbf{x}}^T \bar{\mathbf{A}}^T \mathbf{x}$ by (14b) in Sec. 21.2 again

$= \bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$ because $\mathbf{A}$ is Hermitian by assumption

$= f$ by (15).

10. From (18) and (19), $f = \bar{\tilde{\mathbf{x}}}^T \mathbf{D} \tilde{\mathbf{x}} = (\bar{\tilde{x}}_1, \dots, \bar{\tilde{x}}_n) \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix}$

$= (\bar{\tilde{x}}_1, \dots, \bar{\tilde{x}}_n) \begin{pmatrix} \lambda_1 \tilde{x}_1 \\ \vdots \\ \lambda_n \tilde{x}_n \end{pmatrix} = \lambda_1 \bar{\tilde{x}}_1 \tilde{x}_1 + \cdots + \lambda_n \bar{\tilde{x}}_n \tilde{x}_n = \lambda_1 |\tilde{x}_1|^2 + \cdots + \lambda_n |\tilde{x}_n|^2$, as in (20).

11. $\bar{\underset{\sim}{U}}^T\underset{\sim}{U} = \begin{pmatrix} \bar{\hat{\underset{\sim}{e}}}_1^T \\ \vdots \\ \bar{\hat{\underset{\sim}{e}}}_n^T \end{pmatrix} (\hat{\underset{\sim}{e}}_1, \dots, \hat{\underset{\sim}{e}}_n) = \begin{pmatrix} \bar{\hat{\underset{\sim}{e}}}_1^T \hat{\underset{\sim}{e}}_1 & \cdots & \bar{\hat{\underset{\sim}{e}}}_1^T \hat{\underset{\sim}{e}}_n \\ \vdots & & \vdots \\ \bar{\hat{\underset{\sim}{e}}}_n^T \hat{\underset{\sim}{e}}_1 & & \bar{\hat{\underset{\sim}{e}}}_n^T \hat{\underset{\sim}{e}}_n \end{pmatrix} = \begin{pmatrix} \hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_1 & \cdots & \hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_n \\ \vdots & & \vdots \\ \hat{\underset{\sim}{e}}_n \cdot \hat{\underset{\sim}{e}}_1 & \cdots & \hat{\underset{\sim}{e}}_n \cdot \hat{\underset{\sim}{e}}_n \end{pmatrix} = \underset{\sim}{I}$ ✓

12. The dot product of the 3rd and 4th columns is not zero so the matrix is neither orthogonal nor unitary.

13. $\underset{\sim}{x} \cdot (\underset{\sim}{A}\underset{\sim}{x}) = \underset{\sim}{x} \cdot (\underset{\sim}{B}\underset{\sim}{x})$ is equivalent to $\underset{\sim}{x} \cdot \underset{\sim}{C}\underset{\sim}{x} = 0$ where $\underset{\sim}{C} = \underset{\sim}{A} - \underset{\sim}{B}$ is Hermitian. Then

$$\underset{\sim}{x} \cdot (\underset{\sim}{C}\underset{\sim}{x}) = \underset{\sim}{x}^T \overline{\underset{\sim}{C}\underset{\sim}{x}} = 0 \text{ or, equivalently, } \bar{\underset{\sim}{x}}^T \underset{\sim}{C}\underset{\sim}{x} = 0 \text{ for all } \underset{\sim}{x}\text{'s.}$$

The latter is a quadratic form (actually, a Hermitian form) so a normalized modal transformation $\underset{\sim}{x} = \underset{\sim}{U}\tilde{\underset{\sim}{x}}$ gives

$$\lambda_1 |\tilde{x}_1|^2 + \cdots + \lambda_n |\tilde{x}_n|^2 = 0$$

for all $\tilde{\underset{\sim}{x}}$'s.‡ It follows that $\lambda_1 = \cdots = \lambda_n = 0$: all of $\underset{\sim}{C}$'s eigenvalues are 0. Thus, the question is this: if $\underset{\sim}{C}$ is Hermitian and all of its λ's are 0, then is $\underset{\sim}{C}$ necessarily the zero matrix? Well,

$$\underset{\sim}{U}^* \underset{\sim}{C} \underset{\sim}{U} = \underset{\sim}{D} = \underset{\sim}{O}$$

where $\underset{\sim}{D} = \begin{pmatrix} \lambda_1 & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & \lambda_n \end{pmatrix} = \underset{\sim}{O}$. Then, solving for $\underset{\sim}{C}$ gives

$$\underset{\sim}{C} = \underset{\sim}{U}\underset{\sim}{O}\underset{\sim}{U}^{-1} = \underset{\sim}{O}$$

so the answer is: Yes, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ must be identical.

NOTE: You might consider discussing exercises 13, or 14 and 15 in class.

14. $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{c}$. We want to show that $\underset{\sim}{c} \cdot \underset{\sim}{z} = 0$. Well,

$$\underset{\sim}{c} \cdot \underset{\sim}{z} = (\underset{\sim}{A}\underset{\sim}{x}) \cdot \underset{\sim}{z} = \underset{\sim}{x} \cdot (\underbrace{\underset{\sim}{A}^* \underset{\sim}{z}}_{\underset{\sim}{0} \text{ by assumption}}) = \underset{\sim}{x} \cdot \underset{\sim}{0} = 0 \checkmark$$

15. (a) $\underset{\sim}{A}^* \underset{\sim}{z} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so we need $\underset{\sim}{c} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ or, $\boxed{c_1 - c_2 = 0.}$ Check, by applying Gauss elim. to the original system:

$$\left(\begin{array}{cc|c} 2 & 1 & c_1 \\ 2 & 1 & c_2 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 2 & 1 & c_1 \\ 0 & 0 & c_2 - c_1 \end{array}\right) \text{ so we need } c_2 - c_1 = 0. \checkmark$$

(b) $\underset{\sim}{A}^* \underset{\sim}{z} = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underset{\sim}{0}$, so there is $\boxed{\text{no restriction}}$ on $\underset{\sim}{c}$ (because $\underset{\sim}{c} \cdot \underset{\sim}{0} = 0$ for any $\underset{\sim}{c}$). Check, by applying Gauss elim. to the original system:

$$\left(\begin{array}{cc|c} 2 & 1 & c_1 \\ 4 & 1 & c_2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 1 & c_1 \\ 0 & 1 & 2c_1 - c_2 \end{array}\right)$$

so the system is consistent for any $\underset{\sim}{c}$ and gives $x_2 = 2c_1 - c_2$ and $x_1 = [c_1 - (2c_1 - c_2)]/2$.

---

‡ Watch out, there is a subtle point here: Do "all $\underset{\sim}{x}$'s" correspond to "all $\tilde{\underset{\sim}{x}}$'s"? Yes, because $\underset{\sim}{x} = \underset{\sim}{U}\tilde{\underset{\sim}{x}}$ and $\underset{\sim}{U}$ is nonsingular (indeed, $\underset{\sim}{U}^{-1} = \underset{\sim}{U}^*$), so there is a one-to-one correspondence between the $\underset{\sim}{x}$ vectors and the $\tilde{\underset{\sim}{x}}$ vectors.

(c) $\underset{\sim}{A}^*\underset{\sim}{z} = \begin{pmatrix} 2 & 4 & -i \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underset{\sim}{0}$ gives $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, so we need $\underset{\sim}{c}\cdot\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \boxed{2c_1 - c_2 = 0}$

Check, by applying Gauss elim. to the original system:

$$\begin{pmatrix} 2 & 1 & c_1 \\ 4 & 2 & c_2 \\ i & 1 & c_3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & c_1 \\ 0 & 0 & c_2-2c_1 \\ 0 & 2-i & 2c_3-ic_1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & c_1 \\ 0 & 2-i & 2c_3-ic_1 \\ 0 & 0 & c_2-2c_1 \end{pmatrix} \text{ so we need } c_2-2c_1=0. \checkmark$$

(d) $\underset{\sim}{A}^*\underset{\sim}{z} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ gives $\underset{\sim}{z} = \alpha\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, so we need $\underset{\sim}{c}\cdot\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \boxed{-c_1 - c_2 + c_3 = 0}$

Check, by applying Gauss elim. to the original system:

$$\begin{pmatrix} 1 & 3 & 2 & 1 & c_1 \\ 2 & -1 & 1 & 0 & c_2 \\ 3 & 2 & 3 & 1 & c_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 & 1 & c_1 \\ 0 & -7 & -3 & -2 & c_2-2c_1 \\ 0 & -7 & -3 & -2 & c_3-3c_1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 & 1 & c_1 \\ 0 & -7 & -3 & -2 & c_2-2c_1 \\ 0 & 0 & 0 & 0 & c_3-c_1-c_2 \end{pmatrix} \text{ so we need } c_3-c_1-c_2=0 \ \checkmark$$

(f) $\underset{\sim}{A}^*\underset{\sim}{z} = \begin{pmatrix} 2 & -i & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -3-2i & -4 & -5 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \underset{\sim}{0} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & -2-i & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ gives $z_4 = \alpha$, $z_3 = \beta$,

$z_2 = (3\alpha+3\beta)/(-2-i)$, which is awkward, so let us choose, instead,

$z_4 = (2+i)\alpha$, $z_3 = (2+i)\beta$, $z_2 = -3\alpha-3\beta$, $z_1 = (1-i)\alpha-(1+2i)\beta$,

so

$$\underset{\sim}{z} = \begin{pmatrix} (1-i)\alpha-(1+2i)\beta \\ -3\alpha-3\beta \\ (2+i)\beta \\ (2+i)\alpha \end{pmatrix} = \alpha\begin{pmatrix} 1-i \\ -3 \\ 0 \\ 2+i \end{pmatrix} + \beta\begin{pmatrix} -1-2i \\ -3 \\ 2+i \\ 0 \end{pmatrix} \equiv \alpha\underset{\sim}{z}_1 + \beta\underset{\sim}{z}_2 .$$

note the conjugates

Thus, we need $\underset{\sim}{c}\cdot\underset{\sim}{z}_1 = 0$ and $\underset{\sim}{c}\cdot\underset{\sim}{z}_2 = 0$; i.e., $\boxed{\begin{aligned} &(1+i)c_1 - 3c_2 + (2-i)c_4 = 0 \\ \text{and } &(-1+2i)c_1 - 3c_2 + (2-i)c_3 = 0. \end{aligned}}$

Check, by applying Gauss elim. to original system:

$$\begin{pmatrix} 2 & 1 & 1 & c_1 \\ i & 1 & 2i-3 & c_2 \\ 1 & 2 & -4 & c_3 \\ -1 & 1 & -5 & c_4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & -5 & c_4 \\ 1 & 2 & -4 & c_3 \\ 2 & 1 & 1 & c_1 \\ i & 1 & 2i-3 & c_2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & -5 & c_4 \\ 0 & 3 & -9 & c_4+c_3 \\ 0 & 3 & -9 & 2c_4+c_1 \\ 0 & i+1 & -3i-3 & c_4 i+c_2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & -5 & c_4 \\ 0 & 1 & -3 & (c_4+c_3)/3 \\ 0 & 0 & 0 & c_4-c_3+c_1 \\ 0 & 0 & 0 & 3(c_2+ic_4) \\ & & & -(1+i)(c_3+c_4) \end{pmatrix}$$

so we need $c_1 - c_3 + c_4 = 0$

and $3c_2 - (1+i)c_3 - (1-2i)c_4 = 0$

These two conditions are equivalent to the boxed conditions given above; i.e., their solution sets (for $c_1, \dots, c_4$), as found by Gauss elimination, are found to be the same as each other.

16. Split $\underset{\sim}{A} = \underset{\sim}{B} + i\underset{\sim}{C}$.

Then $\underset{\sim}{A}^* = \underset{\sim}{B}^* - i\underset{\sim}{C}^* = B - iC$ since B, C are to be Hermitian.

Solve for $\underset{\sim}{B}$ and $\underset{\sim}{C}$ by adding and subtracting these equations:

$\underset{\sim}{B} = (\underset{\sim}{A} + \underset{\sim}{A}^*)/2$ and $\underset{\sim}{C} = i(\underset{\sim}{A}^* - \underset{\sim}{A})/2$.

17. (a) $(i\underset{\sim}{A})^* = -i\underset{\sim}{A}^* = -i\underset{\sim}{A} \neq i\underset{\sim}{A}$ (unless $\underset{\sim}{A} = \underset{\sim}{0}$), so $i\underset{\sim}{A}$ is not Hermitian.

(b) $(\underset{\sim}{A}^2)^* = (\underset{\sim}{A}\underset{\sim}{A})^* = \underset{\sim}{A}^*\underset{\sim}{A}^*$ per (6f)

$= \underset{\sim}{A}\underset{\sim}{A} = \underset{\sim}{A}^2$, so $\underset{\sim}{A}^2$ is Hermitian.

CHAPTER 13

Section 13.2

1. (b) $d(P,P') = \sqrt{(1-0)^2+(-1-4)^2+(5-3)^2+(0-2)^2} = \sqrt{34}$

   (c) $d(P,P') = \sqrt{(1-6)^2+(-4-5)^2} = \sqrt{106}$

3. $d(P,P_0) = \sqrt{(4.3-4.2)^2+(1.1-1)^2} = \sqrt{0.02} \approx 0.141 < r$, so P is in N.

4. $d(P,P_0) = \sqrt{(2-3)^2+(5-4)^2+(7-5)^2} = \sqrt{6} \approx 2.45 < r$, so P is in N.

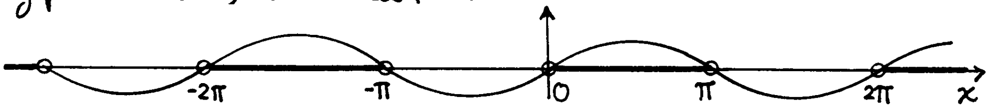
5. $d(P,P_0) = \sqrt{(.01)^2+(.04)^2+(.05)^2+(.03)^2} = \sqrt{.0051} \approx 0.071 > r$, so P is not in N.

6. (b) boundary points are $-3, 5$; connected; neither

   (c) bdy pts are $x=0$; connected; open

   (d) bdy pts are $x=0$; connected; closed

   (e)



   The set is $0<x<\pi$, $-2\pi<x<-\pi$, $2\pi<x<3\pi$, $-4\pi<x<-3\pi$, etc.
   Bdy pts are $n\pi$ $(n=0,\pm1,\pm2,\ldots)$; not connected; open

   (f) The set is $\{x \mid x=n\pi,\ n=0,\pm1,\pm2,\ldots\}$. Each pt. is a bdy pt; not connected; closed

   (g) The set is a rectangle. Bdy pts are $\{(x_1,x_2) \mid x_1=2,\ 1\le x_2\le 5\}$, $\{(x_1,x_2) \mid x_1=3,\ 1\le x_2\le 5\}$, $\{(x_1,x_2) \mid 2\le x_1\le 3,\ x_2=1\}$, $\{(x_1,x_2) \mid 2\le x_1\le 3,\ x_2=5\}$; connected; open

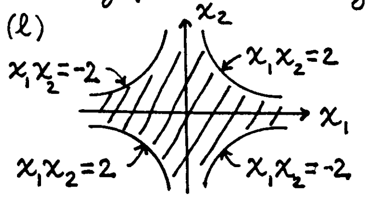
   (h) Bdy pts are $\{(x_1,x_2) \mid -\infty<x_1<\infty,\ x_2=0\}$ and $\{(x_1,x_2) \mid -\infty<x_1<\infty,\ x_2=1\}$; connected; open

   (i) Bdy pts are $\{(x_1,x_2) \mid 6\le x_1\le 8,\ x_2=0\}$; connected; neither

   (j) Bdy pts are the two circles $\{(x_1,x_2) \mid x_1^2+x_2^2=1\}$ and $\{(x_1,x_2) \mid x_1^2+x_2^2=4\}$; connected; neither

   (k) Bdy pts are the origin and the circle $\{(x_1,x_2) \mid x_1^2+x_2^2=1\}$; connected; open

   (ℓ)



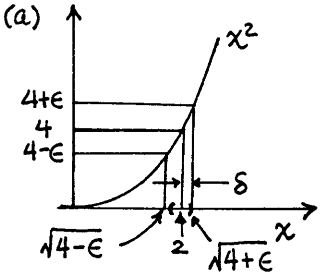
   Bdy pts are all pts on the four branches of the hyperbolas $x_1x_2=\pm2$; connected; open

   (m) Bdy pts are $\{(x_1,x_2,x_3) \mid 1\le x_1^2+x_3^2\le 2,\ x_2=0\}$; connected; neither

   (o) Bdy pts are $\{(x_1,\ldots,x_4) \mid x_1=0,\ 0\le x_2\le1,\ 0\le x_3\le1,\ 0\le x_4\le1\}$,
   $\{(\quad " \quad) \mid x_1=1,\quad " \quad " \quad " \}$,
   $\{(\quad " \quad) \mid 0\le x_1\le1,\ x_2=0,\ 0\le x_3\le1,\ 0\le x_4\le1\}$,
   $\{(\quad " \quad) \mid \quad " \quad,\ x_2=1,\quad " \quad " \}$,
   $\{(\quad " \quad) \mid \quad " \quad,\ 0\le x_2\le1,\ x_3=0,\ 0\le x_4\le1\}$,

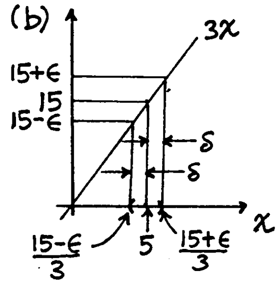
$\{(x_1,\dots,x_4) \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ x_3 = 1,\ 0 \le x_4 \le 1\}$,
$\{(\ \ "\ \ ) \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1,\ x_4 = 0\}$,
and $\{(\ \ "\ \ ) \mid \ \ "\ \ \ \ "\ \ \ \ "\ \ ,\ x_4 = 1\}$; connected; open

7. (a)



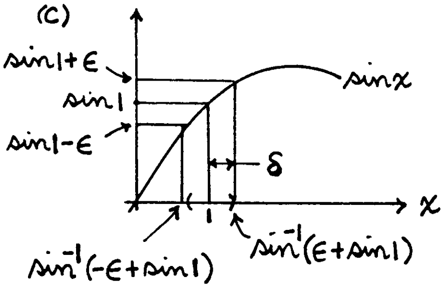
Can choose $\delta(\epsilon) = \sqrt{4+\epsilon} - 2$ (or smaller).

(b)



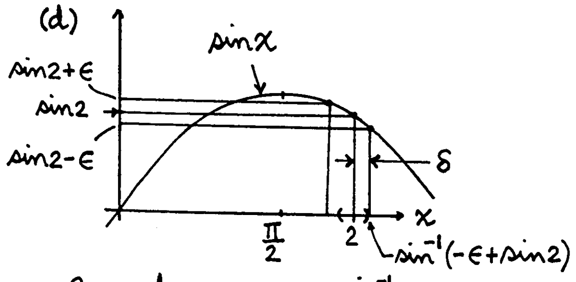
Can choose $\delta(\epsilon) = \frac{15+\epsilon}{3} - 5 = \frac{\epsilon}{3}$ (or smaller)

(c)



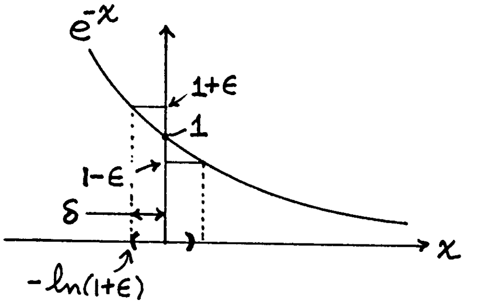
Can choose $\delta(\epsilon) = \sin^{-1}(\epsilon+\sin 1) - 1$ (or smaller).

(d)



Can choose $\delta(\epsilon) = \sin^{-1}(-\epsilon+\sin 2) - 2$ (or smaller).

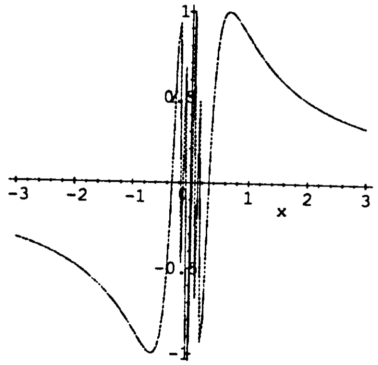
(f)



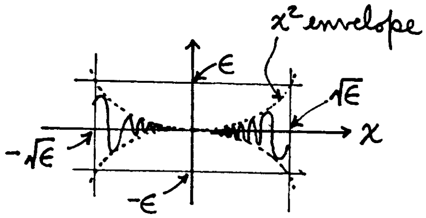
Can choose $\delta(\epsilon) = \ln(1+\epsilon)$ (or smaller).

8. (a) Follow the hints. Since $\omega = 1/x^2 \to \infty$ as $x \to 0$ and $\to 0$ as $x \to \infty$, the graph of $\sin(1/x) = \sin[(\frac{1}{x^2})x]$ is given at the right. We used the Maple command

```
with(plots):
implicitplot(y=sin(1/x),
  x=-3..3, y=-1..1, numpoints=15000);
```



Even with 15000 points the graph is inaccurate for small x since the frequency $\to 0$ as $x \to 0$. Naturally, successive peaks should be at ± 1. As $x \to 0$, from the left or from the right, the values merely oscillate and do not approach a limit. A simpler approach is to let $1/x = t$. Then $\lim\limits_{x\to 0} \sin\frac{1}{x} = \lim\limits_{t\to\infty} \sin t$ which does not exist.



(b) This time the x^2 factor "channels" the function values to the limit 0. We can make $|x^2 \sin 1/x - 0| < \epsilon$ by keeping x within $|x - 0| < \sqrt{\epsilon}$. That is, $\delta(\epsilon) = \sqrt{\epsilon}$ (or smaller).

9. (d) Follow the hint:

$$\begin{aligned}|[f(x)+g(x)]-(A+B)| &= |[f(x)-A]+[g(x)-B]| \\ &\le |f(x)-A|+|g(x)-B| \\ &< \quad \epsilon \quad + \quad \epsilon \quad = 2\epsilon \equiv \epsilon', \text{ say}.\end{aligned}$$

Since $|[f(x)+g(x)]-(A+B)| < \epsilon'$, for arbitrarily small ϵ', whenever $0<|x-a| < \underbrace{\min\{\delta_1, \delta_2\}}_{\delta}$, it follows that (9.4) is true.

(e) Let $\lim_{x\to a} f(x) = A$. Then, for arbitrarily small $\epsilon > 0$ there is a $\delta(\epsilon)$ such that $|f(x)-A| < \epsilon$ for all x in $0<|x-a|<\delta$. Next,

$$|Cf(x)-CA| = |C||f(x)-A| < |C|\epsilon \equiv \epsilon', \text{ say},$$

for all x in $0<|x-a|<\delta$, so (9.5) is true.

(f)
$$\begin{aligned}\lim_{x\to a}[\alpha f(x)+\beta g(x)] &= \lim_{x\to a}[\alpha f(x)] + \lim_{x\to a}[\beta g(x)] && \text{per (9.4)} \\ &= \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x) && \text{per (9.5)}\end{aligned}$$

(g) Following the hint, $|f(x)g(x)-AB| \le (|A|+1)\epsilon + |B|\epsilon \equiv \epsilon'$ for arbitrarily small ϵ and hence for arbitrarily small ϵ', for all x in $0<|x-a| < \min\{\delta_1,\delta_2\} \equiv \delta$.

10. Let $f(x) = 1/x$, $g(x) = -1/x$, $a=0$. Then

$$\text{LHS} = \lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x}\right] = \lim_{x\to 0} 0 = 0$$

but

$$\text{RHS} = \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x}$$ fails to exist because each of the two limits fails to exist.

11. Let $C=0$, $f(x)=1/x$, $a=0$. Then $\text{LHS} = \lim_{x\to 0}(0\frac{1}{x}) = \lim_{x\to 0} 0 = 0$, but $\text{RHS} = 0 \lim_{x\to 0}\frac{1}{x}$ does not exist because the limit does not exist.

12. (a) $x=0$ and $x = (-1\pm\sqrt{1+8})/2 = 1, -2$

(b) $x =$ the zeros of $\cos x$, namely, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

(c) All points, in the x,y plane, on the lines $x+y = \pm\pi/2, \pm 3\pi/2, \ldots$

(d) All points, in the x,y plane, on the line $y=-3$

(e) " " " " " " " " curve $y=-x^2$

(f) All points, in x,y,z space, on the planes $y=0$ and $z=0$

(g) " " " " " " sphere $x^2+y^2+z^2=1$

Section 13.3

1. (e) $f=(x^2+y^2)^{1/2}$, $f_x=\frac{1}{2}2x(\ )^{-1/2}=x(x^2+y^2)^{-1/2}$, $f_{xx}=(\ )^{-1/2}+x(-\frac{1}{2})(2x)(\ )^{-3/2}=y^2(x^2+y^2)^{-3/2}$, $f_{xy}=-\frac{1}{2}(2y)x(\ )^{-3/2}=-xy(x^2+y^2)^{-3/2}$, $f_y=\frac{1}{2}(2y)(x^2+y^2)^{-1/2}=y(x^2+y^2)^{-1/2}$, $f_{yx}=-\frac{1}{2}y(2x)(\ )^{-3/2}=-xy(x^2+y^2)^{-3/2}$. We see $f_{xy}=f_{yx}$ everywhere (except at 0,0, where they don't exist).

2. (a) $f=x^4y^3$, $f_x=4x^3y^3$, $f_{xx}=12x^2y^3$, $f_{xxy}=36x^2y^2$,
$f_y=3x^4y^2$, $f_{yx}=12x^3y^2$, $f_{yxx}=36x^2y^2$,
$f_{xy}=12x^3y^2$, $f_{xyx}=36x^2y^2$
> all equal, everywhere in the plane.

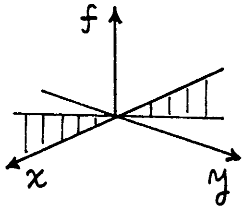
3. (a) $f=\ln(x^2+y^2)$, $f_x=2x(x^2+y^2)^{-1}$, $f_{xx}=2(\ )^{-1}-4x^2(\ )^{-2}=2(y^2-x^2)(x^2+y^2)^{-2}$
$f_y=2y(x^2+y^2)^{-1}$, $\underline{f_{yy}=2(\ )^{-1}-4y^2(\ )^{-2}=2(x^2-y^2)(x^2+y^2)^{-2}}$
so $f_{xx}+f_{yy}=0$ ✓ (except at 0,0, where f_{xx} and f_{yy} fail to exist.

(c) $f=r^n\sin n\theta$, $f_r=nr^{n-1}\sin n\theta$, $f_{rr}=n(n-1)r^{n-2}\sin n\theta$, $f_\theta=nr^n\cos n\theta$, $f_{\theta\theta}=-n^2r^n\sin n\theta$, so $r^2f_{rr}+rf_r+f_{\theta\theta}=n(n-1)r^n\sin n\theta+nr^n\sin n\theta-n^2r^n\sin n\theta=0$, except at $r=0$ for $n<0$, where the r^n's are undefined.

(f) $f=\sin kx\,e^{-k^2t}$, $f_x=k\cos kx\,e^{-k^2t}$, $f_{xx}=-k^2\sin kx\,e^{-k^2t}$,
$f_t=-k^2\sin kx\,e^{-k^2t}$, so $f_{xx}=f_t$ ✓

(h) $f=\sin(x-ct)$, $f_x=\cos(x-ct)\frac{\partial}{\partial x}(x-ct)=\cos(x-ct)$, $f_{xx}=-\sin(x-ct)$,
$f_t=\cos(x-ct)\frac{\partial}{\partial t}(x-ct)=-c\cos(x-ct)$, $f_{tt}=-c^2\sin(x-ct)$
> $c^2f_{xx}=f_{tt}$

5.



(a) $f_x(0,0)=1$ (b) $f_x(0,2)=0$ (c) $f_x(2,0)=1$
(d) $f_x(3,4)=0$ (e) $f_y(6,0)$ doesn't exist (f) $f_y(0,2)=0$
(g) $f_y(0,0)=0$ (h) $f_{yx}(0,0)$ doesn't exist because $f_{yx}(0,0)=\lim_{x\to0}\frac{f_y(x,0)-f_y(0,0)}{x}$, but $f_y(x,0)$ doesn't exist for $x\neq0$.

(j) $f_{xy}(3,0)=\lim_{y\to0}\frac{f_x(3,y)-f_x(3,0)}{y}=\lim_{y\to0}\frac{0-1}{y}$ doesn't exist

(k) $f_{yxx}(3,0)=\lim_{\Delta x\to0}\frac{f_{yx}(3+\Delta x,0)-f_{yx}(3,0)}{\Delta x}$ doesn't exist because the f_{yx} derivatives in the numerator fail to exist. For example,

$$f_{yx}(3,0)=\lim_{\Delta x\to0}\frac{f_y(3+\Delta x,0)-f_y(3,0)}{\Delta x}$$

fails to exist because the f_y derivatives in the numerator do not exist.

(l) $f_{xxy}(3,0)$ does not exist.

6. (a) From (4a), $f_x=y(x^4+4x^2y^2-y^4)/(x^2+y^2)^2$ (for $(x,y)\neq(0,0)$)
so $f_{xy}=(x^4+4x^2y^2-y^4)(x^2+y^2)^{-2}+y(8x^2y-4y^3)/(x^2+y^2)^2-2(2y)y(x^4+4x^2y^2-y^4)/(x^2+y^2)^3$
and, putting $y=\alpha x$ in the latter gives (6.1). That expression for f_{xy} is a

constant along each $y=\alpha x$ ray but varies from ray to ray, so f_{xy} is not continuous at (0,0) since $\lim f_{xy}(x,y)$ does not exist as $(x,y)\to(0,0)$.

(c) (4a) gives $f_x(x,y)=\dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$. Putting in $y=\alpha x$ gives

$f_x=\alpha x(1+4\alpha^2-\alpha^4)/(1+\alpha^2)^2$ which $\to 0$ as we approach the origin along any such ray, because of the x factor. Since (4b) gives $f_x(0,0)=0$, it appears that we have established that f_x is continuous at the origin. However, it is conceivable that there might exist paths of approach leading to different limits or even to no limit, so let us proceed in a more general way. Letting $x=r\cos\theta$ and $y=r\sin\theta$ gives

$$f_x=\frac{rS(r^4C^4+4r^4C^2S^2-r^4S^4)}{r^4}=rS(C^4+4\overset{\cos\theta}{C^2}\overset{\sin\theta}{S^2}-S^4)\to 0 \text{ as}$$

$r\to 0$ in <u>any</u> manner. Further, $f_x(0,0)=0$, so f_x is continuous at (0,0).

7. If $f'(x)$ exists at x_0 then $f(x)$ must be continuous at x_0, but the existence of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ does not imply that $f(x,y)$ is continuous at (x_0,y_0), as we will prove by consider the function $f(x,y)=\begin{cases}1, & x=0 \text{ or } y=0\\ 0, & \text{elsewhere}\end{cases}$

Then $f_x(0,0)$ exists (and equals 0) and $f_y(0,0)$ exists (and equals 0), yet f is surely not continuous at (0,0).

Section 13.4

2. (b) $f=e^{xy}$; $x=\sqrt{t+1}$, $y=\cos t$. $F'(t)=f_x x'(t)+f_y y'(t)=ye^{xy}\frac{1}{2}(t+1)^{-1/2}+xe^{xy}(-\sin t)$
$=\left[\frac{1}{2}\cos t\,(t+1)^{-1/2}-\sqrt{t+1}\sin t\right]\exp\left[\sqrt{t+1}\cos t\right]$

3. (b) $g=\sqrt{u^2-v}$; $u=4s^2$, $v=e^{-3s}$.
Chain rule: $G'(s)=G_u u'(s)+G_v v'(s)=\frac{1}{2}2u(u^2-v)^{-1/2}8s-\frac{1}{2}(u^2-v)^{-1/2}(-3e^{-3s})$
$=\left[8(4s^2)s+\frac{3}{2}e^{-3s}\right](16s^4-e^{-3s})^{-1/2}$
$=(32s^3+\frac{3}{2}e^{-3s})(16s^4-e^{-3s})^{-1/2}$. same ✓
Directly: $G(s)=(16s^4-e^{-3s})^{1/2}$ so $G'(s)=\frac{1}{2}(64s^3-3e^{-3s})(16s^4-e^{-3s})^{-1/2}$

4. (b) $R'(t)=R_x x'(t)+R_y y'(t)+R_z z'(t)$
$=2x(x^2+y^2+z^2)^{-1/2}(2)+2y(x^2+y^2+z^2)^{-1/2}(2t)+2z(x^2+y^2+z^2)^{-1/2}(3t^2)$
$=\left[2(2t)(2)+2(t^2)(2t)+2(t^3+1)(3t^2)\right]\left[(2t)^2+(t^2)^2+(t^3+1)^2\right]^{-1/2}$
so $R'(2)=264/\sqrt{113}$

5. (a) $T'(t)=T_x x'+T_y y'+T_z z'=-2x(x^2+y^2+z^2)^{-2}e^{-z}(6)-2y(x^2+y^2+z^2)^{-2}e^{-z}(1)$
$-2z(\quad " \quad)^{-2}e^{-z}(-e^{-t})+\left[5+(x^2+y^2+z^2)^{-1}\right](-e^{-z})(-e^{-t})$

at $t=0$, $x(0)=0, y(0)=2, z(0)=1$ so $T'(0) = 0 - 4(5)^{-2}e^{-1} - 2(5)^{-2}e^{-1}(-1) + [5+(5)^{-1}](-e^{-1})(-1)$

$$= \frac{128}{25}e^{-1}$$

(b) This time chain differentiation gives a T_t term as well:

$$T'(t) = T_t + T_x x' + T_y y' + T_z z'$$

$$= \frac{60e^z(.1)\cos t}{x^2+y^2+2} + (-2x)\frac{60e^z(1+.1\sin t)}{(x^2+y^2+2)^2}(6) + (-2y)\frac{60e^z(1+.1\sin t)}{(x^2+y^2+2)^2}(1)$$

$$+ \frac{60e^z(1+.1\sin t)}{x^2+y^2+2}(-e^{-t})$$

so $T'(0) = \frac{(60)e(.1)}{6} - 0 - 4\frac{(60)e}{36} + \frac{60e}{6}(-1) = -\frac{47}{3}e$

6. (a) $f(\lambda x, \lambda y) = \lambda^2x^2 + 3\lambda x\lambda y = \lambda^2(x^2+3xy) = \lambda^2 f(x,y)$ so f is homo. of degree 2.

$g(\quad " \quad) = \ln(\lambda^2x^2+\lambda^2y^2) = 2\ln\lambda + \ln(x^2+y^2)$ so g is not homo.

$h(\quad " \quad) = (\lambda^2x^2 - \lambda x\lambda y)/(2\lambda x + \lambda y) = \lambda(x^2-xy)/(2x+y) = \lambda h(x,y)$ so h is homo. of degree 1.

$p(\quad " \quad) = \lambda^3x^3e^{\lambda x/(2\lambda y)} = \lambda^3x^3e^{x/(2y)} = \lambda^3 p(x,y)$ so p is homo. of degree 3.

(b) $f(u_1(\lambda x_1,\dots,\lambda x_n),\dots,u_m(\lambda x_1,\dots,\lambda x_n))$

$= f(\lambda^q u_1(x_1,\dots,x_n),\dots,\lambda^q u_m(x_1,\dots,x_n))$

$= (\lambda^q)^p f(u_1(x_1,\dots,x_n),\dots,u_m(x_1,\dots,x_n)) = \lambda^{qp} f(u_1,\dots,u_m)$,

so f is a homo. function of $x_1,\dots,x_n$ of degree qp.

(c) $\frac{\partial f}{\partial x} = \lim_{x'\to x}\frac{f(x',y,z)-f(x,y,z)}{x'-x}$

so $\frac{\partial f}{\partial x}(\lambda x, \lambda y, \lambda z) = \lim_{x'\to x}\frac{f(\lambda x', \lambda y, \lambda z) - f(\lambda x, \lambda y, \lambda z)}{\lambda x' - \lambda x}$

$$= \frac{\lambda^k}{\lambda}\lim_{x'\to x}\frac{f(x',y,z)-f(x,y,z)}{x'-x} = \lambda^{k-1}\frac{\partial f}{\partial x},$$

so $\partial f/\partial x$ (likewise $\partial f/\partial y$, $\partial f/\partial z$) is homo. of degree $k-1$.

(d) $f(\lambda x_1,\dots,\lambda x_n) = \lambda^k f(x_1,\dots,x_n)$.

$d/d\lambda$ gives $x_1\frac{\partial f}{\partial x_1}(\lambda x_1,\dots,\lambda x_n) + \cdots + x_n\frac{\partial f}{\partial x_n}(\lambda x_1,\dots,\lambda x_n) = k\lambda^{k-1}f(x_1,\dots,x_n)$

$$\cancel{\lambda^{k-1}}x_1\frac{\partial f}{\partial x_1}(x_1,\dots,x_n) + \cdots + \cancel{\lambda^{k-1}}x_n\frac{\partial f}{\partial x_n}(x_1,\dots,x_n) = k\cancel{\lambda^{k-1}}f(x_1,\dots,x_n)$$

so $x_1\partial f/\partial x_1 + \cdots + x_n\partial f/\partial x_n = kf$.

7. (a) f is homogeneous of degree 2. We obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x\left[\frac{1}{2}4x^3(x^4+2y^4)^{-1/2}\sin\frac{3x}{y} + \cancel{\sqrt{x^4+2y^4}\,\frac{3}{y}}\cos\frac{3x}{y}\right]$$

$$+y\left[\frac{1}{2}8y^3(x^4+2y^4)^{-1/2}\sin\frac{3x}{y} + \cancel{\sqrt{x^4+2y^4}\left(-\frac{3x}{y^2}\right)}\cos\frac{3x}{y}\right]$$

$$= 2(x^4+2y^4)^{1/2}\sin\frac{3x}{y} = 2f \quad ✓$$

8. $u = f(x+ct) + g(x-ct)$

$u_x = f'(x+ct)(1) + g'(x-ct)(1)$, $u_{xx} = f''(x+ct) + g''(x-ct)$

$u_t = f'(x+ct)(+c) + g'(x-ct)(-c)$, $u_{tt} = f''(x+ct)c^2 + g''(x-ct)(-c)^2 = c^2 u_{xx}$ ✓

9. (a) $xy'' + y' + k^2 xy = 0$. With $x = \alpha t$, $\alpha t \frac{d}{d(\alpha t)} \frac{d}{d(\alpha t)} y + \frac{d}{d(\alpha t)} y + k^2 \alpha t y = 0$,

or, multiplying by α and writing $y(x(t)) = u(t)$, $tu'' + u' + k^2\alpha^2 tu = 0$.

Then, setting $\alpha = 1/k$ gives $u(t) = AJ_0(t) + BY_0(t)$,

or, $y(x) = AJ_0(kx) + BY_0(kx)$

(b) $xy'' + y' + k^2 xy = 0$. $x = \alpha t^\beta$, $dx = \alpha\beta t^{\beta-1} dt$, so

$$\alpha t^\beta \frac{d}{\alpha\beta t^{\beta-1} dt} \frac{d}{\alpha\beta t^{\beta-1} dt} y + \frac{d}{\alpha\beta t^{\beta-1} dt} y + k^2 y = 0,$$

$$\frac{\alpha}{\alpha^2\beta^2} \frac{t^\beta}{t^{\beta-1}} \frac{d}{dt} t^{1-\beta} \frac{d}{dt} u + \frac{1}{\alpha\beta} t^{1-\beta} \frac{du}{dt} + k^2 u = 0,$$

$$\frac{t}{\alpha\beta^2}(t^{1-\beta} u'' + (1-\beta) t^{-\beta} u') + \frac{t^{1-\beta}}{\alpha\beta} u' + k^2 u = 0,$$

$$t^{3-\beta} u'' + (1-\beta+\beta) t^{2-\beta} u' + k^2 \alpha\beta^2 u = 0$$

so set $\beta = 2$, $\alpha = 1/4k^2$, so $x = t^2/4k^2$, $t = 2k\sqrt{x}$.

$tu'' + u' + tu = 0$, $u(t) = AJ_0(t) + BY_0(t)$, $y(x) = AJ_0(2k\sqrt{x}) + BY_0(2k\sqrt{x})$.

Section 13.5

1. (b) $f = e^{-2x}$, $f' = -2e^{-2x}$, $f'' = 4e^{-2x}$, $f''' = -8e^{-2x}$

so $f(x) = e^{-10} - 2e^{-10}(x-5) + \frac{4}{2!} e^{-10}(x-5)^2 - \frac{8}{3!} e^{-10}(x-5)^3 + \cdots$

2. (b) The idea is to let $x^{10}/2 = t$, say, and then do a Taylor series in t:

$$f(x) = \frac{1}{2+x^{10}} = \frac{1}{2} \frac{1}{1 + \boxed{x^{10}/2} \leftarrow t} = \frac{1}{2}(1 - t + t^2 - t^3 + \cdots)$$
$$= \frac{1}{2}\left(1 - \frac{x^{10}}{2} + \frac{x^{20}}{4} - \frac{x^{30}}{8} + \cdots\right),$$

for if we used $f(x) = f(0) + f'(0)x + \cdots$ directly, then most of the terms would turn out to be zero, leaving only the $1, x^{10}, x^{20}, \ldots$ powers. That is, the process would be very *inefficient*.

(c) Let $x^{20} = t$. Then $f(x) = \sin x^{20} = \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$

$$= x^{20} - \frac{x^{60}}{3!} + \frac{x^{100}}{5!} - \frac{x^{140}}{7!} + \cdots$$

(d) Let $x^{20} = t$. Then $f(x) = \cos x^{20} = \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$

$$= 1 - \frac{x^{40}}{2!} + \frac{x^{80}}{4!} - \cdots$$

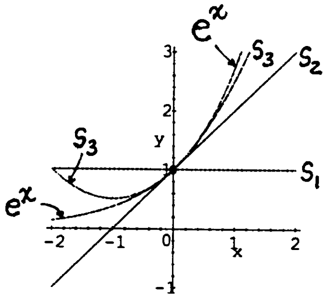
3\. (a)

```
> taylor(exp(x),x=0,5);
```

$$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+O(x^5)$$

```
> with(plots):
> implicitplot({y=exp(x),y=1,y=1+x,y=1+x+x^2/2},x=-2..2,y=-3..3,nump
oints=500);
```

Don't give these separate names (such as S1, S2, S3) because the command can handle only 2-dim. plots (namely, x and y).



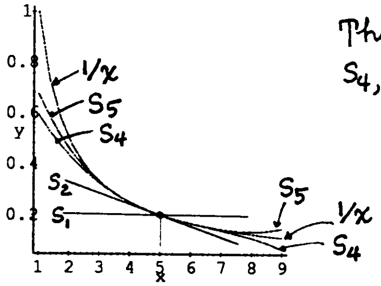
Note: I didn't plot S_4 and S_5 simply because the plot would be too crowded; could do e^x, S_4, and S_5 in a separate plot if we wish.

(d)

```
> taylor(1/x,x=5,5);
```

$$\frac{1}{5}-\frac{1}{25}(x-5)+\frac{1}{125}(x-5)^2-\frac{1}{625}(x-5)^3+\frac{1}{3125}(x-5)^4+O((x-5)^5)$$

```
> implicitplot({y=1/x,y=1/5-(1/25)*(x-5)+(1/125)*(x-5)^2-(1/625)*(x-
5)^3,y=1/5-(1/25)*(x-5)+(1/125)*(x-5)^2-(1/625)*(x-5)^3+(1/3125)*(
x-5)^4},x=1..9,y=-2..2,numpoints=500);
```



This time I plotted only $1/x$, S_4, and S_5, and added S_1 and S_2 by hand.

4\. (a) $\sin x = (\sin 0) + (\cos 0)x - \left(\frac{\sin 0}{2!}\right)x^2 - \frac{\cos\mu}{3!}x^3$

$= x - \frac{\cos\mu}{3!}x^3$

$|\sin x - x| = \left|\frac{\cos\mu}{6}x^3\right| < \frac{\cos 0}{6}(.5)^3 \approx 0.0208$, so $|\sin x - x| < 0.021$. ✓

(b) $\sin x = x - \frac{x^3}{3!} + \frac{\cos\mu}{5!}x^5$, so $\left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| = \left|\frac{\cos\mu}{120}x^5\right|$

$\leq \frac{\cos 0}{120}(.5)^5 \approx 0.0002604$

so $\left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| < 0.000261$ ✓

(c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\sin\mu}{6!}x^6$ so

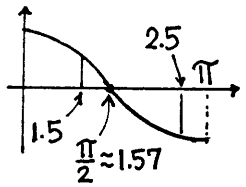
$\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\right| = \left|\frac{\sin\mu}{6!}x^6\right| \leq \frac{(\sin 0.5)(.5)^6}{720} = 0.0000104$

so $\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\right| < 0.000011$ ✓

5. Taylor's formula:

$$\sin x = (\sin 2) + (\cos 2)(x-2) + \left(-\frac{\sin 2}{2}\right)(x-2)^2 + \left(-\frac{\cos\mu}{6}\right)(x-2)^3 .$$

Over $1.5 \leq x \leq 2.5$,
$\left|-\frac{\cos\mu}{6}(x-2)^3\right| \leq \frac{1}{6}(0.5)^3$ by using $|\cos\mu| \leq 1$, but we can do better:



so, clearly, $|\cos\mu| \leq |\cos 2.5| = 0.8011$,
so $\left|-\frac{\cos\mu}{6}(x-2)^3\right| \leq \frac{0.801}{6}(0.5)^3 = 0.01669$
and
$\left|\sin x - [\sin 2 + (\cos 2)(x-2) - (\frac{\sin 2}{2})(x-2)^2]\right| < 0.0167$

over $1.5 \leq x \leq 2.5$.

6. $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Thus, for $x \neq 0$, $g'(x) = \frac{d}{dx}(x^2 \sin\frac{1}{x}) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$. Thus,

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

does not exist, since $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist. Yet $g'(0)$ does exist since, by the limit quotient definition,

$$g'(0) = \lim_{x\to 0} \frac{x^2 \sin 1/x - 0}{x} = \lim_{x\to 0} x \sin\frac{1}{x} = 0$$ because $\sin\frac{1}{x}$ is bounded

and the x factor → 0. Thus, $g'(0) = 0$ but $\lim_{x\to 0} g'(x) \neq 0$ (in fact it does not exist), so $g'(x)$ is not continuous at $x=0$ even though $g'(0)$ exists.

7. We must use the limit-of-the-difference-quotient formula because $f(x) = e^{-1/x^2}$ holds only for $x \neq 0$.

(a) $f'(0) = \lim_{x\to 0} \frac{e^{-1/x^2} - 0}{x} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$ (by l'Hôpital's rule)

(b) $f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0} \frac{2x^{-3}e^{-1/x^2} - 0}{x} = \lim_{x\to 0} \frac{2e^{-1/x^2}}{x^4} = \lim_{t\to\infty} \frac{2t^2}{e^t}$

$= \lim_{t\to\infty} \frac{4t}{e^t} = \lim_{t\to\infty} \frac{4}{e^t} = 0$

(c) $f'''(0) = \lim_{x\to 0} \frac{f''(x) - f''(0)}{x} = \lim_{x\to 0} \frac{(\frac{4}{x^6} - \frac{6}{x^4})e^{-1/x^2} - 0}{x}$ = etc = 0, as above.

8. $\frac{x^n}{n!} \sim \frac{x^n}{\sqrt{2\pi n}\, n^n e^{-n}} = \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{n}\right)^n \to (0)(0) = 0$, as $n \to \infty$

9. (b) $xyz + 6 = (3)(-2)(x-1) + (1)(-2)(y-3) + (1)(3)(z+2) + \cdots = 0$
so tangent plane is $-6(x-1) - 2(y-3) + 3(z+2) = 0$ at $(1,3,-2)$

(c) $3x^2 + y^2 + z^2 - 16 = 6(1)(x-1) + 2(3)(y-3) + 2(-2)(z+2) + \cdots = 0$
so tangent plane is $6(x-1) + 6(y-3) - 4(z+2) = 0$ or $3(x-1) + 3(y-3) - 2(z+2) = 0$. Of course we can write the latter as $3x + 3y - 2z = 16$; similarly in (b).

10. $f(x,y) = f[a+(x_0-a)t,\, b+(y_0-b)t^2] = F(t)$

$$F'(t) = f_x x'(t) + f_y y'(t) = (x_0-a)f_x + 2(y_0-b)t f_y$$

$$F''(t) = (x_0-a)[f_{xx}x' + f_{xy}y'] + 2(y_0-b)f_y + 2(y_0-b)t[f_{yx}x' + f_{yy}y']$$

$$= (x_0-a)^2 f_{xx} + 2(x_0-a)(y_0-b)t f_{xy} + 2(y_0-b)f_y + 2(x_0-a)(y_0-b)t f_{yx} + 4(y_0-b)^2 t^2 f_{yy}$$

$$F'''(t) = 2(x_0-a)(y_0-b)f_{xy} + 2(x_0-a)(y_0-b)f_{yx} + 4(y_0-b)^2 t f_{yy} + 2(x_0-a)(y_0-b)f_{yx} + 8(y_0-b)^2 t f_{yy}$$

+ 3rd order terms in $(x_0-a), (y_0-b)$, in which we're not interested since we are only going as far as 2nd order

$$F''''(t) = 4(y_0-b)^2 f_{yy} + 8(y_0-b)^2 f_{yy}$$
$$\vdots$$

Then $F(t) = F(0) + F'(0)t + \frac{1}{2!}F''(0)t^2 + \frac{1}{3!}F'''(0)t^3 + \frac{1}{4!}F''''(0)t^4 + \cdots$

$$= f + [(x_0-a)f_x]t + \tfrac{1}{2}[(x_0-a)^2 f_{xx} + 2(y_0-b)f_y]t^2$$
$$+ \tfrac{1}{6}[2(x_0-a)(y_0-b)f_{xy} + 2(x_0-a)(y_0-b)f_{yx} + 2(x_0-a)(y_0-b)f_{yx}]t^3$$
$$+ \tfrac{1}{24}[12(y_0-b)^2 f_{yy}]t^4 + \cdots$$

Then, assuming that $f_{xy} = f_{yx}$, we have

$$TSf\big|_{a,b} = F(1) = f(a,b) + (x_0-a)f_x(a,b) + (y_0-b)f_y(a,b)$$
$$+ \tfrac{1}{2}(x_0-a)^2 f_{xx} + (x_0-a)(y_0-b)f_{xy} + \tfrac{1}{2}(y_0-b)^2 f_{yy}(a,b) + \cdots,$$

as in (41).

11. (b) $e^{xy} = e^y + ye^y(x-1) + \frac{1}{2}y^2e^y(x-1)^2 + \frac{1}{6}y^3e^y(x-1)^3 + \cdots$

$$= [e^2 + e^2(y-2) + \tfrac{1}{2}e^2(y-2)^2 + \tfrac{1}{6}e^2(y-2)^3 + \cdots]$$
$$+ [2e^2 + 3e^2(y-2) + \tfrac{1}{2}(4)e^2(y-2)^2 + \cdots](x-1)$$
$$+ [2e^2 + 4e^2(y-2) + \cdots](x-1)^2$$
$$+ [\tfrac{1}{6}8e^2 + \cdots](x-1)^3 + \cdots$$
$$= e^2 + 2e^2(x-1) + e^2(y-2) + 2e^2(x-1)^2 + 3e^2(x-1)(y-2) + \tfrac{1}{2}e^2(y-2)^2$$
$$+ \tfrac{4}{3}e^2(x-1)^3 + 4e^2(x-1)^2(y-2) + 2e^2(x-1)(y-2)^2 + \tfrac{1}{6}e^2(y-2)^3 + \cdots$$

(c) $\sin 3xy = \sin(3y) + 3y\cos(3y)(x-1) - \frac{9}{2}y^2\sin(3y)(x-1)^2 - \frac{9}{2}y^3\cos(3y)(x-1)^3 + \cdots$

$$= [-\sin 3 + 3\cos 3\,(y+1) + \tfrac{9}{2}\sin 3\,(y+1)^2 - \tfrac{9}{2}\cos 3\,(y+1)^3 + \cdots]$$
$$+ [-3\cos 3 + (3\cos 3 - 9\sin 3)(y+1) + (9\sin 3 + \tfrac{27}{2}\cos 3)(y+1)^2 + \cdots](x-1)$$
$$+ [\tfrac{9}{2}\sin 3 - (9\sin 3 + \tfrac{27}{2}\cos 3)(y+1) + \cdots](x-1)^2$$
$$- [\tfrac{9}{2}(-1)\cos 3 + \cdots](x-1)^3 + \cdots$$
$$= -\sin 3 - 3\cos 3\,(x-1) + 3\cos 3\,(y+1) + \tfrac{9}{2}\sin 3\,(x-1)^2 + (3\cos 3 - 9\sin 3)(x-1)(y+1)$$
$$+ \tfrac{9}{2}\sin 3\,(y+1)^2 + \tfrac{9}{2}\cos 3\,(x-1)^3 - (9\sin 3 + \tfrac{27}{2}\cos 3)(x-1)^2(y+1)$$
$$+ (9\sin 3 + \tfrac{27}{2}\cos 3)(x-1)(y+1)^2 - \tfrac{9}{2}\cos 3\,(y+1)^3 + \cdots$$

12. (a) The Maple commands

```
readlib(mtaylor):
mtaylor(x^5*y^4-y, [x=1, y=2], 4);
```

gives the result

$$x^5y^4-y = -128+31y+80x+160(x-1)^2+160(x-1)(y-2)+24(y-2)^2+160(x-1)^3 + 320(x-1)^2(y-2)+120(x-1)(y-2)^2+8(y-2)^3+\cdots$$

Actually, that result is not quite correct until we re-express the first three terms as $-128+31(y-2)+80(x-1)+62+80 = 14+80(x-1)+31(y-2)$. I don't know why that happened. This bug did not occur in the other cases. For example, in part (c) the commands

```
readlib(mtaylor);
mtaylor(sin(3*x*y), [x=1, y=-1], 4);
```

does give the same result that we obtained above.

Section 13.6

1. (b) $f(x,y) = x^2+4y^2-4$ is C^1 everywhere, and $f_y(0,1) = 8 \neq 0$, so the conditions of Theorem 13.6.1 are met. To develop Taylor series of y about $x=0$, start with

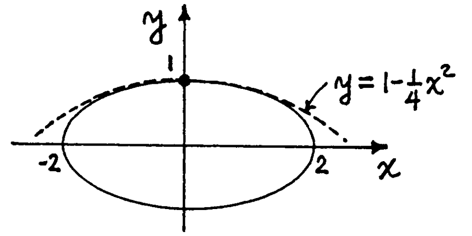
$$f_x + f_y y' = 2x + 8yy' = 0.$$

Thus, $y' = -\frac{x}{4y}$, $y'' = -\frac{1}{4y} + \frac{x}{4y^2}y'$, $\ldots$

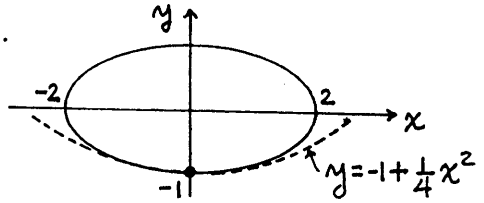
so $y'(0) = 0$, $y''(0) = -\frac{1}{4} + 0$, $\ldots$

so $y(x) = 1 - \frac{1}{4}x^2 + \cdots$.

Graphically, $1-\frac{1}{4}x^2$ is a parabolic "fit" at (0,1), as sketched at the right.



(c) As in (b), we obtain $y' = -\frac{x}{4y}$, $y'' = -\frac{1}{4y} + \frac{x}{4y^2}y'$, $\ldots$ so $y'(0)=0$, $y''(0) = +1/4$, so $y(x) = -1+\frac{1}{4}x^2+\cdots$. Naturally, $-1+\frac{1}{4}x^2$ gives the parabolic fit shown at the right.



(e) $f(x,y) = x(\cos\pi y+1)+(x^3+8)y$ is C^1 everywhere, but $f_y = -\pi x\sin\pi y + x^3+8 = 0$ at (-2,1), so the conditions of the theorem are not met.

(f) $f(x,y) = x-y-\sin y$ is C^1 everywhere and $f_y = -1-\cos y = -2 \neq 0$ at (0,0), so the conditions of the theorem are met.

$$f_x + f_y y' = 1-(1+\cos y)y' = 0$$

gives $y' = (1+\cos y)^{-1} = 1/2$ at (0,0)

$y'' = -1(-\sin y)(1+\cos y)^{-2}y' = 0$ at (0,0)

so $y(x) = 0+\frac{1}{2}x+0x^2+\cdots$

(g) $f(x,y) = x-y+\sin y$ is C^1 everywhere, but $f_y = -1+\cos y = 0$ at (0,0), so the conditions of the theorem are not met.

(h) $f(x,y) = (y-1)e^{y} - x^2 + 1$ is C^1 everywhere and $f_y = ye^{y} = e \neq 0$ at $(1,1)$, so the conditions of the theorem are met. Then

$$f_x + f_y y' = -2x + ye^{y}y' = 0$$

gives $y' = \frac{2x}{y}e^{-y} = 2e^{-1}$ at $(1,1)$

$$y'' = \frac{2}{y}e^{-y} + \frac{2x(-1)y'}{y^2}e^{-y} + \frac{2x}{y}(-y')e^{-y} = \left(2 - 2\tfrac{2}{e} - 2\tfrac{2}{e}\right)e^{-1} = \frac{2e-8}{e^2} \text{ at } (1,1)$$

so $y(x) = 1 + 2e^{-1}(x-1) + \frac{e-4}{e^2}(x-1)^2 + \cdots$

2. (b) d/dx gives $e^{y} + xe^{y}y' + y' = 0$ so $y' = -(x+e^{-y})^{-1}$

$$y'' = +1(1 - e^{-y}y')(x+e^{-y})^{-2}$$
$$= \frac{xe^{y}+2}{(xe^{y}+1)(x+e^{-y})^2}$$

(c) d/dx gives $(1+2y+3y^2)y' = 1$ so $y' = (1+2y+3y^2)^{-1}$

$$y'' = -1(2y' + 6yy')(1+2y+3y^2)^{-2}$$
$$= -\frac{2(1+3y)}{(1+2y+3y^2)^3}$$

(d) d/dx gives $ye^{y} + xy'e^{y} + xye^{y}y' = 0$ so $y' = -\frac{y}{x(1+y)}$

$$y'' = -\frac{y'}{x(1+y)} + \frac{y}{x^2(1+y)} - \frac{y}{x}\frac{(-1)y'}{(1+y)^2}$$
$$= \frac{2y+2y^2+y^3}{x^2(1+y)^3}$$

3. (b) z fixed: d/dx gives $e^{y} + xe^{y}y_x - 2yy_x = 0$, $y_x|_z = e^{y}/(2y - xe^{y})$

y fixed: d/dx gives $e^{y} - 2zz_x \sin z - z^2(\cos z)z_x = 0$,

$$z_x|_y = e^{y}/(2z\sin z + z^2\cos z)$$

(c) z fixed: d/dx gives $e^{x} + e^{y}y_x = 0$, $y_x|_z = -e^{x-y}$

y fixed: d/dx gives $e^{x} + e^{z}z_x = 0$, $z_x|_y = -e^{x-z}$

4. (b) $\left.\begin{aligned} f_1 &= x - u\cos v = 0 \\ f_2 &= y - u\sin v = 0 \end{aligned}\right\}$ In Thm 13.6.2 x_1 is x, x_2 is y, u_1 is u, u_2 is v, say. Surely, f_1 and f_2 are C^1 everywhere in x,y,u,v space.

Next, $\begin{vmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{vmatrix} = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{vmatrix} = \begin{vmatrix} -\cos v & u\sin v \\ -\sin v & -u\cos v \end{vmatrix} = u = 2 \neq 0$ at the point in question. Thus the theorem applies, and there do exist implicit functions $u(x,y)$ and $v(x,y)$ in some neighborhood of $(x,y) = (0,2)$.

(c) $\left.\begin{aligned} f_1 &= x\sin u + y^2 - v^2 = 0 \\ f_2 &= (x+v)^2 - \sin(uy) = 0 \end{aligned}\right\}$ In Thm 13.6.2, x_1 is x, x_2 is y, u_1 is u, u_2 is v, say. Surely, f_1 and f_2 are C^1 everywhere in x,y,u,v space. Next,

$$\begin{vmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{vmatrix} = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{vmatrix} = \begin{vmatrix} x\cos u & -2v \\ -y\cos(uy) & 2(x+v) \end{vmatrix}$$

$$= \begin{vmatrix} (1)\cos 0 & -2(-1) \\ -(1)\cos(0) & 2(1-1) \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0,$$ so the theorem applies, and there do exist implicit functions $u(x,y)$ and $v(x,y)$ in some neighborhood of $(x,y)=(1,1)$.

(d) $\left.\begin{aligned} f_1 &= x\cos u + y + v^2 = 0 \\ f_2 &= x - y + \sin(u^2 v) = 0 \end{aligned}\right\}$ In Thm 13.6.2, x_1 is x, x_2 is y, u_1 is u, u_2 is v, say. Surely, f_1 and f_2 are C^1 everywhere in x,y,u,v space. Next,

$$\begin{vmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{vmatrix} = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{vmatrix} = \begin{vmatrix} -x\sin u & 2v \\ 2uv\cos(u^2 v) & u^2\cos(u^2 v) \end{vmatrix}$$

$$= \begin{vmatrix} -(1)\sin\pi & 0 \\ \text{etc} & \text{etc} \end{vmatrix} = 0.$$ Thus, the conditions of Thm 13.6.2 are not met, and that theorem provides no information.

5. (b) $x - u^2\cos v = 0 \quad \partial/\partial y \to 0 - 2u u_y \cos v + u^2(\sin v)v_y = 0$

$y - ue^v = 6 \quad \partial/\partial y \to 1 - u_y e^v - ue^v v_y = 0$

gives, by Cramer's rule,

$$u_y = \frac{\begin{vmatrix} 0 & u^2\sin v \\ 1 & ue^v \end{vmatrix}}{\begin{vmatrix} -2u\cos v & u^2\sin v \\ e^v & ue^v \end{vmatrix}} = \frac{(\sin v)e^{-v}}{2\cos v + \sin v}$$

(c) $xe^u - uy + v = 0 \quad \partial/\partial y \to xe^u u_y - u - yu_y + v_y = 0$

$x^2 u - y^3 + v^3 = 1 \quad \partial/\partial y \to x^2 u_y - 3y^2 + 3v^2 v_y = 0$

gives, by Cramer's rule,

$$u_y = \frac{\begin{vmatrix} u & 1 \\ 3y^2 & 3v^2 \end{vmatrix}}{\begin{vmatrix} xe^u - y & 1 \\ x^2 & 3v^2 \end{vmatrix}} = \frac{3(uv^2 - y^2)}{3v^2(xe^u - y) - x^2}$$

6. (b) $$\frac{\partial(f,g,h)}{\partial(u,v,w)} = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix} = \begin{vmatrix} w^3 & 0 & 3uw^2 \\ 0 & 2 & -1 \\ ve^{uv} & ue^{uv} & 0 \end{vmatrix} = (uw^3 - 6uvw^2)e^{uv}$$
$$= uw^2(w - 6v)e^{uv}$$

(d) $$\frac{\partial(P,Q)}{\partial(x,y)} = \begin{vmatrix} P_x & P_y \\ Q_x & Q_y \end{vmatrix} = \begin{vmatrix} 0 & 3y^2 \\ 2x & -2y \end{vmatrix} = -6xy^2$$

$$\frac{\partial(P,Q)}{\partial(y,x)} = \begin{vmatrix} P_y & P_x \\ Q_y & Q_x \end{vmatrix} = \begin{vmatrix} 3y^2 & 0 \\ -2y & 2x \end{vmatrix} = 6xy^2$$

7. (a) $$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = -\begin{vmatrix} f_y & f_x \\ g_y & g_x \end{vmatrix} = -\frac{\partial(f,g)}{\partial(y,x)}$$

(b) $$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = -\begin{vmatrix} f_y & f_x \\ g_y & g_x \end{vmatrix} = +\begin{vmatrix} g_y & g_x \\ f_y & f_x \end{vmatrix} = \frac{\partial(g,f)}{\partial(y,x)}$$

8\. (a) Regard u as a function of x. Then d/dx gives

$$e^u + xe^u u' - 3x^2u - x^3u' = 0 \quad \text{so} \quad u'(x) = (3x^2u - e^u)/(xe^u - x^3)$$

Regard x as a function of u. Then d/du gives

$$x'e^u + xe^u - 3x^2x'u - x^3 = 0 \quad \text{so} \quad x'(u) = (xe^u - x^3)/(3x^2u - e^u)$$

We see from these results that $\frac{du}{dx}\frac{dx}{du}$ does $=1$.

9\. In these cases (48) reduces to (47): $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$ ☆

(a) $x=u+v$, $y=u-v$ so $u=(x+y)/2$, $v=(x-y)/2$

Thus, ★ becomes

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-\tfrac{1}{2})(-2) = 1 \checkmark$$

(b) $x=u+v$, $y=u^3-v^3$ Can't solve (easily) for $u(x,y)$ and $v(x,y)$, as we were able to do in (a), but we don't need $u(x,y)$ and $v(x,y)$, we merely need the derivatives u_x, u_y, v_x, v_y. Well,

$\partial/\partial x$ gives $1 = u_x + v_x$

$$0 = 3u^2u_x - 3v^2v_x \quad \text{so} \quad u_x = \frac{\begin{vmatrix} 1 & 1 \\ 0 & -3v^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3u^2 & -3v^2 \end{vmatrix}} = \frac{-3v^2}{-3v^2-3u^2} = \frac{v^2}{u^2+v^2}$$

$$\text{and} \quad v_x = 1 - u_x = \frac{u^2}{u^2+v^2}$$

$\partial/\partial y$ gives $0 = u_y + v_y$

$$1 = 3u^2u_y - 3v^2v_y \quad \text{so} \quad u_y = \frac{\begin{vmatrix} 0 & 1 \\ 1 & -3v^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3u^2 & -3v^2 \end{vmatrix}} = \frac{-1}{-3v^2-3u^2} = \frac{1}{3(u^2+v^2)}$$

$$\text{and} \quad v_y = -u_y = -\frac{1}{3(u^2+v^2)}$$

Then ★ becomes

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v^2/(u^2+v^2) & 1/3(u^2+v^2) \\ u^2/(u^2+v^2) & -1/3(u^2+v^2) \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 3u^2 & -3v^2 \end{vmatrix} = \text{etc.} = 1 \checkmark$$

10\. $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_xv_y - u_yv_x$, $\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix} = x_ry_s - x_sy_r$

so $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,s)} = u_xv_yx_ry_s - u_xv_yx_sy_r - u_yv_xx_ry_s + u_yv_xx_sy_r$ ♀

Now working from the other end,

$$\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} = u_rv_s - u_sv_r = (u_xx_r + u_yy_r)(v_xx_s + v_yy_s) - (u_xx_s + u_yy_s)(v_xx_r + v_yy_r)$$

$$= u_x\cancel{x_rv_x}x_s + u_xx_rv_yy_s + u_yy_rv_xx_s + \cancel{u_yy_rv_y}y_s$$
$$- u_x\cancel{x_sv_x}x_r - u_xx_sv_yy_r - u_yy_sv_xx_r - \cancel{u_yy_sv_y}y_r$$

$=$ ♀. $\checkmark$

11. I intended to give u and v as functions of x and y, and x and y as functions of r and s, as stated in Exercise 10. Nevertheless, we can still verify (10.2).

(a) $$x = u\cos v \qquad y = u\sin v \quad \text{and} \quad u = r+s \qquad v = r^2+s^2$$

For the first Jacobian in (10.2) we need u_x, u_y, v_x, v_y. These are given by (41a,b) and (42a,b) with $r \to u$ and $\theta \to v$:

$$u_x = \cos v, \quad u_y = \sin v$$
$$v_x = -\frac{\sin v}{u}, \quad v_y = \frac{\cos v}{u}$$

For the second Jacobian in (10.2) we need x_r, x_s, y_r, y_s. From the four given formulas,
$$x = (r+s)\cos(r^2+s^2)$$
$$y = (r+s)\sin(r^2+s^2)$$
so
$$x_r = \cos(r^2+s^2) - 2r(r+s)\sin(r^2+s^2)$$
$$x_s = \cos(\ \text{"}\ ) - 2s(\ \text{"}\ )\sin(\ \text{"}\ )$$
$$y_r = \sin(\ \text{"}\ ) + 2r(\ \text{"}\ )\cos(\ \text{"}\ )$$
$$y_s = \sin(\ \text{"}\ ) + 2s(\ \text{"}\ )\cos(\ \text{"}\ )$$

For the third Jacobian in (10.2) we need u_r, u_s, v_r, v_s. From the given formulas, these are
$$u_r = 1, \quad u_s = 1$$
$$v_r = 2r, \quad v_s = 2s.$$

Then (10.2) becomes

$$\text{LHS} = \left(\frac{\cos^2 v}{u} + \frac{\sin^2 v}{u}\right)\Big([\cos(r^2+s^2) - 2r(r+s)\sin(r^2+s^2)][\sin(r^2+s^2)$$
$$+2s(r+s)\cos(r^2+s^2)] - [\cos(r^2+s^2) - 2s(r+s)\sin(r^2+s^2)][\sin(r^2+s^2)$$
$$+2r(r+s)\cos(r^2+s^2)]\Big) = [\overset{\cos(r^2+s^2)}{C} - 2r(r+s)\overset{\sin(r^2+s^2)}{S}][S + 2s(r+s)C]/u$$
$$-[C - 2s(r+s)S][S + 2r(r+s)C]/u$$
$$= (\cancel{CS} + 2s(r+s)C^2 - 2r(r+s)S^2 - 4rs\cancel{(s+r)^2}CS$$
$$- \cancel{CS} - 2r(r+s)C^2 + 2s(r+s)S^2 + 4rs\cancel{(s+r)^2}CS)/u$$
$$= 2(s^2-r^2)/u = 2(s-r)$$

and

$$\text{RHS} = (1)(2s) - (1)(2r) = 2(s-r)$$ ← these agree ✓

12. (a) $f[p,T,v(p,T)] = 0$.

$\partial/\partial p$ gives $f_p + f_v v_p = 0$, so $\dfrac{\partial v}{\partial p} = -\dfrac{f_p}{f_v}$ if $f_v \neq 0$

$\partial/\partial T$ gives $f_T + f_v v_T = 0$, so $\dfrac{\partial v}{\partial T} = -\dfrac{f_T}{f_v}$ if $f_v \neq 0$

(b) In general, such relations hold only for ordinary derivatives, not for partial derivatives (see the discussion following Example 7).

However, they do hold in the present case: In the first of equations (12.3) it is clear that each partial derivative is with T fixed. But with T fixed in (12.1) there are only the <u>two</u> variables p and v, so $\partial v/\partial p$ and $\partial p/\partial v$ behave like ordinary derivatives. Similarly, in the second of equations (12.3) each partial derivative is with p fixed.

(c) $f(p,T,v) = (p+\frac{a}{v^2})(v-b)-RT=0$

so (12.2) gives
$$v_p = -\frac{v-b}{-\frac{2a}{v^3}(v-b)+(p+\frac{a}{v^2})} = \frac{-v^3(v-b)}{pv^3-av+2ab} \quad ①$$
$$v_T = -\frac{-R}{-\frac{2a}{v^3}(v-b)+(p+\frac{a}{v^2})} = \frac{Rv^3}{pv^3-av+2ab} \quad ②$$

Now,
$$p=\frac{RT}{v-b}-\frac{a}{v^2} \text{ gives } p_v = \frac{-RT}{(v-b)^2}+\frac{2a}{v^3} = -\frac{(p+\frac{a}{v^2})(v-b)}{(v-b)^2}+\frac{2a}{v^3}$$
and
$$= -\frac{pv^3-av+2ab}{v^3(v-b)},$$

in agreement with ①, and
$$T=\frac{1}{R}(p+\frac{a}{v^2})(v-b) \text{ gives } T_v = \frac{1}{R}\left[(-\frac{2a}{v^3})(v-b)+p+\frac{a}{v^2}\right] = \frac{pv^3-av+2ab}{Rv^3},$$

in agreement with ②.

13. $$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} = \overset{\sin\theta}{\overset{\downarrow}{s}}\frac{\partial}{\partial r}+\frac{\overset{\cos\theta}{\overset{\downarrow}{c}}}{r}\frac{\partial}{\partial\theta}$$

$$\begin{aligned}\text{so } T_{yy} &= (s\tfrac{\partial}{\partial r}+\tfrac{c}{r}\tfrac{\partial}{\partial\theta})(s\tfrac{\partial}{\partial r}+\tfrac{c}{r}\tfrac{\partial}{\partial\theta})T = (s\tfrac{\partial}{\partial r}+\tfrac{c}{r}\tfrac{\partial}{\partial\theta})(sT_r+\tfrac{c}{r}T_\theta)\\ &= s^2T_{rr}+s\tfrac{\partial}{\partial r}(\tfrac{c}{r}T_\theta)+\tfrac{c}{r}\tfrac{\partial}{\partial\theta}(sT_r)+\tfrac{c}{r}\tfrac{\partial}{\partial\theta}(\tfrac{c}{r}T_\theta)\\ &= s^2T_{rr}-\tfrac{sc}{r^2}T_\theta+\tfrac{sc}{r}T_{\theta r}+\tfrac{c^2}{r}T_r+\tfrac{cs}{r}T_{r\theta}-\tfrac{cs}{r^2}T_\theta+\tfrac{c^2}{r^2}T_{\theta\theta},\end{aligned}$$

as in (57). ✓

14. (b) $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial u}+3\frac{\partial}{\partial v} \qquad \text{since } u=-x+y \text{ and } v=3x-2y$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial}{\partial u}-2\frac{\partial}{\partial v}$$

so
$$\begin{aligned}T_{xx} &= (-\tfrac{\partial}{\partial u}+3\tfrac{\partial}{\partial v})(-\tfrac{\partial}{\partial u}+3\tfrac{\partial}{\partial v})T = (-\tfrac{\partial}{\partial u}+3\tfrac{\partial}{\partial v})(-T_u+3T_v)\\ &= T_{uu}-3T_{vu}-3T_{uv}+9T_{vv}\end{aligned}$$

$$\begin{aligned}T_{yy} &= (\tfrac{\partial}{\partial u}-2\tfrac{\partial}{\partial v})(\tfrac{\partial}{\partial u}-2\tfrac{\partial}{\partial v})T = (\tfrac{\partial}{\partial u}-2\tfrac{\partial}{\partial v})(T_u-2T_v)\\ &= T_{uu}-2T_{vu}-2T_{uv}+4T_{vv}\end{aligned}$$

so (assuming $T_{uv}=T_{vu}$),

(c) $x = e^{v}$, $y = u - v^2$ can be solved for u and v: $v = \ln x$, $u = y + (\ln x)^2$
so $u_x = 2\frac{\ln x}{x} = 2ve^{-v}$, $v_x = \frac{1}{x} = e^{-v}$, $u_y = 1$, $v_y = 0$
Thus,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial v}{\partial x} = 2ve^{-v}\frac{\partial}{\partial u} + e^{-v}\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial}{\partial u}$$

so

$$\begin{aligned} T_{xx} &= \left(2ve^{-v}\frac{\partial}{\partial u} + e^{-v}\frac{\partial}{\partial v}\right)\left(2ve^{-v}T_u + e^{-v}T_v\right) \\ &= 4v^2e^{-2v}T_{uu} + 2ve^{-2v}T_{vu} + e^{-v}\frac{\partial}{\partial v}(2ve^{-v}T_u) + e^{-v}\frac{\partial}{\partial v}(e^{-v}T_v) \\ &= 4v^2e^{-2v}T_{uu} + 2ve^{-2v}T_{vu} + 2e^{-2v}(1-v)T_u + 2ve^{-2v}T_{uv} - e^{-2v}T_v + e^{-2v}T_{vv} \end{aligned}$$

and

$$T_{yy} = \frac{\partial}{\partial u}\frac{\partial}{\partial u}T = T_{uu}$$

so (if $T_{uv} = T_{vu}$)

$$4v^2e^{-2v}T_{uu} + 4ve^{-2v}T_{uv} + 2(1-v)e^{-2v}T_u - e^{-2v}T_v + e^{-2v}T_{vv} + T_{uu} = 0$$

or

$$(4v^2 + e^{2v})T_{uu} + 4vT_{uv} + 2(1-v)T_u - T_v + T_{vv} = 0$$

(d) $x = u^2 + v^2$, $y = u + v$. We'll need u_x, v_x, u_y, v_y so let us compute these first.

$$\left.\begin{array}{ll} \partial/\partial x \text{ of first gives} & 1 = 2uu_x + 2vv_x \\ \partial/\partial x \text{ of second "} & 0 = u_x + v_x \end{array}\right\} \text{give } u_x = \frac{1}{2(u-v)},\ v_x = -\frac{1}{2(u-v)}$$

$$\left.\begin{array}{ll} \partial/\partial y \text{ of first "} & 0 = 2uu_y + 2vv_y \\ \partial/\partial y \text{ of second "} & 1 = u_y + v_y \end{array}\right\} \text{give } u_y = \frac{v}{v-u},\ v_y = -\frac{u}{v-u}$$

Thus,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2(u-v)}\frac{\partial}{\partial u} - \frac{1}{2(u-v)}\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial v}{\partial y} = \frac{v}{v-u}\frac{\partial}{\partial u} - \frac{u}{v-u}\frac{\partial}{\partial v}$$

so

$$\begin{aligned} T_{xx} &= \frac{1}{2(u-v)}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\frac{1}{2(u-v)}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)T = \frac{1}{4(u-v)}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left[(u-v)^{-1}(T_u - T_v)\right] \\ &= \frac{1}{4(u-v)}\left\{\left[-\frac{1}{(u-v)^2} - \frac{(-1)(-1)}{(u-v)^2}\right](T_u - T_v) + (u-v)^{-1}(T_{uu} - 2T_{uv} + T_{vv})\right\} \\ &= \frac{1}{4(u-v)^2}(T_{uu} - 2T_{uv} + T_{vv}) - \frac{1}{2(u-v)^3}(T_u - T_v) \end{aligned}$$

$$\begin{aligned} T_{yy} &= \left(\frac{v}{v-u}\frac{\partial}{\partial u} - \frac{u}{v-u}\frac{\partial}{\partial v}\right)\left(\frac{v}{v-u}T_u - \frac{u}{v-u}T_v\right) \\ &= \frac{v}{v-u}\frac{v}{(v-u)^2}T_u + \frac{v^2}{(v-u)^2}T_{uu} + \frac{v}{v-u}\left[-\frac{1}{v-u} - \frac{u(-1)(-1)}{(v-u)^2}\right]T_v - \frac{uv}{(v-u)^2}T_{uv} \end{aligned}$$

$$-\frac{u}{v-u}\left[\frac{1}{v-u}+\frac{v(-1)}{(v-u)^2}\right]T_u-\frac{uv}{(v-u)^2}T_{uv}$$

$$+\frac{u}{v-u}\left[-\frac{u}{(v-u)^2}T_v+\frac{u}{v-u}T_{vv}\right]$$

$$=\frac{1}{(v-u)^2}\left(v^2T_{uu}+u^2T_{vv}-2uvT_{uv}\right)+\frac{u^2+v^2}{(v-u)^3}\left(T_u-T_v\right)$$

Finally,

$$T_{xx}+T_{yy}=\frac{1}{4(v-u)^2}\left[(4v^2+1)T_{uu}+(4u^2+1)T_{vv}-(8uv+2)T_{uv}\right]$$

$$+\frac{2u^2+2v^2+1}{2(v-u)^3}\left(T_u-T_v\right)$$

(f) $x=u^2+v$, $y=u^2-v$. We'll need u_x, u_y, v_x, v_y so let us compute these first. Adding and subtracting gives

$$u=\sqrt{(x+y)/2}, \quad v=(x-y)/2$$

so $u_x=\frac{1}{2\sqrt{2}}(x+y)^{-1/2}=\frac{1}{4u}$, $u_y=\frac{1}{2\sqrt{2}}(x+y)^{-1/2}=\frac{1}{4u}$, $v_x=\frac{1}{2}$, $v_y=-\frac{1}{2}$

Thus, $\frac{\partial}{\partial x}=\frac{\partial}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\frac{\partial v}{\partial x}=\frac{1}{4u}\frac{\partial}{\partial u}+\frac{1}{2}\frac{\partial}{\partial v}$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{4u}\frac{\partial}{\partial u}-\frac{1}{2}\frac{\partial}{\partial v}$$

so $T_{xx}=\left(\frac{1}{4u}\frac{\partial}{\partial u}+\frac{1}{2}\frac{\partial}{\partial v}\right)\left(\frac{1}{4u}T_u+\frac{1}{2}T_v\right)=\frac{1}{4u}\left(-\frac{1}{4u^2}\right)T_u+\frac{1}{16u^2}T_{uu}+\frac{1}{8u}T_{uv}$

$$+\frac{1}{8u}T_{uv}+\frac{1}{4}T_{vv}$$

$$T_{yy}=\left(\frac{1}{4u}\frac{\partial}{\partial u}-\frac{1}{2}\frac{\partial}{\partial v}\right)\left(\frac{1}{4u}T_u-\frac{1}{2}T_v\right)=\frac{1}{4u}\left(-\frac{1}{4u^2}\right)T_u+\frac{1}{16u^2}T_{uu}-\frac{1}{8u}T_{uv}$$

$$-\frac{1}{8u}T_{uv}+\frac{1}{4}T_{vv}$$

Finally,

$$T_{xx}+T_{yy}=\frac{1}{8u^2}T_{uu}+\frac{1}{2}T_{vv}-\frac{1}{8u^3}T_u$$

15. $\frac{\partial}{\partial x}=\frac{\partial}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\frac{\partial v}{\partial x}$ so $T_{xx}=\left(u_x\frac{\partial}{\partial u}+v_x\frac{\partial}{\partial v}\right)\left(u_xT_u+v_xT_v\right)$

$$=\left(T_{uu}u_x+T_{uv}v_x\right)u_x+T_uu_{xx}+\left(T_{vu}u_x+T_{vv}v_x\right)v_x+T_vv_{xx}$$

$\frac{\partial}{\partial y}=\frac{\partial}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\frac{\partial v}{\partial y}$ so $T_{yy}=\left(u_y\frac{\partial}{\partial u}+v_y\frac{\partial}{\partial v}\right)\left(u_yT_u+v_yT_v\right)$

$$=\left(T_{uu}u_y+T_{uv}v_y\right)u_y+T_uu_{yy}+\left(T_{vu}u_y+T_{vv}v_y\right)v_y+T_vv_{yy}$$

so $T_{xx}+T_{yy}=\left(u_x^2+u_y^2\right)T_{uu}+\left(u_xv_x+u_yv_y\right)\left(T_{uv}+T_{vu}\right)+\left(v_x^2+v_y^2\right)T_{vv}$

$$+\left(u_{xx}+u_{yy}\right)T_u+\left(v_{xx}+v_{yy}\right)T_v$$

Now suppose that the change of variables is restricted so that $u_x = v_y$ and $u_y = -v_x$. Then $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$, $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$, $u_x v_x + u_y v_y = u_x(-u_y) + u_y(u_x) = 0$, and $v_x^2 + v_y^2 = u_x^2 + u_y^2$, so the second, fourth, and fifth terms are 0 and we have

$$T_{xx} + T_{yy} = (u_x^2 + u_y^2)(T_{uu} + T_{vv}) = 0$$

and if $u_x^2 + u_y^2 \neq 0$ in the x,y domain of interest then $T_{uu} + T_{vv} = 0$.

Section 13.7

1. (b) $TSf|_{x=1} = 5 + 3(x-1)^4$ so $f'(1) = f''(1) = f'''(1) = 0$, $f^{(iv)}(1) > 0$, so f has a min at $x=1$
   (d) $TSf|_{x=1} = 4(x-1)^3 + \cdots \sim 4(x-1)^3$, so f has a horiz. infl. pt. at $x=1$
   (f) $TSf|_{x=1} = 1 + 8(x-1)^5 + \cdots \sim 1 + 8(x-1)^5$, so f has a horiz. infl. pt. at $x=1$
   (g) $TSf|_{x=1} = -8(x-1)^4 + \cdots$, so f has a max. at $x=1$
   (h) $TSf|_{x=1} = 1 - (x-1)^3 + \cdots$, so f has a horiz. infl. pt. at $x=1$.

2. (b) $f = (x^2 - 4x + 5)^{-1}$, $f' = -(2x-4)(x^2-4x+5)^{-2} = 0$ only at $x = 2$.
   $$f'' = -2(x^2-4x+5)^{-2} - (2x-4)(-2)(2x-4)(x^2-4x+5)^{-3} = -1 \text{ at } x=2$$
   so $f(x) = 1 - \frac{1}{2}(x-1)^2 + \cdots$, so f has a max at $x=2$.
   (d) $f = -3(\ln x)^3$, $f' = -9(\ln x)^2/x = 0$ only at $x=1$. Since $\ln x = (x-1) + \cdots$, we can see that $f = -3(\ln x)^3 = -3(x-1)^3 + \cdots$ has a horiz. infl. pt. at $x=1$.
   (e) $f = (\ln x)^4$, $f' = 4(\ln x)^3/x = 0$ only at $x=1$. Since $\ln x = (x-1) + \cdots$, we can see that $f = (x-1)^4 + \cdots$ so f has a min. at $x=1$.
   (f) $f = e^{-\sin x}$, $f' = e^{-\sin x}(-\cos x) = 0$ at $n\pi/2$ $(n = \pm 1, \pm 3, \ldots)$. At those points
   $$f'' = e^{-\sin x}(\cos^2 x + \sin x) = \begin{cases} +e^{-1} & @\ x = \pi/2 + 2n\pi \quad (n = 0, \pm 1, \pm 2, ..) \\ -e & @\ x = -\pi/2 + 2n\pi \quad (\quad '' \quad) \end{cases}$$
   so f has maxima at $x = -\pi/2 + 2n\pi$ and minima at $x = \pi/2 + 2n\pi$ for any $n = 0, \pm 1, \pm 2, \ldots$
   (g) $f = x^2 e^{-x}$, $f' = (2x - x^2)e^{-x} = 0$ at $x = 0, 2$. There,
   $$f'' = (2-2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x} = \begin{cases} 2 & \text{at } x=0 \\ -2e^{-2} & \text{at } x=2 \end{cases}$$
   so f has a min at $x=0$ and a max at $x=2$.
   (h) $f = 4\sqrt{x} - x^2$, $f' = 2/\sqrt{x} - 2x = 0$ only at $x=1$. There,
   $$f'' = -x^{-3/2} - 2 = -3,$$ so f has a max. at $x=1$.

3. Let $h(x) = f(x)g(h)$. Then $h' = f'g + fg'$ so $h'(X) = 0 + 0 = 0$. Hence, fg does have a critical point at $x = X$. To show that that critical point

can (perhaps surprisingly) turn out to be a max, a min, or a horiz. inflec. pt, it will suffice to put forward examples of each. For this purpose, let

$$f(x) = 1 - x^2,$$
$$g(x) = a - x^2 + bx^3,$$

each of which does have a max at $x=0$. Now,

$$h' = f'g + fg'$$
$$h'' = f''g + 2\cancelto{0 \text{ at } x=0}{f'g'} + fg'' \quad \text{so} \quad h''(0) = (-2)(a) + (1)(-2).$$

If $a=0$, say, then $h''(0)<0$ and h has a max at $x=0$. If $a=-5$, say, then $h''(0)<0$ and h has a min at $x=0$. If $a=-1$, then $h''(0)=0$ and we need to look to h''':

$$h''' = \cancelto{0}{f'''}g + 3f''\cancelto{0}{g'} + 3\cancelto{0}{f'}g'' + fg''' = (1)(6b).$$

Thus, if we choose any $b \neq 0$ then $h'(0) = h''(0) = 0$ and $h'''(0) \neq 0$, so h has a horiz. infl. pt. at $x=0$.

4. $f_x = 3x^2y - y^3 = (3x^2 - y^2)y = 0$ along the lines $y=0$ and $y = \pm\sqrt{3}\,x$ —(A)
$f_y = x^3 - 3xy^2 = (x^2 - 3y^2)x = 0$ " " " $x=0$ " $y = \pm\frac{1}{\sqrt{3}}x$ —(B)
Any of the three (A) lines and any of the three (B) lines can intersect only at (0,0).

5. (b) $\left.\begin{array}{l} f_x = 2x + y = 0 \\ f_y = x + 2y + 6 = 0 \end{array}\right\}$ give $x=2, y=-4$.
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} f_{xx}(2,-4) & f_{xy}(2,-4) \\ f_{yx}(2,-4) & f_{yy}(2,-4) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has $\lambda = 3, 1$, hence f has a minimum at $(2,-4)$

(d) $\left.\begin{array}{l} f_x = -6y + 5x^4 = 0 \\ f_y = -6x = 0 \end{array}\right\}$ give $x=y=0$.
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$ has $\lambda = \pm 6$, hence f has a saddle at (0,0).

(e) $\left.\begin{array}{l} f_x = 4x - y = 0 \\ f_y = -x = 0 \end{array}\right\}$ give $x=y=0$.
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} 4 & -1 \\ -1 & 0 \end{pmatrix}$ has $\lambda = 2 \pm \sqrt{5}$, hence f has a saddle at (0,0).

(f) $\left.\begin{array}{l} f_x = -2x \exp[-(x^2+y^2+1)] = 0 \\ f_y = -2y \exp[\quad " \quad] = 0 \end{array}\right\}$ give $x=y=0$
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} -2e^{-1} & 0 \\ 0 & -2e^{-1} \end{pmatrix}$ has $\lambda = -2e^{-1}, -2e^{-1}$, hence f has a max at (0,0).

(h) $\left.\begin{array}{l} f_x = 2x \exp(x^2-y^2) = 0 \\ f_y = -2y \exp(\;"\;) = 0 \end{array}\right\}$ give $x=y=0$.
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ has $\lambda = 2, -2$, hence f has a saddle at (0,0).

(i) $\left.\begin{array}{l} f_x = 2y - y\cos xy = 0 \\ f_y = 2x - x\cos xy = 0 \end{array}\right\}$ give only $x=y=0$ (because $2 - \cos xy$ cannot $=0$).
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ has $\lambda = 2, -2$, hence f has a saddle at (0,0).

(j) $\left.\begin{array}{l} f_x = 3x^2 - y + 5x^4 = 0 \\ f_y = -x = 0 \end{array}\right\}$ give $x=y=0$.
In Thm 13.7.4, $\underset{\sim}{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ has $\lambda = 1, -1$, hence f has a saddle at (0,0).

6. (b) $$\left.\begin{aligned} f_x &= (4x+z)\exp(2x^2+xz-5z^2)=0\\ f_y &= 0\\ f_z &= (x-10z)\exp(\quad \text{"} \quad)=0 \end{aligned}\right\}$$ give $x=z=0$. Thus, every point on the y axis is a critical point.

In Thm 13.7.4, $\underset{\sim}{A}=\begin{pmatrix} 4 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & -10 \end{pmatrix}$ gives $\lambda=0,-10.07,+4.07$, hence f has a saddle at each point on the y axis.

(c) $$\left.\begin{aligned} f_x &= 2x+y+z=0\\ f_y &= x+2y+z=0\\ f_z &= x+y+2z=0 \end{aligned}\right\}$$ gives $x=y=z=0$.

In Thm 13.7.4, $\underset{\sim}{A}=\begin{pmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2 \end{pmatrix}$ gives $\lambda=4,1,1$, hence f has a min at $(0,0,0)$.

(e) $$\left.\begin{aligned} f_w &= (2w-x)\exp(w^2-wx-yz+z^2)=0\\ f_x &= (-w)\exp(\quad \text{"} \quad)=0\\ f_y &= (-z)\exp(\quad \text{"} \quad)=0\\ f_z &= (-y+2z)\exp(\quad \text{"} \quad)=0 \end{aligned}\right\}$$ give $w=x=y=z=0$

In Thm 13.7.4,
$\underset{\sim}{A}=\begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 2 \end{pmatrix}$ gives $\lambda=1+\sqrt{2},1+\sqrt{2},1-\sqrt{2},1-\sqrt{2}$, which are of mixed sign. Hence, f has a saddle at $w=x=y=z=0$.

(f) $$\left.\begin{aligned} f_w &= -8w+4x+2z=0\\ f_x &= 4w-8x=0\\ f_y &= -8y=0\\ f_z &= 2w-8z=0 \end{aligned}\right\}$$ gives $w=x=y=z=0$

In Thm 13.7.4,
$\underset{\sim}{A}=\begin{pmatrix} -8 & 4 & 0 & 2\\ 4 & -8 & 0 & 0\\ 0 & 0 & -8 & 0\\ 2 & 0 & 0 & -8 \end{pmatrix}$ gives $\lambda=-8,-8,-12.47,-3.53$, which are all <0. Hence, f has a max at $(0,0,0,0)$.

7. $$\left.\begin{aligned} f_x &= 2x+2(z-1)\cos[x(z-1)]=0\\ f_y &= 10y=0\\ f_z &= 4(z-1)+2x\cos[x(z-1)]=0 \end{aligned}\right\}$$ is satisfied by $x=0, y=0, z=1$.

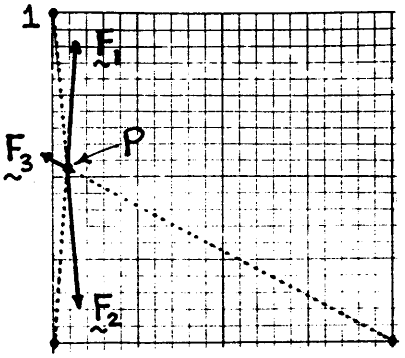
In Thm 13.7.4,
$\underset{\sim}{A}=\begin{pmatrix} 2 & 0 & 2\\ 0 & 10 & 0\\ 2 & 0 & 4 \end{pmatrix}$ gives $\lambda=10,\ 0.764,\ 5.24$. All are positive, so f has a min (i.e., a local min) at $(0,0,1)$. ✓

9. $f_x = 2x\cos(x^2+y^2+z^2)+y+z=0$
$f_y = x+2y\cos(\text{"})+z=0$
$f_z = x+y+2z\cos(\text{"})=0$

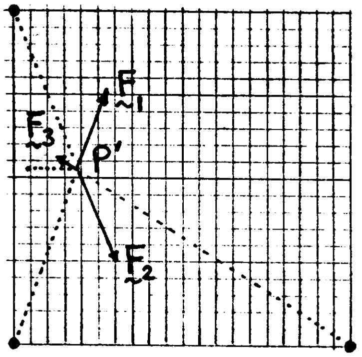
By inspection we can see that if $2\cos(x^2+y^2+z^2)=1$, then all three equations reduce to $x+y+z=0$. Thus, all points on the circle of intersection of the sphere $x^2+y^2+z^2=\cos^{-1}\frac{1}{2}=1.0472$ with the plane $x+y+z=0$ are critical points of f.

11. (a) From Fig. 6 we see that at any point (x,y) where $x<0$ there will be a nonzero x-component of force, so such a point cannot be an equilibrium point. Similarly if $x>1$. Similarly, if $y<0$ or $y>1$ there will be a nonzero y-component of force.

(b)

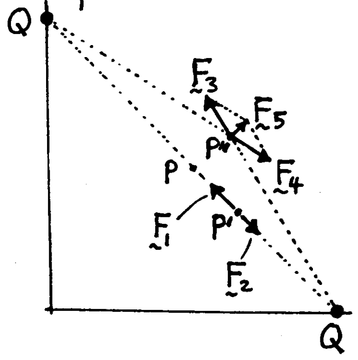


At the left we have plotted the repulsive forces $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3$ induced (per unit positive charge) at P. We've shown them to scale, $\underset{\sim}{F}_1$ be proportional to the inverse-square distance $1/(.047^2+.513^2) = 3.77$, $\underset{\sim}{F}_2$ to $1/(.047^2+.487^2) = 4.18$, and $\underset{\sim}{F}_3$ to $1/(.953^2+.513^2)=0.85$. We can see, if only approximately, how the sum $\underset{\sim}{F}_1+\underset{\sim}{F}_2$ is the negative of $\underset{\sim}{F}_3$.

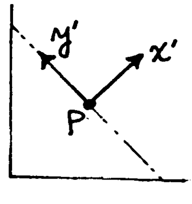


The question also asks why, in physical terms, the equilibrium is unstable. Here, my argument will not be as compelling. If we approximate $\underset{\sim}{A}$ in (34) as $\underset{\sim}{A} \approx \begin{pmatrix} -14 & 0 \\ 0 & 31 \end{pmatrix}$, then we can see that V has a "mountain" in the x direction and a "valley" in the y. Thus, we should be able to "see" the instability if we displace the "test charge" from P to the right (or left), say to P'. Indeed, it seems clear that the sum of the x-components of $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ will easily exceed the x-component of $\underset{\sim}{F}_3$, so that the charge will be pushed rightward rather than back toward the equilibrium point P.

(c) This time let us use our knowledge from Example 6 and parts (a) and (b) above to anticipate the outcome. Clearly, there will be an equilibrium point at P. If we move to a point P' then the charge on the x axis causes a force $\underset{\sim}{F}_1$ that is larger (because of the inverse-square law) than the force $\underset{\sim}{F}_2$ induced by the charge on the Q axis so their resultant tends to return us to the equil. pt. P. However, a lateral displacement, say to P'', leads to a force $\underset{\sim}{F}_5$ that tends to move us away from P. Thus, the equilibrium at P will be <u>unstable</u>. Further, we anticipate that in terms of local rotated coordinates x', y' (shown at the right) the potential should be of the form



$$V \sim \lambda_1 x'^2 + \lambda_2 y'^2$$



where $\lambda_1 < 0$ (so V has a "mountain at P in the x'-direction) and $\lambda_2 > 0$ (so V has a "valley" in the y'-direction through P). Let us see. We follow the Maple steps given in "Computer software", but omitting the first term in V, which is due to a charge at the origin. Enter

```
V: =((x-1)^2+y^2)^(-1/2)+(x^2+(y-1)^2)^(-1/2)):
Vx: = diff(V,x):
Vy: = diff(V,y):
XY: = fsolve({Vx=0,Vy=0},{x,y},{x=0..1,y=0..1});
```

and obtain the output

```
XY: = {y=.5000000000, x=.5000000000},
```

as anticipated. Next

```
Vxx: = diff(V,x,x);
Vxy: = diff(V,x,y);
Vyy: = diff(V,y,y);
```

gives $\partial^2V/\partial x^2$, $\partial^2V/\partial x\partial y$, $\partial^2V/\partial y^2$. Then

```
a11: = eval(subs(XY,Vxx));
a12:= eval(subs(XY,Vxy));
a22: = eval(subs(XY,Vyy));
```

give

```
a11: = 2.828427119, a12:= -8.485281364, a22: = 2.828427119
```

Then,

```
with(linalg):
A:= array([[a11,a12],[a12,a22]]);
eigenvects(");
```

give $\lambda_1 = -5.656854245$, $\underset{\sim}{e}_1 = (.7071067809, .7071067809)^T$,

$\lambda_2 = 11.31370848$, $\underset{\sim}{e}_2 = (-.7071067809, .7071067809)^T$.

Thus, under the change of variables $\underset{\sim}{x} = \underset{\sim}{Q}\underset{\sim}{x}' = (\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2)\underset{\sim}{x}'$ we obtain the canonical form $V \sim -5.66x'^2 + 11.31y'^2$ where the x', y' axes are as shown above since they are in the $\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2$ directions, respectively.

(d) Modify V to be

```
V: =((x-1)^2+y^2)^(-1/2)+((x-1)^2+(y-1)^2)^(-1/2)
        +(x^2+(y-1)^2)^(-1/2)+(x^2+y^2)^(-1/2)
```

then proceed as in (c). The result is

$$a_{11} = a_{22} \approx 5.65685, \quad a_{12} = .1 \times 10^{-8}$$

so V is already in canonical form, $V \approx 5.65685x^2 + 5.65685y^2$ which gives a minimum and stable equilibrium, again at the point (0.5, 0.5).

12. (a) $E(a,b) = \sum (ax_j + b - f_j)^2 = \min$

so $\partial E/\partial a = \sum 2x_j(ax_j + b - f_j) = 0$

$\partial E/\partial b = \sum 2(ax_j + b - f_j) = 0$

or, $(2\sum x_j^2)a + (2\sum x_j)b = 2\sum x_j f_j$

$$(2\sum x_j)a + (2\underbrace{\sum 1}_{=N})b = 2\sum f_j$$

so, by Cramer's rule, $a = \dfrac{\begin{vmatrix} 2\sum x_j f_j & 2\sum x_j \\ 2\sum f_j & 2N \end{vmatrix}}{\begin{vmatrix} 2\sum x_j^2 & 2\sum x_j \\ 2\sum x_j & 2N \end{vmatrix}}$, $b = \dfrac{\begin{vmatrix} 2\sum x_j^2 & 2\sum x_j f_j \\ 2\sum x_j & 2\sum f_j \end{vmatrix}}{\begin{vmatrix} 2\sum x_j^2 & 2\sum x_j \\ 2\sum x_j & 2N \end{vmatrix}}$, which

results do give (12.2) and (12.3).

(b) $E_{aa} = 2\sum x_j^2 > 0$ ✓

and

$$D = \begin{vmatrix} 2\sum x_j^2 & 2\sum x_j \\ 2\sum x_j & 2N \end{vmatrix} = 4[N\sum x_j^2 - (\sum x_j)^2].$$

If $\underset{\sim}{u} \equiv (x_1, \dots, x_N)$ and $\underset{\sim}{v} \equiv (1, 1, \dots, 1)$, then $|\underset{\sim}{u} \cdot \underset{\sim}{v}| \leq \|\underset{\sim}{u}\| \|\underset{\sim}{v}\|$ becomes

$$|\sum x_j| \leq \sqrt{\sum x_j^2}\sqrt{N} \quad \text{or} \quad (\sum x_j)^2 \leq N\sum x_j^2,$$

so $D \geqslant 0$. However, the Thm. 13.7.5 calls for $E_{aa} > 0$ and $D > 0$, so we would like to discard of the case where $D = 0$. Claim: That case occurs if and only if $x_1 = x_2 = \cdots = x_N$, whereas we have assumed that the x_j points are distinct; hence $D > 0$ and it follows from Thm 13.7.5 that $E(a,b)$'s critical point is a local minimum. Proof of "claim": Let us merely verify it for $N=2$ and $N=3$ and leave it for you to generalize to $N \geqslant 4$.

$N=2$: $N\sum x_j^2 - (\sum x_j)^2 = 0$ becomes $2x_1^2 + 2x_2^2 - (x_1^2 + 2x_1x_2 + x_2^2) = 0$

or, $x_1^2 - 2x_1x_2 + x_2^2 = 0$

or, $(x_1 - x_2)^2 = 0$, so for $N=2$ $D=0$ implies that $x_1 = x_2$.

$N=3$: $N\sum x_j^2 - (\sum x_j)^2 = 0$ becomes

$$3x_1^2 + 3x_2^2 + 3x_3^2 - (x_1^2 + x_2^2 + x_3^2 + 2x_1x_3 + 2x_1x_2 + 2x_2x_3) = 0$$

or, $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_3 - 2x_1x_2 - 2x_2x_3 = 0$

or, $(x_1^2 - 2x_1x_2 + x_2^2) + (x_1^2 - 2x_1x_3 + x_3^2) + (x_2^2 - 2x_2x_3 + x_3)^2 = 0$

or, $(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 = 0$, so for $N=3$ $D=0$ implies that $x_1 = x_2 = x_3$.

This result, that $D \neq 0$, also shows that the denominators in (12.2) and (12.3) are nonzero so that those results are meaningful.

(c) To see that this local minimum of E is actually an absolute minimum we merely need to observe that $E(a,b)$ given by (12.1) is a quadratic function of a, b, so a local min is also an absolute min.

14. (a) $f(x,y) = \alpha(2\pi x^2 + 2\pi xy) = \text{min}$

$\pi x^2 y = V$

The constraint gives $y = V/\pi x^2$ so $f(x,y) = \alpha(2\pi x^2 + \frac{2\pi x V}{\pi x^2}) \equiv F(x)$. Then $F'(x) = 2\alpha(2\pi x - \frac{V}{x^2}) = 0$, so $x = \sqrt[3]{V/2\pi}$ and then $y = V/\pi x^2 = 2\sqrt[3]{V/2\pi}$.

To verify that these dimensions give a minimum, compute

$$F''(x) = 2\alpha\left(2\pi + \frac{2V}{x^3}\right) = 2\alpha\left(2\pi + \frac{2V}{V/2\pi}\right) = 12\pi\alpha > 0. \checkmark$$

(b) The two terms in (14.3a) are supposed to be "penalties", and the idea is to minimize the penalty. We want the penalty associated with the deviation from $y=kx$ to increase as y increases beyond kx or decreases below kx; $y-kx$ does increase " " " " " but it becomes a "negative penalty" as y decreases below kx whereas it should be a positive penalty, just as if y were greater than kx. Simple ways to fix that problem are to use $(y-kx)^2$ or $|y-kx|$ or $(y-kx)^4$, and so on. Of these, the most convenient and natural one to use is $(y-kx)^2$.

15. (a) $f(x_1,x_2) = x_1^2 + x_2^2 = \text{ext}$

$g(x_1,x_2) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = c$

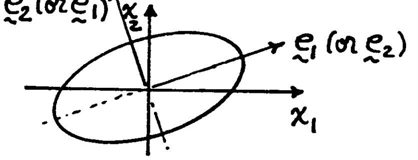
Lagrange: Form $f^* = f - \lambda g$ and extremize subject to no constraints.

Thus, set
$$f^*_{x_1} = f_{x_1} - \lambda g_{x_1} = 2x_1 - \lambda(2a_{11}x_1 + 2a_{12}x_2) = 0$$
$$f^*_{x_2} = f_{x_2} - \lambda g_{x_2} = 2x_2 - \lambda(2a_{22}x_2 + 2a_{12}x_1) = 0$$

or, in matrix form,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \Lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$ where Λ is an eigenvalue.

(b)



(c) $a_{11}=1,\ a_{22}=2,\ a_{12}=1/2$ so $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, which gives

$$\lambda_1 = (3+\sqrt{2})/2 \approx 2.207, \quad \underset{\sim}{e}_1 \approx (0.414, 1)^T$$
$$\Lambda_2 = (3-\sqrt{2})/2 \approx 0.793, \quad \underset{\sim}{e}_2 \approx (-2.414, 1)^T$$

The first eigenvector gives $x_1 = 0.414\alpha$, $x_2 = \alpha$ and to find α we put these into the constraint equation $g(x_1,x_2)=c$:

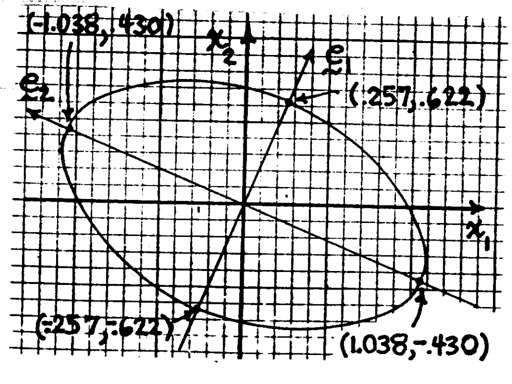
$$x_1^2 + 2x_2^2 + x_1x_2 = 0,$$
$$0.171\alpha^2 + 2\alpha^2 + 0.414\alpha^2 = 1, \text{ so } \underline{\alpha = \pm 0.622}$$

and $x_1 = \pm 0.414(0.622) = \pm 0.257$, $x_2 = 1(\pm 0.622) = \pm 0.622$.

The second eigenvector gives $x_1 = -2.414\beta$, $x_2 = \beta$ so the constraint equation gives

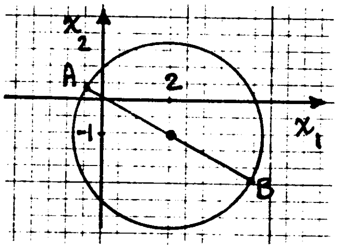
$$5.827\beta^2 + 2\beta^2 - 2.414\beta^2 = 1, \text{ so } \beta = \pm .430$$

and $x_1 = -2.414(\pm 0.430) = \mp 1.038$, $x_2 = \pm 0.430$



(e) NOTE: The equation $x_1^2+x_2^2-4x_1+2x_2=2$ is not of the form $a_{11}x_1^2+a_{22}x_2^2+2a_{12}x_1x_2=$ constant. Nevertheless, we can solve as usual by elimination or Lagrange multipliers. Actually, it is simplest to solve by "inspection" as follows. First, complete the squares,

$$(x_1^2-4x_1+4)+(x_2^2+2x_2+1)=2+4+1$$
$$(x_1-2)^2+(x_2+1)^2=7$$



which is a circle of radius $\sqrt{7}$, centered at $(2,-1)$. Thus, the nearest and farthest points (from the origin) on the circle are on a line through the origin and the center of the circle, $(2,-1)$, namely, the points A,B shown in the figure. Since that line is $x_2=-\frac{1}{2}x_1$ we have

$$x_1^2+(-\tfrac{1}{2}x_1)^2-4x_1+2(-\tfrac{1}{2}x_1)=2$$
$$5x_1^2-20x_1-8=0$$
$$x_1=\frac{20\pm\sqrt{400+160}}{10}=4.366,\,-0.366$$
and $x_2=-\frac{1}{2}x_1=-2.183,\ 0.183$

so nearest $=A=(-0.366, 0.183)$, farthest $=B=(4.366,-2.183)$

Alternatively, let us solve using a Lagrange multiplier:

$$f=x_1^2+x_2^2=\text{extremum}$$
$$g=x_1^2+x_2^2-4x_1+2x_2=2. \qquad ①$$

Form $f^*=f-\lambda g=x_1^2+x_2^2-\lambda(x_1^2+x_2^2-4x_1+2x_2)$

and set $f^*_{x_1}=2x_1-\lambda(2x_1-4)=0 \longrightarrow x_1=2\lambda/(\lambda-1)$ ②

$f^*_{x_2}=2x_2-\lambda(2x_2+2)=0 \longrightarrow x_2=\lambda/(1-\lambda)$ ③

Putting ② and ③ into ① gives $7\lambda^2-14\lambda+2=0$ so $\lambda=1.845, 0.1548$. Finally, putting these λ's into ② and ③ give the same points A and B that we found above.

16. (b) $f=(x-1)^2+(y+2)^2=\min$,

$g=x+y=-5$.

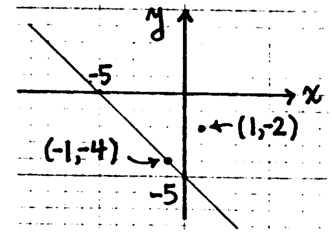
Using elimination, $y=-x-5$ so $f(x,y)=(x-1)^2+(-x-3)^2=2x^2+4x+10\equiv F(x)$ and $F'(x)=4x+4=0$ gives $x=-1$, $y=-x-5=-4$: $(x,y)=(-1,-4)$

Using Lagrange multipliers, instead,

set $f^*=f-\lambda g=(x-1)^2+(y+2)^2-\lambda(x+y)=\min$,

$f^*_x=2(x-1)-\lambda=0 \rightarrow x=1+\lambda/2$

$f^*_y=2(y+2)-\lambda=0 \rightarrow y=(\lambda-4)/2$



and putting these into the constraint equation $x+y=-5$ gives

$$1+\frac{\lambda}{2}+\frac{\lambda}{2}-2=-5$$

so $\lambda=-4$. Hence, $x=1+(-4)/2=-1$ and $y=(-4-4)/2=-4$, as above.

(c) $f=(x-1)^2+(y+2)^2=\min$,

$g = x-3y=2$

Using elimination, $x=3y+2$ so $f(x,y)=(3y+1)^2+(y+2)^2=10y^2+10y+5$ $\equiv F(y)$ and $F'(y)=20y+10=0$ gives $y=-1/2$, $x=3(-1/2)+2=1/2$: $(x,y)=(\frac{1}{2},-\frac{1}{2})$

17. (b) $f=(x-2)^2+y^2+(z+1)^2=\min$,

$g=x-y+2z=4$

Using elimination, $x=y-2z+4$ so

$f=(y-2z+2)^2+y^2+(z+1)^2=2y^2+5z^2-4yz+4y-6z+5.$

Since, with x eliminated, y and z are independent variables, set

$f_y=4y-4z+4=0$

$f_z=10z-4y-6=0.$

These give $z=1/3, y=-2/3$ and then $g=x-y+2z=4$ gives $x=8/3$.

(c) $f=(x-2)^2+y^2+(z+1)^2=\min$

$g=2x-y+z=3$

Using elimination, $y=2x+z-3$ so

$f=(x-2)^2+(2x+z-3)^2+(z+1)^2$

Then $f_x=2(x-2)+2(2)(2x+z-3)=0$

$f_z=2(2x+z-3)+2(z+1)=0$

These give $x=2, z=-1$, and then $g=2x-y+z=3$ gives $y=0$.

20. $f=z=\max$ ①

$g=x+y+z=1$ ②

$h=z^2/(xy^3)=3$ ③

Solve ③ for x and put that expression into ②:

$$\frac{z^2}{3y^3}+y+z=1$$

Taking d/dy of the latter gives $z'(y)=\dfrac{9y^2(1-z)-12y^3}{2z+3y^3}=0$ (to maximize z)

so $z=1-\frac{4}{3}y$. Putting that and $x=z^2/(3y^3)$ into ② gives

$$\frac{(1-\frac{4}{3}y)^2}{3y^3}+y+(1-\tfrac{4}{3}y)=1 \quad \text{or} \quad y^2=\pm(1-\tfrac{4}{3}y).$$

$y^2+\frac{4}{3}y-1=0$ gives $y=(-2\pm\sqrt{13})/3$

$y^2-\frac{4}{3}y+1=0$ " complex roots, which we disallow.

Finally, $z=1-\frac{4}{3}y=1-\frac{4}{3}\left(\frac{-2\pm\sqrt{13}}{3}\right)$ so $z_{max}=1-\frac{4}{3}\left(\frac{-2-\sqrt{13}}{3}\right)=\frac{17+4\sqrt{13}}{9}$.

21. Actually, Exercise 20 involved two constraints, $g=1$ and $h=3$. We were able to solve by elimination. In this exercise we show how to solve such problems by the method of Lagrange multipliers.

(a) $f(x,y,z) = \text{ext.}$
$g_1(\ \text{"}\ ) = c_1$
$g_2(\ \text{"}\ ) = c_2$

Taking differentials,
$$df = f_x dx + f_y dy + f_z dz = 0 \quad (\text{since } f = \text{extremum})$$
$$dg_1 = g_{1x} dx + g_{1y} dy + g_{1z} dz = 0 \quad (\text{since } g_1 = \text{const.})$$
$$dg_2 = g_{2x} dx + g_{2y} dy + g_{2z} dz = 0 \quad (\ \text{"} \quad g_2 = \ \text{"}\ )$$
from which it follows that

$$df - \lambda_1 dg_1 - \lambda_2 dg_2 = (f_x - \lambda_1 g_{1x} - \lambda_2 g_{2x})dx + (f_y - \lambda_1 g_{1y} - \lambda_2 g_{2y})dy + (f_z - \lambda_1 g_{1z} - \lambda_2 g_{2z})dz = 0.$$

We cannot argue that the coefficients of dx, dy, dz must be 0 because x, y, z are independent variables (and hence dx, dy, dz are independent increments), because they are not. Nor can we argue that if we choose λ_1 and λ_2 such that $f_z - \lambda_1 g_{1z} - \lambda_2 g_{2z} = 0$ (at the point of extremum) then the coefficients of dx and dy can be set equal to 0 because dx, dy are arbitrary increments, because they are not — since there are two constraints. Rather, suppose

$$\begin{vmatrix} g_{1y} & g_{2y} \\ g_{1z} & g_{2z} \end{vmatrix} \neq 0$$

so that we can choose λ_1, λ_2 such that
$$g_{1y}\lambda_1 + g_{2y}\lambda_2 = f_y$$
$$g_{1z}\lambda_1 + g_{2z}\lambda_2 = f_z$$
at the ext.[‡] Then

$$df - \lambda_1 dg_1 - \lambda_2 dg_2 = (f_x - \lambda_1 g_{1x} - \lambda_2 g_{2x})dx + 0dy + 0dz = 0$$

and since x — all by itself — is an independent variable (hence dx an independent increment), we can assume that $dx \neq 0$, in which case we must have $f_x - \lambda_1 g_{1x} - \lambda_2 g_{2x} = 0$ at the ext.

The upshot is that if we define
$$f^* = f - \lambda_1 g_1 - \lambda_2 g_2,$$
then the system
$$f^*_x = 0, \ f^*_y = 0, \ f^*_z = 0$$
plus the constraints $g_1 = c_1$, $g_2 = c_2$
provides 5 eqns. on the 5 unknowns $x, y, z, \lambda_1, \lambda_2$.

(b) To solve
$$f = x^2 + y^2 + z^2 = \text{ext}$$
$$g_1 = x^2 + 4y^2 + 4z^2 = 4$$
$$g_2 = x + y + z = 0,$$
set $f^* = f - \lambda_1 g_1 - \lambda_2 g_2 = x^2 + y^2 + z^2 - \lambda_1(x^2 + 4y^2 + 4z^2) - \lambda_2(x + y + z)$ and solve the system

---

[‡] Even if $\begin{vmatrix} g_{1y} & g_{2y} \\ g_{1z} & g_{2z} \end{vmatrix}$ does $= 0$ at the ext., we're still OK if either $\begin{vmatrix} g_{1x} & g_{2x} \\ g_{1z} & g_{2z} \end{vmatrix} \neq 0$ or $\begin{vmatrix} g_{1y} & g_{2y} \\ g_{1x} & g_{2x} \end{vmatrix} \neq 0$ at the ext.

$$\left.\begin{aligned} f^*_x &= 2x-2\lambda_1 x-\lambda_2=0\\ f^*_y &= 2y-8\lambda_1 y-\lambda_2=0\\ f^*_z &= 2z-8\lambda_1 z-\lambda_2=0\\ g_1 &= x^2+4y^2+4z^2=4\\ g_2 &= x+y+z=0 \end{aligned}\right\}$$

Solving, we obtain $\lambda_1=1/2, \lambda_2=\pm 2/\sqrt{3}$; thus, $x=-2/\sqrt{3}, y=z=1/\sqrt{3}$ and $x=+2/\sqrt{3}, y=z=-1/\sqrt{3}$

(c) $f^*=(x+2z)-\lambda_1(x^2+y^2+z^2)-\lambda_2(y-z)$

$$\left.\begin{aligned} f^*_x &= 1-2\lambda_1 x=0\\ f^*_y &= -2\lambda_1 y-\lambda_2=0\\ f^*_z &= 2-2\lambda_1 z+\lambda_2=0\\ g_1 &= x^2+y^2+z^2=1\\ g_2 &= y-z=0 \end{aligned}\right\}$$

Solving, $x=y=z=+1/\sqrt{3}$ and $x=y=z=-1/\sqrt{3}$

Alternatively, let us use elimination: $z=y$, $x^2+2y^2=1$, $x=\sqrt{1-2y^2}$ so $f=\sqrt{1-2y^2}+2y=$ ext, $f'=\frac{1}{2}(-4y)(1-2y^2)^{-1/2}+2=0$, $3y^2=1$, $y=\pm 1/\sqrt{3}$ (and then $z=\pm 1/\sqrt{3}$, $x=\pm 1/\sqrt{3}$). Is the extremum a max, min, or saddle?

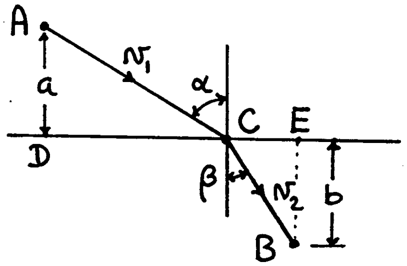
$f''=-\frac{2}{\sqrt{1-2y^2}}-\frac{4y^2}{(1-2y^2)^{3/2}}=-\frac{2(1-2y^2)+4y^2}{(1-2y^2)^{3/2}}=-\frac{2}{(1-2y^2)^{3/2}}<0$, hence a maximum.

(d) $f^*=x+y+z-\lambda_1(x^2+y^2+z^2)-\lambda_2(x+y)$

$$\left.\begin{aligned} f^*_x &= 1-2\lambda_1 x-\lambda_2=0\\ f^*_y &= 1-2\lambda_1 y-\lambda_2=0\\ f^*_z &= 1-2\lambda_1 z=0\\ g_1 &= x^2+y^2+z^2=1\\ g_2 &= x+y=0 \end{aligned}\right\}$$

gives $x=y=0$ and $z=\pm 1$

22. (a) Time from A to C is $=\frac{AC}{N_1}=\frac{a}{N_1\cos\alpha}$



Time from C to B is $=\frac{CB}{N_2}=\frac{b}{N_2\cos\beta}$

so $T=\frac{a}{N_1\cos\alpha}+\frac{b}{N_2\cos\beta}=\min$

As C moves to left and right, seeking to minimize T, DC+CE remains constant:

$$g=DC+CE=a\tan\alpha+b\tan\beta=\text{const.}$$

Proceeding,

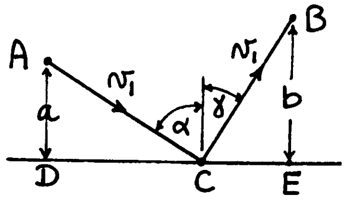
$$dT=\frac{a}{N_1}\frac{\sin\alpha}{\cos^2\alpha}d\alpha+\frac{b}{N_2}\frac{\sin\beta}{\cos^2\beta}d\beta=0$$

$$dg=\frac{a}{\cos^2\alpha}d\alpha+\frac{b}{\cos^2\beta}d\beta=0$$

Since $d\alpha, d\beta$ are not necessarily both zero, it follows that

$$\begin{vmatrix} \frac{a}{v_1}\frac{\sin\alpha}{\cos^2\alpha} & \frac{b}{v_2}\frac{\sin\beta}{\cos^2\beta} \\ \frac{a}{\cos^2\alpha} & \frac{b}{\cos^2\beta} \end{vmatrix} = \frac{ab\sin\alpha}{v_1\cos^2\alpha\cos^2\beta} - \frac{ab\sin\beta}{v_2\cos^2\alpha\cos^2\beta} = 0, \text{ so } \boxed{\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}.}$$

(b) $T(\alpha,\gamma) = \frac{AC}{v_1} + \frac{CB}{v_1} = \frac{a}{v_1\cos\alpha} + \frac{b}{v_1\cos\gamma} = \text{min}$



$g(\alpha,\gamma) = DC + CE = a\tan\alpha + b\tan\gamma = \text{cnst}$, which equations are the same as in (a), but with $v_2 \to v_1$ and $\beta \to \delta$. Thus, change $v_2 \to v_1$ and $\beta \to \gamma$ in the law of refraction and obtain

$$\frac{\sin\alpha}{\sin\gamma} = \frac{v_1}{v_1} \text{ so } \sin\alpha = \sin\gamma \text{ so } \boxed{\alpha=\gamma.}$$

Section 13.8

1. (b) $\frac{d}{dt}\int_3^t x^t \sin x\,dx = \frac{d}{dt}\int_3^t e^{t\ln x}\sin x\,dx = \int_3^t (\ln x)e^{t\ln x}\sin x\,dx + \frac{d(t)}{dt}(t^t\sin t)$

$$= \int_3^t x^t \ln x \sin x\,dx + t^t \sin t$$

(c) $\frac{d}{d\alpha}\int_\alpha^2 e^{x^2}dx = -\frac{d(\alpha)}{d\alpha}e^{\alpha^2} = -e^{\alpha^2}$

(e) $\frac{d}{dx}\int_x^{2x}\ln(u^2+x^2)du = \int_x^{2x}\frac{2x}{u^2+x^2}du + (2)\ln(4x^2+x^2) - (1)\ln(x^2+x^2)$

$$= \int_x^{2x}\frac{2x}{u^2+x^2}du + 2\ln x + \ln\frac{25}{2}$$

$$\frac{d^2}{dx^2}\int_x^{2x}\ln(u^2+x^2)du = \int_x^{2x}\left(\frac{2}{u^2+x^2} - \frac{4x^2}{(u^2+x^2)^2}\right)du + (2)\frac{2x}{4x^2+x^2} - (1)\frac{2x}{x^2+x^2} + \frac{2}{x}$$

$$= \int_x^{2x}\frac{2(u^2-x^2)}{(u^2+x^2)^2}du + \frac{9}{5x}$$

2. (b) $f(x) = \int_{-x}^{\cos x} dt/(t^3+1)$, $f'(x) = (-\sin x)(1+\cos^3 x)^{-1} - (-1)(1-x^3)^{-1}$,

$$f''(x) = -\cos x(1+\cos^3 x)^{-1} - \sin x(-1)(3)(-\sin x)\cos^2 x(1+\cos^3 x)^{-2} + (-1)(-3x^2)(1-x^3)^{-2}$$

so $f(0) = \int_0^1 dt/(t^3+1)$, $f'(0) = 1$, $f''(0) = -1/2$,

so $f(x) = \left(\int_0^1 dt/(t^3+1)\right) + x - \frac{1}{4}x^2 + \cdots$

NOTE: The Maple command int(1/(t^3+1), t=0..1); gives

$$\int_0^1 dt/(t^3+1) = \frac{\ln 2}{3} + \frac{\sqrt{3}\pi}{9}.$$

(c) $f(x) = \int_0^{2\sin x}\ln(t^2+1)dt$, $f'(x) = (2\cos x)\ln(4\sin^2 x+1)$,

$$f''(x) = -2\sin x\ln(4\sin^2 x+1) + 2\cos x\frac{8\cos x\sin x+1}{4\sin^2 x+1}$$

so $f(0)=0, f'(0)=0, f''(0)=2$, so $f(x) = 0 + 0x + \frac{2}{2!}x^2 + \cdots$

4. Consider $J(a) = \int_0^\infty x^2 e^{-ax^2}\,dx \overset{ax^2=t^2}{\overset{\downarrow}{=}} \int_0^\infty \frac{t^2}{a} e^{-t^2} \frac{dt}{\sqrt{a}} = \frac{\sqrt{\pi}}{4a^{3/2}}$ from (20)

$$J'(a) \underset{\overset{\uparrow}{\text{Leibniz}}}{=} \int_0^\infty -x^4 e^{-ax^2}\,dx = \frac{d}{da}\left(\frac{\sqrt{\pi}}{4a^{3/2}}\right) = \frac{\sqrt{\pi}}{4}\left(-\frac{3}{2}\right)a^{-5/2}$$

Now set $a=1$: $\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8}$

5. (a) $J(a) = \int_0^\infty \frac{x^a}{1+x^3}\,dx = \frac{\pi}{3\sin\left(\frac{a+1}{3}\pi\right)}$

$$J'(a) = \int_0^\infty \frac{x^a \ln x}{1+x^3}\,dx = -\frac{\pi^2}{9}\frac{\cos\left(\frac{a+1}{3}\pi\right)}{\sin^2\left(\frac{a+1}{3}\pi\right)}$$

$$J''(a) = \int_0^\infty \frac{x^a(\ln x)^2}{1+x^3}\,dx = -\frac{\pi^2}{9}\left\{\frac{\frac{\pi}{3}\left[-\sin\left(\frac{a+1}{3}\pi\right)\right]}{\sin^2\left(\frac{a+1}{3}\pi\right)} + \frac{\cos\left(\frac{a+1}{3}\pi\right)(-2)\frac{\pi}{3}\cos\left(\frac{a+1}{3}\pi\right)}{\sin^3\left(\frac{a+1}{3}\pi\right)}\right\}$$

Now set $a=0$: $\int_0^\infty \frac{(\ln x)^2}{1+x^3}\,dx = \frac{10\pi^3}{81\sqrt{3}}$

(b) From second line of (a), with $a=1$, $\int_0^\infty \frac{x\ln x}{1+x^3}\,dx = -\frac{\pi^2}{9}\frac{(-1/2)}{(3/4)} = \frac{2\pi^2}{27}$

6. (a) $y(x) = \frac{1}{6}\int_a^x (x-t)^3 f(t)\,dt$, $\quad y(a) = \frac{1}{6}\int_a^a \cdots = 0$ ✓

$y'(x) = \frac{1}{6}\int_a^x 3(x-t)^2 f(t)\,dt + (1)\cancel{\frac{1}{6}(x-x)^3 f(x)}$, $\quad y'(a) = \frac{1}{6}\int_a^a \cdots = 0$ ✓

$y''(x) = \frac{1}{2}\int_a^x 2(x-t)f(t)\,dt + \frac{1}{2}\cancel{(x-x)^2 f(x)}$, $\quad y''(a) = \frac{1}{2}\int_{-a}^a \cdots = 0$ ✓

$y'''(x) = \int_a^x f(t)\,dt + \cancel{(x-x)}f(x)$, $\quad y'''(a) = \int_{-a}^a \cdots = 0$ ✓

$y''''(x) = f(x)$ ✓

(b) $xy(x) = \int_a^x (x-t)f(t)\,dt$, $\quad y(a) = \int_a^a \cdots = 0$ ✓

$(xy)' = \int_a^x f(t)\,dt + (1)\cancel{(x-x)}f(x)$, $\quad y'(a) = \int_a^a \cdots = 0$ ✓

$(xy)'' = f(x)$ ✓

(c) $y(x) = e^x + \int_0^x t^2\cosh(x-t)\,dt$, $\quad y(0) = e^0 + \int_0^0 \cdots = 1$ ✓

$y'(x) = e^x + \int_0^x t^2\sinh(x-t)\,dt + x^2\cosh(x-x)$, $\quad y'(0) = e^0 + 0 = 1$ ✓

$y''(x) = e^x + \int_0^x t^2\cosh(x-t)\,dt + x^2\cancel{\sinh(x-x)} + 2x$

$= y(x) + 2x$ ✓

7. $J(a) = \int_0^1 x^a\,dx = \frac{1}{a+1}$ gives $J'(a) = \int_0^1 x^a \ln x\,dx = -\frac{1}{(a+1)^2}$
and $a = 0.7$ gives $\int_0^1 x^{0.7}\ln x\,dx = -1/2.89 \approx -0.34602$

8. $I(a) = \underset{\uparrow}{\int_0^1} \frac{x^a - 1}{\ln x}\,dx$, $\quad I'(a) = \int_0^1 \frac{x^a \cancel{\ln x}}{\cancel{\ln x}}\,dx = \frac{1}{a+1}$, $\quad I(a) = \ln(a+1) + C$

We can see that $I(0) = 0$, so $I(0) = 0 = \ln 1 + C$ gives $C = 0$. Hence, $I(a) = \ln(a+1)$
so
$$\int_0^1 \frac{x^3-1}{\ln x}\,dx = I(3) = \ln 4$$

11. (b) $y(x,t) = \frac{F(x-ct)+F(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\alpha)\,d\alpha$, $\quad y(x,0) = F(x) + \frac{1}{2c}\int_x^x = F(x)$ ✓

$y_x = [F'(x-ct) + F'(x+ct)]/2 + \frac{1}{2c}[G(x+ct) - G(x-ct)]$

NOTE: $F'(\text{arg})$ means $dF/d\text{arg}$. For ex., $F'(x-ct) = dF(x-ct)/d(x-ct)$

$y_{xx} = [F''(x-ct) + F''(x+ct)]/2 + [G'(x+ct) - G'(x-ct)]/2c$

$y_t = [-cF'(x-ct) + cF'(x+ct)]/2 + c[G(x+ct) + G(x-ct)]/2c, \quad y(x,0) = G(x)$ ✓

$y_{tt} = [-c(-c)F''(x-ct) + c(c)F''(x+ct)]/2 + [c^2G'(x+ct) - c^2G'(x-ct)]/2c$

and we see that $y_{tt} = c^2 y_{xx}$. ✓

(c) $u(x,y) = \frac{y}{\pi}\int_{-1}^{1}\frac{f(\xi)}{(\xi-x)^2+y^2}\,d\xi = \frac{y}{\pi}\int_{-1}^{1}\frac{f(\xi)}{(x-\xi)^2+y^2}\,d\xi$

$u_x = \frac{y}{\pi}\int_{-1}^{1}\frac{(-1)(2)(x-\xi)f(\xi)}{[(x-\xi)^2+y^2]^2}\,d\xi, \quad u_{xx} = -\frac{2y}{\pi}\int_{-1}^{1}\frac{f(\xi)}{[\;\;]^2}\,d\xi + \frac{y}{\pi}\int_{-1}^{1}\frac{4(2)(x-\xi)^2 f(\xi)}{[\;\;]^3}\,d\xi$

$u_y = \frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi)}{[\;\;]}\,d\xi + \frac{y}{\pi}\int_{-1}^{1}\frac{(-1)2y}{[\;\;]^2}f(\xi)\,d\xi$

$u_{yy} = \frac{1}{\pi}\int_{-1}^{1}\frac{(-1)2y}{[\;\;]^2}f(\xi)\,d\xi - \frac{4y}{\pi}\int_{-1}^{1}\frac{f(\xi)}{[\;\;]^2}\,d\xi - \frac{2y^2}{\pi}\int_{-1}^{1}\frac{(-2)(2y)}{[\;\;]^3}f(\xi)\,d\xi$

so $u_{xx} + u_{yy} = -\cancel{\frac{8y}{\pi}\int_{-1}^{1}\frac{f(\xi)}{[\;\;]^2}}\,d\xi + \frac{8y}{\pi}\cancel{\int_{-1}^{1}\frac{(x-\xi)^2+y^2}{[\;\;]^3}}f(\xi)\,d\xi = 0$ ✓

12.(a) $u(x,y) = \int_0^1\int_0^1 \ln[(x-\xi)^2+(y-\eta)^2]\,f(\xi,\eta)\,d\xi\,d\eta$

$u_x = \int_0^1\int_0^1 \frac{2(x-\xi)f(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]}\,d\xi\,d\eta, \quad u_y = \int_0^1\int_0^1\frac{2(y-\eta)f(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]}\,d\xi\,d\eta$

$u_{xx} = \int_0^1\int_0^1\frac{2f(\xi,\eta)}{[\;\;]}\,d\xi\,d\eta + \int_0^1\int_0^1\frac{2(-1)(2)(x-\xi)^2 f(\xi,\eta)}{[\;\;]^2}\,d\xi\,d\eta$

$u_{yy} = \int_0^1\int_0^1\frac{2f(\xi,\eta)}{[\;\;]}\,d\xi\,d\eta + \int_0^1\int_0^1\frac{2(-1)(2)(y-\eta)^2 f(\xi,\eta)}{[\;\;]^2}\,d\xi\,d\eta$

so $u_{xx} + u_{yy} = 4\int_0^1\int_0^1\cancel{\frac{f(\xi,\eta)}{[\;\;]}}\,d\xi\,d\eta - 4\int_0^1\int_0^1\cancel{\frac{(x-\xi)^2+(y-\eta)^2}{[\;\;]^2}}f(\xi,\eta)\,d\xi\,d\eta = 0.$ ✓

CHAPTER 14

Section 14.2

1. $\underset{\sim}{u} \times \underset{\sim}{v}$ is perpendicular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$, so it cannot equal $2\underset{\sim}{u}$ — unless $\underset{\sim}{u} = \underset{\sim}{0}$. We conclude that $\underset{\sim}{u} = \underset{\sim}{0}$, but $\underset{\sim}{v}$ is in no way restricted.

2. $\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{0}$ implies that $\underset{\sim}{u}$ and/or $\underset{\sim}{v}$ is $\underset{\sim}{0}$ or $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are collinear and nonzero. The latter is not consistent with $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ but the former is. We conclude that $\underset{\sim}{u}$ and/or $\underset{\sim}{v} = \underset{\sim}{0}$.

3. $\underset{\sim}{c} \cdot \underset{\sim}{c} = (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} - 2\|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos\theta$
   $\|\underset{\sim}{c}\|^2 = \|\underset{\sim}{a}\|^2 + \|\underset{\sim}{b}\|^2 - 2\|a\|\|b\| \cos\theta$ or, simply, $c^2 = a^2 + b^2 - 2ab\cos\theta$.

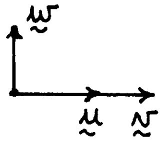
4. $\underset{\sim}{c} \times (\underset{\sim}{b} - \underset{\sim}{c}) = \underset{\sim}{c} \times \underset{\sim}{a}$, so $\underset{\sim}{c} \times \underset{\sim}{b} = \underset{\sim}{c} \times \underset{\sim}{a}$, $cb \sin\alpha \, \underset{\sim}{\hat{e}} = ca \sin(\pi - \beta) \underset{\sim}{\hat{e}}$ where $\underset{\sim}{\hat{e}}$ is out of the paper, so $b \sin\alpha = a \sin(\pi - \beta)$
   $= a \sin\beta$, so $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

5. $\underset{\sim}{u} \times \underset{\sim}{v} = \|\underset{\sim}{u}\|\|\underset{\sim}{v}\| \sin\theta \, \underset{\sim}{\hat{e}}$ so $\|\underset{\sim}{u} \times \underset{\sim}{v}\|^2 = \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 \sin^2\theta$
   $= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 (1 - \cos^2\theta)$
   $= \|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2$

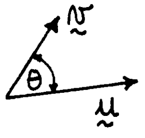
6. The RHS of (3c) can be interpreted in any of these ways: $\underset{\sim}{u} \cdot (\underset{\sim}{w} + \underset{\sim}{v} \cdot \underset{\sim}{w})$, $\underset{\sim}{u} \cdot (\underset{\sim}{w} + \underset{\sim}{v}) \cdot \underset{\sim}{w}$, $(\underset{\sim}{u} \cdot \underset{\sim}{w} + \underset{\sim}{v}) \cdot \underset{\sim}{w}$, $(\underset{\sim}{u} \cdot \underset{\sim}{w}) + (\underset{\sim}{v} \cdot \underset{\sim}{w})$. However, only the last of these four interpretations is meaningful [in the first, for instance, $\underset{\sim}{w} + \underset{\sim}{v} \cdot \underset{\sim}{w}$ is a vector plus a scalar, which is not defined] so it can safely be assumed that the interpretation $(\underset{\sim}{u} \cdot \underset{\sim}{w}) + (\underset{\sim}{v} \cdot \underset{\sim}{w})$ is intended even if the parentheses are omitted. In the RHS of (8c), however, there are four meaningful interpretations of $\underset{\sim}{u} \times \underset{\sim}{w} + \underset{\sim}{v} \times \underset{\sim}{w}$, so parentheses are needed — to indicate which interpretation is intended.

7. Yes; $\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{w} \Rightarrow \underset{\sim}{u} \cdot \underset{\sim}{w} = 0$ and $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$, and we are given that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$.

8. No. For example, if $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ are in the plane of the paper and are as shown at the right, then $(\underset{\sim}{u} \times \underset{\sim}{v}) \times \underset{\sim}{w} = \underset{\sim}{0} \times \underset{\sim}{w} = \underset{\sim}{0}$ whereas $\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w})$ is nonzero (and directed this way: ↓).



9. $(\alpha \underset{\sim}{u}) \times \underset{\sim}{v} = \|\alpha \underset{\sim}{u}\| \|\underset{\sim}{v}\| \sin\theta \, \underset{\sim}{\hat{e}}$ ($\hat{e}$ normal to paper, toward viewer)
   $= \alpha \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \sin\theta \, \underset{\sim}{\hat{e}}$



if $\alpha>0$, and
$$\begin{aligned}(\alpha\underset{\sim}{u})\times\underset{\sim}{v} &= \|\alpha\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\sin(\pi-\theta)(-\hat{\underset{\sim}{e}})\\ &= |\alpha|\,\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\sin\theta\,(-\hat{\underset{\sim}{e}})\\ &= \alpha\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\sin\theta\,\hat{\underset{\sim}{e}}\end{aligned}$$
if $\alpha<0$. Of course, if $\alpha=0$ then (8b) is simply $\underset{\sim}{0}=\underset{\sim}{0}$. Thus, $(\alpha\underset{\sim}{u})\times\underset{\sim}{v}=\alpha(\underset{\sim}{u}\times\underset{\sim}{v})$ holds in all cases.

10. (a) It holds according to (4) if $n=2$. Next, assume it holds for $n=k$. Then
$$\begin{aligned}(\overbrace{\alpha_1\underset{\sim}{u}_1+\cdots+\alpha_k\underset{\sim}{u}_k}^{\underset{\sim}{w}}+\alpha_{k+1}\underset{\sim}{u}_{k+1})\cdot\underset{\sim}{v} &= (1\underset{\sim}{w}+\alpha_{k+1}\underset{\sim}{u}_{k+1})\cdot\underset{\sim}{v}\\ &= 1(\underset{\sim}{w}\cdot\underset{\sim}{v})+\alpha_{k+1}(\underset{\sim}{u}_{k+1}\cdot\underset{\sim}{v}) \quad \text{per (4)}\\ &= \alpha_1(\underset{\sim}{u}_1\cdot\underset{\sim}{v})+\cdots+\alpha_k(\underset{\sim}{u}_k\cdot\underset{\sim}{v})+\alpha_{k+1}(\underset{\sim}{u}_{k+1}\cdot\underset{\sim}{v})\end{aligned}$$
by assumption, so it holds for $n=k+1$. Thus, by induction, it holds for all $n=2,3,\dots$.

(b) First, observe that (8b) and (8c) are equivalent to
$$(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v})\times\underset{\sim}{w}=\alpha(\underset{\sim}{u}\times\underset{\sim}{w})+\beta(\underset{\sim}{v}\times\underset{\sim}{w}). \quad \text{(i.e., linearity)}$$
From here, proceed as in part (a).

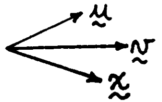
11. If $\underset{\sim}{u},\underset{\sim}{v}$ are LD then we can express $\underset{\sim}{v}$ as some scalar multiple of $\underset{\sim}{u}$ and/or $\underset{\sim}{u}$ as some scalar multiple of $\underset{\sim}{v}$. Suppose the former: $\underset{\sim}{v}=\alpha\underset{\sim}{u}$. Then $\underset{\sim}{u}\times\underset{\sim}{v}=\underset{\sim}{u}\times(\alpha\underset{\sim}{u})=\alpha(\underset{\sim}{u}\times\underset{\sim}{u})=\alpha\underset{\sim}{0}=\underset{\sim}{0}$. ✓ Conversely, suppose $\underset{\sim}{u}\times\underset{\sim}{v}=\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\sin\theta\,\hat{\underset{\sim}{e}}=\underset{\sim}{0}$. Then $\|\underset{\sim}{u}\|=0$ and/or $\|\underset{\sim}{v}\|=0$ and/or $\theta=0,\pi$. If $\|\underset{\sim}{u}\|=0$ then $\underset{\sim}{u}=\underset{\sim}{0}$ so $\underset{\sim}{u},\underset{\sim}{v}$ are LD. Likewise if $\|\underset{\sim}{v}\|=0$. If $\theta=0$ or π (and $\underset{\sim}{v}\neq\underset{\sim}{0}$, $\underset{\sim}{u}\neq\underset{\sim}{0}$) then surely $\underset{\sim}{v}$ is a multiple of $\underset{\sim}{u}$ (a positive multiple if $\theta=0$ and a negative multiple if $\theta=\pi$). Thus, $\underset{\sim}{u}\times\underset{\sim}{v}=\underset{\sim}{0}$ implies that $\underset{\sim}{u},\underset{\sim}{v}$ are LD.

12. If $\|\underset{\sim}{u}\|\neq0$ and $\|\underset{\sim}{v}\|\neq0$, then $\underset{\sim}{u}\times\underset{\sim}{v}=\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\sin\theta\,\hat{\underset{\sim}{e}}=0 \Rightarrow \sin\theta=0$ so $\theta=0$ or π.
$$\left.\begin{array}{ll}\theta=0: \;\xrightarrow{\;\underset{\sim}{u}\;\;\underset{\sim}{v}\;} & \text{and } \underset{\sim}{u}=\alpha\underset{\sim}{v} \text{ for some } \alpha>0\\ \theta=\pi: \;\xleftarrow{\;\underset{\sim}{v}\;}\xrightarrow{\;\underset{\sim}{u}\;} & \text{and } \underset{\sim}{u}=\alpha\underset{\sim}{v} \text{ for some } \alpha<0\end{array}\right\}\text{ Either way, } \underset{\sim}{u}=\alpha\underset{\sim}{v}. ✓$$

Section 14.3

1. (b) $$(\underset{\sim}{u}+3\underset{\sim}{w})\times\underset{\sim}{v}=\begin{vmatrix}\hat{\underset{\sim}{\imath}} & \hat{\underset{\sim}{\jmath}} & \hat{\underset{\sim}{k}}\\ 11 & -1 & 3\\ 1 & 1 & -1\end{vmatrix}=-2\hat{\underset{\sim}{\imath}}+14\hat{\underset{\sim}{\jmath}}+12\hat{\underset{\sim}{k}}$$
$$\|(\underset{\sim}{u}+3\underset{\sim}{w})\times\underset{\sim}{v}\|=\sqrt{(-2)^2+(14)^2+(12)^2}=\sqrt{344}=2\sqrt{86}.$$
$$\underset{\sim}{u}\cdot\underset{\sim}{w}=(2)(3)+(-1)(0)+(-3)(2)=0\;,\quad |\underset{\sim}{u}\cdot\underset{\sim}{w}|=|0|=0.$$

(f) Schematically,



Now, $\underset{\sim}{v}\times\underset{\sim}{x}$ is $\perp$ to $\underset{\sim}{v},\underset{\sim}{x}$ plane (if it is not $\underset{\sim}{0}$), so if $\underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{x})=0$ then $\underset{\sim}{u}$ is in the $\underset{\sim}{v},\underset{\sim}{x}$ plane. Let us see:

$$\underset{\sim}{v} \times \underset{\sim}{x} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 8 & -1 & -11 \end{vmatrix} = -12\underset{\sim}{\hat{\imath}} - (-3)\hat{\jmath} + (-9)\underset{\sim}{\hat{k}} \text{ and } \underset{\sim}{u} \cdot \underset{\sim}{v} \times \underset{\sim}{x} = -24 - 3 + 27 = 0 \checkmark$$

Alternatively, we see by inspection that $\underset{\sim}{x} = 3\underset{\sim}{u} + 2\underset{\sim}{v}$ so $\underset{\sim}{x}$ is in the $\underset{\sim}{u}, \underset{\sim}{v}$ plane.

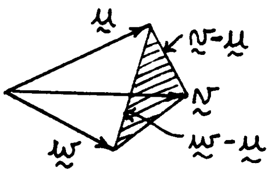
(g) This time, $\underset{\sim}{u} \cdot \underset{\sim}{v} \times \underset{\sim}{w} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 3 & 0 & 2 \end{vmatrix} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) \cdot (2\hat{\imath} - 5\hat{\jmath} - 3\hat{k}) = 18 \neq 0$, so $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ are <u>not</u> coplanar.

(h) $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 4\underset{\sim}{\hat{\imath}} - \hat{\jmath} + 3\underset{\sim}{\hat{k}}$. Then that vector is ⊥ to $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Check: $\underset{\sim}{u} \cdot (4\underset{\sim}{\hat{\imath}} - \hat{\jmath} + 3\underset{\sim}{\hat{k}}) = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) \cdot (4\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 8 + 1 - 9 = 0 \checkmark$

and $\underset{\sim}{v} \cdot (\quad " \quad) = (\hat{\imath} + \hat{\jmath} - \hat{k}) \cdot (\quad " \quad) = 4 - 1 - 3 = 0 \checkmark$

(n)



$(\underset{\sim}{w} - \underset{\sim}{u}) \times (\underset{\sim}{v} - \underset{\sim}{u}) = -8\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$ is ⊥ to the crosshatched triangle, so the head of $\underset{\sim}{x}$ lies in that plane if and only if $(\underset{\sim}{x} - \underset{\sim}{u}) \cdot (-8\underset{\sim}{\hat{\imath}} - 7\hat{\jmath} + 3\underset{\sim}{\hat{k}})$ is 0. Well, it $= -48 + 0 - 24 = -72 \neq 0$, so the four heads do <u>not</u> lie in a common plane.

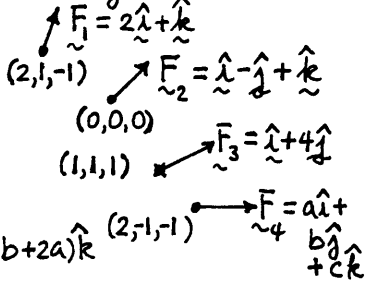
2. (a) $$\underset{\sim}{M} = \Sigma \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix}$$
$$= \hat{\imath} - 4\hat{\jmath} - 2\hat{k} + 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} - 4\hat{\imath} + \hat{\jmath} + 3\hat{k} = -3\hat{\imath} - 3\hat{\jmath} + \hat{k}$$

(b) $$\underset{\sim}{M} = \Sigma \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 0 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 4 & 0 \end{vmatrix}$$
$$= -3\hat{\jmath} - \hat{\imath} + \hat{\jmath} + 2\hat{k} - 4\hat{\imath} + \hat{\jmath} = -5\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

(e) Let $\underset{\sim}{F}_4 = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. Then

$$\underset{\sim}{M} = \Sigma \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$
$$+ \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & -2 \\ a & b & c \end{vmatrix} = \underset{\sim}{0}$$

Schematically,



$$= 0\hat{\imath} - 5\hat{\jmath} + 0\hat{k} - 2\hat{\imath} + 0\hat{\jmath} + 2\hat{k} + 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + (-2c + 2b)\hat{\imath} - (c + 2a)\hat{\jmath} + (b + 2a)\hat{k}$$
$$= (-2 - 2c + 2b)\hat{\imath} + (-5 - c - 2a)\hat{\jmath} + (2 + b + 2a)\hat{k}$$

so $-2 - 2c + 2b = 0$
$-5 - c - 2a = 0$
$2 + b + 2a = 0$, which, by Gauss elimination, is seen to have no solution. How can we understand that result geometrically? Let us denote the vector sum of the moments of $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3$ about (1,1,1) as "M_{123}". Then, we wish to choose $\underset{\sim}{F}_4$ so that

$$\underset{\sim}{r}_4 \times \underset{\sim}{F}_4 + \underset{\sim}{M}_{123} = \underset{\sim}{0} \quad \text{, or, } \quad \underset{\sim}{r}_4 \times \underset{\sim}{F}_4 = -\underset{\sim}{M}_{123} \quad \text{— ♀}$$

Since $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4$ is necessarily ⊥ to $\underset{\sim}{r}_4$, ♀ will have no solution (for $\underset{\sim}{F}_4$) unless $\underset{\sim}{M}_{123}$ happens to be ⊥ to $\underset{\sim}{r}_4$. In this example it is <u>not</u>, since $\underset{\sim}{r}_4 \cdot \underset{\sim}{M}_{123} = (1, -2, -2) \cdot (-2, -5, 2) = 4 \neq 0$. This result is interesting since one would think that since we have the three parameters a, b, c available, and we need to cancel the three components of $\underset{\sim}{M}_{123}$, then we should – typically – be able to find a suitable $\underset{\sim}{F}_4$ and should fail only in an exceptional circumstance. Yet, the opposite is true: typically we are <u>not</u> able to find a suitable $\underset{\sim}{F}_4$; a suitable $\underset{\sim}{F}_4$ can be found <u>only</u> if the point of action of $\underset{\sim}{F}_4$ lies in a plane [through the point (1,1,1)] that is ⊥ to $\underset{\sim}{M}_{123}$.

NOTE: The foregoing might make a good examination question.

(f) Let $\underset{\sim}{F}_4 = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. Then

$$\underset{\sim}{M} = \sum \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 5 & -2 \\ a & b & c \end{vmatrix}$$

$$= (-2+5c+2b)\hat{\imath} + (-5-2c-2a)\hat{\jmath} + (2+2b-5a)\hat{k} = \underset{\sim}{0}$$

so $\left.\begin{aligned} 2b+5c &= 2 \\ 2a \quad +2c &= -5 \\ 5a-2b \quad &= 2 \end{aligned}\right\} \rightarrow$ gives no solution. As discussed in (e), the reason there does not exist a suitable $\underset{\sim}{F}_4$ so that $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4 + \underset{\sim}{M}_{123} = \underset{\sim}{0}$, where "$\underset{\sim}{M}_{123}$" $= \sum_1^3 \underset{\sim}{r}_j \times \underset{\sim}{F}_j$, is that $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4$ is necessarily ⊥ to $\underset{\sim}{r}_4$. Thus, $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4 = -\underset{\sim}{M}_{123}$ will not be solvable for $\underset{\sim}{F}_4$ unless $\underset{\sim}{M}_{123}$ is ⊥ to $\underset{\sim}{r}_4$. Let us see: $\underset{\sim}{M}_{123} \cdot \underset{\sim}{r}_4 = (-2,-5,2)\cdot(2,5,-2) = -33 \neq 0$.

(g) Let $\underset{\sim}{F}_4 = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. Then

$$\underset{\sim}{M} = \sum \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -5 & 2 \\ a & b & c \end{vmatrix}$$

$$= (-2-5c-2b)\hat{\imath} + (-5+2c+2a)\hat{\jmath} + (2-2b+5a)\hat{k} = \underset{\sim}{0}$$

so set $-2-5c-2b=0$

$-5+2c+2a=0$

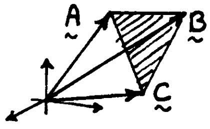
$2-2b+5a=0$. This system has no solution. As discussed in (e), the reason there does not exist a suitable $\underset{\sim}{F}_4$ such that $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4 + \underset{\sim}{M}_{123} = \underset{\sim}{0}$, where "$\underset{\sim}{M}_{123}$" $= \sum_1^3 \underset{\sim}{r}_j \times \underset{\sim}{F}_j$, is that $\underset{\sim}{r}_4 \times \underset{\sim}{F}_4$ is ⊥ to $\underset{\sim}{r}_4$, yet $\underset{\sim}{M}_{123}$ is <u>not</u>, since $\underset{\sim}{M}_{123} \cdot \underset{\sim}{r}_4 = (-2,-5,2)\cdot(-2,-5,2) = 33 \neq 0$.

NOTE: I didn't realize that all three cases (parts (e),(f),(g)) all give the same result. To make up a problem where a suitable $\underset{\sim}{F}_4$ <u>does</u> exist we first find any $\underset{\sim}{r}_4$ that <u>is</u> ⊥ to $\underset{\sim}{M}_{123} = (-2,-5,-2)$, such as $\underset{\sim}{r}_4 = (5,-2,0)$. Thus, the tail of $\underset{\sim}{F}_4$ is at $(1,1,1)+(5,-2,0) = (6,-1,1)$. In that case, with $\underset{\sim}{F}_4 = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ again,

$$\underset{\sim}{M} = \sum \underset{\sim}{r}_j \times \underset{\sim}{F}_j = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -2 & 0 \\ a & b & c \end{vmatrix}$$

$$= (-2-2c)\hat{\imath} + (-5-5c)\hat{\jmath} + (-2+5b+2a)\hat{k} = \underset{\sim}{0}$$

so set $-2-2c=0$, $-5-5c=0$, and $(-2+5b+2a)=0$. Thus, $a=\alpha$, $b=\frac{2}{5}-\frac{2}{5}\alpha$, $c=-1$, where α is arbitrary, so $\underset{\sim}{F}_4 = (\alpha, \frac{2-\alpha}{5}, -1)$. The nonuniqueness will always occur (when we do have existence) since the component of $\underset{\sim}{F}_4$ <u>along</u> $\underset{\sim}{r}_4$ produces no moment and is arbitrary.



3. Given 3 points located by position vectors $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$ it should be evident that they lie on a straight line if and only if the area of the cross-hatched triangle is 0. From (10),

$$\text{area} = \|(\underset{\sim}{B}-\underset{\sim}{A})\times(\underset{\sim}{C}-\underset{\sim}{A})\| \ *$$

(b) $\underset{\sim}{A}=(1,3,0)$, $\underset{\sim}{B}=(2,-1,1)$, $\underset{\sim}{C}=(3,-5,2)$ so area $=\left\|\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ 1 & -4 & 1\\ 2 & -8 & 2\end{vmatrix}\right\| = \|0\hat\imath - 0\hat\jmath + 0\hat k\| = 0$, so the points lie on a straight line.

(d) $\underset{\sim}{A}=(1,0,2)$, $\underset{\sim}{B}=(-4,3,0)$, $\underset{\sim}{C}=(12,-6,6)$ so area $=\left\|\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ -5 & 3 & -2\\ 11 & -6 & 4\end{vmatrix}\right\| = \|0\hat\imath - 2\hat\jmath - 3\hat k\| \neq 0$, so the points do not lie on a straight line.

4. (b) $\underset{\sim}{A}=(2,-2,1)$, $\underset{\sim}{B}=(4,0,3)$, $\underset{\sim}{C}=(2,3,5)$ so

$$\text{area} = \|(\underset{\sim}{B}-\underset{\sim}{A})\times(\underset{\sim}{C}-\underset{\sim}{A})\| = \left\|\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ 2 & 2 & 2\\ 0 & 5 & 4\end{vmatrix}\right\| = \|-2\hat\imath - 8\hat\jmath + 10\hat k\| = \sqrt{168} = 2\sqrt{42}.$$

(c) $\text{area} = \left\|\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ 0 & 4 & -2\\ 3 & 1 & -1\end{vmatrix}\right\| = \|-2\hat\imath + 6\hat\jmath - 12\hat k\| = \sqrt{184} = 2\sqrt{46}.$

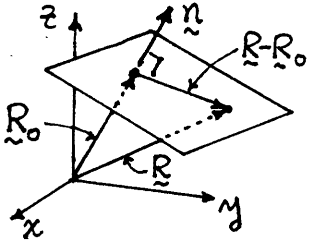
5. (b) $\underset{\sim}{A}=(0,0,0)$, $\underset{\sim}{B}=(1,0,2)$, $\underset{\sim}{C}=(0,1,1)$, say. Then $\underset{\sim}{n} = \underset{\sim}{A}\underset{\sim}{B}\times\underset{\sim}{A}\underset{\sim}{C}$ [where $\underset{\sim}{A}\underset{\sim}{B}$ means the vector $\underset{\sim}{B}-\underset{\sim}{A}$ from A to B, and similarly for $\underset{\sim}{A}\underset{\sim}{C}$] so

$$\underset{\sim}{n} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ 1 & 0 & 2\\ 0 & 1 & 1\end{vmatrix} = -2\hat\imath - \hat\jmath + \hat k, \quad \underset{\sim}{\hat n} = \pm\frac{1}{\sqrt6}(2\hat\imath + \hat\jmath - \hat k)$$

(c) $\underset{\sim}{A}=(-2,0,0)$, $\underset{\sim}{B}=(1,1,0)$, $\underset{\sim}{C}=(-6,0,1)$, say. Then $\underset{\sim}{n} = \underset{\sim}{A}\underset{\sim}{B}\times\underset{\sim}{A}\underset{\sim}{C} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ 3 & 1 & 0\\ -4 & 0 & 1\end{vmatrix} = \hat\imath - 3\hat\jmath + 4\hat k$ so $\underset{\sim}{\hat n} = \pm\frac{1}{\sqrt{26}}(\hat\imath - 3\hat\jmath + 4\hat k)$

6. (b) $\underset{\sim}{R}_0 = (2,5,-1)$, $\underset{\sim}{v} = 4\hat\imath - \hat k$, so (6.2) gives $\underset{\sim}{R} = (2,5,-1) + (4\hat\imath - \hat k)t$ so $x = 2+4t$, $y=5$, $z=-1-t$ $(-\infty<t<\infty)$

7.



$$\begin{aligned}(\underset{\sim}{R}-\underset{\sim}{R}_0)\cdot\underset{\sim}{\hat n} &= 0\\ \underset{\sim}{R}\cdot\underset{\sim}{\hat n} &= \underset{\sim}{R}_0\cdot\underset{\sim}{\hat n}\\ &= \|\underset{\sim}{R}_0\|\,\|\underset{\sim}{\hat n}\|\cos 0\\ &= \|\underset{\sim}{R}_0\|\\ &= \text{perpendicular distance from origin to plane}\\ &= \text{shortest distance from origin to plane}\end{aligned}$$

(b) $x-y-z=5$ is $(x\hat\imath + y\hat\jmath + z\hat k)\cdot(\hat\imath - \hat\jmath - \hat k) = 5$

$$\text{or,}\quad \underset{\sim}{R}\ \cdot\ \underbrace{\frac{\hat\imath-\hat\jmath-\hat k}{\sqrt3}}_{\underset{\sim}{\hat n}} = \underbrace{\frac{5}{\sqrt3}}$$

↑shortest distance from origin to plane

(c) $3x+y-z=0$ is $(x\hat\imath + y\hat\jmath + z\hat k)\cdot(3\hat\imath + \hat\jmath - \hat k) = 0$

$$\text{or,}\quad \underset{\sim}{R}\ \cdot\ \underbrace{\frac{3\hat\imath+\hat\jmath-\hat k}{\sqrt{11}}}_{\underset{\sim}{\hat n}} = 0$$

↑shortest distance = 0; the plane passes through the origin

---

* Of course we could use $\|(\underset{\sim}{C}-\underset{\sim}{B})\times(\underset{\sim}{A}-\underset{\sim}{B})\|$ or $\|(\underset{\sim}{A}-\underset{\sim}{C})\times(\underset{\sim}{B}-\underset{\sim}{C})\|$; they are all equal.

9. (a) $\underset{\sim}{x}\cdot\underset{\sim}{w} = [a(4,-1,1)+b(1,1,2)]\cdot(3,0,5) = 0$ gives

$(4a+b,-a+b,a+2b)\cdot(3,0,5) = 17a+13b=0$. Choose $b=-17, a=13$, say. Then $\underset{\sim}{x} = \alpha[13(4,-1,1)-17(1,1,2)] = \alpha(35,-30,-21)$, where α is an arbitrary (but nonzero) constant.

(b) $\underset{\sim}{x}\cdot\underset{\sim}{w} = [a(1,2,3)+b(3,2,1)]\cdot(1,2,4) = 0$ gives

$(a+3b, 2a+2b, 3a+b)\cdot(1,2,4) = 17a+11b=0$. Choose $b=-17, a=11$, say. Then $\underset{\sim}{x} = \alpha[11(1,2,3)-17(3,2,1)] = \alpha(-40,-12,16) = \beta(10,3,-4)$, where β is an arbitrary (but nonzero) constant.

Section 14.4

1. (b) $\underset{\sim}{v}\times\underset{\sim}{w} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & 3\\ 0 & 1 & -1\end{vmatrix} = -4\hat{\imath}+2\hat{\jmath}+2\hat{k}$, $\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -1 & 0\\ -4 & 2 & 2\end{vmatrix} = -2\hat{\imath}-2\hat{\jmath}-2\hat{k}$

$\underset{\sim}{u}\times\underset{\sim}{v} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -1 & 0\\ 2 & 1 & 3\end{vmatrix} = -3\hat{\imath}-3\hat{\jmath}+3\hat{k}$, $(\underset{\sim}{u}\times\underset{\sim}{v})\times\underset{\sim}{w} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -3 & -3 & 3\\ 0 & 1 & -1\end{vmatrix} = 0\hat{\imath}-3\hat{\jmath}-3\hat{k} \neq \underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w})$.

2. (a) $\underset{\sim}{A}\cdot\underset{\sim}{B}\times\underset{\sim}{C} \underset{\substack{\uparrow\\ \text{per (7)}}}{=} \underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C} = \begin{vmatrix}1 & 1 & -1\\ 0 & 3 & -1\\ 1 & 0 & 2\end{vmatrix} = 8$

3. (b) $\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) = \hat{\imath}\times(\hat{\imath}\times\hat{\jmath}) = \hat{\imath}\times\hat{k} = -\hat{\jmath}$

$(\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w} = 0\hat{\imath} - 1\hat{\jmath} = -\hat{\jmath}$ ✓

(c) $\underset{\sim}{v}\times\underset{\sim}{w} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 3 & 0\\ 1 & 0 & 2\end{vmatrix} = 6\hat{\imath}-2\hat{\jmath}-3\hat{k}$, $\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 1 & -1\\ 6 & -2 & -3\end{vmatrix} = -5\hat{\imath}-6\hat{\jmath}-6\hat{k}$

$(\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w} = -2(\hat{\imath}+3\hat{\jmath}) - 3(\hat{\imath}+2\hat{k}) = -5\hat{\imath}-6\hat{\jmath}-6\hat{k}$ ✓

4. Use (5).

(b) Volume $= \left|\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{vmatrix}\right| = |1| = 1$

(c) Volume $= \left|\begin{vmatrix}1 & 2 & 0\\ 3 & 0 & 0\\ 0 & 1 & 1\end{vmatrix}\right| = |-6| = 6$

5. $(\underset{\sim}{u}\times\underset{\sim}{v})\times\underset{\sim}{w} = -\underset{\sim}{w}\times(\underset{\sim}{u}\times\underset{\sim}{v})$ and now use (9) with "$\underset{\sim}{u}$"→$\underset{\sim}{w}$, "$\underset{\sim}{v}$"→$\underset{\sim}{u}$, "$\underset{\sim}{w}$"→$\underset{\sim}{v}$

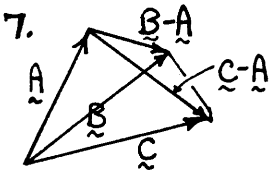
$= -(\underset{\sim}{w}\cdot\underset{\sim}{v})\underset{\sim}{u} + (\underset{\sim}{w}\cdot\underset{\sim}{u})\underset{\sim}{v}$. ✓

6. $\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) + \underset{\sim}{w}\times(\underset{\sim}{u}\times\underset{\sim}{v}) + \underset{\sim}{v}\times(\underset{\sim}{w}\times\underset{\sim}{u}) = \cancel{(\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v}} - \cancel{(\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w}}$ by (9)

$+\cancel{(\underset{\sim}{w}\cdot\underset{\sim}{v})\underset{\sim}{u}} - \cancel{(\underset{\sim}{w}\cdot\underset{\sim}{u})\underset{\sim}{v}}$ by (9)

$+\cancel{(\underset{\sim}{v}\cdot\underset{\sim}{u})\underset{\sim}{w}} - \cancel{(\underset{\sim}{v}\cdot\underset{\sim}{w})\underset{\sim}{u}}$ by (9)

7.

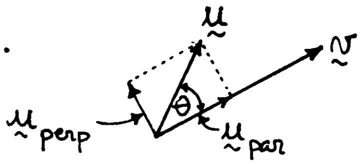


The given vector will be perpendicular to the plane if it is perp. to each nonzero vector in the plane or, equivalently, if it is perp. to each member of a basis for the plane, say $\underset{\sim}{B}-\underset{\sim}{A}$ and $\underset{\sim}{C}-\underset{\sim}{A}$. Let's see:

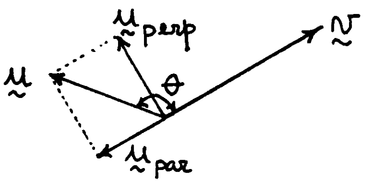
$\underset{\sim}{A}\times\underset{\sim}{B}$ is ⊥ to $\underset{\sim}{B}$, so $\underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{B}=0$

$$\begin{aligned}(\underset{\sim}{A}\times\underset{\sim}{B}+\underset{\sim}{B}\times\underset{\sim}{C}+\underset{\sim}{C}\times\underset{\sim}{A})\cdot(\underset{\sim}{B}-\underset{\sim}{A}) &= 0-0+0-\underset{\sim}{B}\times\underset{\sim}{C}\cdot\underset{\sim}{A}+\underset{\sim}{C}\times\underset{\sim}{A}\cdot\underset{\sim}{B}-0\\ &= -\underset{\sim}{B}\cdot\underset{\sim}{C}\times\underset{\sim}{A}+\underset{\sim}{C}\times\underset{\sim}{A}\cdot\underset{\sim}{B}=0\ ✓\\ (\underset{\sim}{A}\times\underset{\sim}{B}+\underset{\sim}{B}\times\underset{\sim}{C}+\underset{\sim}{C}\times\underset{\sim}{A})\cdot(\underset{\sim}{C}-\underset{\sim}{A}) &= \underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C}-0+0-\underset{\sim}{B}\times\underset{\sim}{C}\cdot\underset{\sim}{A}+0-0\\ &= \underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C}+\underset{\sim}{C}\times\underset{\sim}{B}\cdot\underset{\sim}{A}\\ &= \underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C}+\underset{\sim}{C}\cdot\underset{\sim}{B}\times\underset{\sim}{A}=\underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C}-\underset{\sim}{C}\cdot\underset{\sim}{A}\times\underset{\sim}{B}=0\ ✓\end{aligned}$$

8.



or



$\underset{\sim}{u}_{par}$: $\underset{\sim}{u}\cdot\underset{\sim}{v}=\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|\cos\theta$

$$=\|\underset{\sim}{u}_{par}\|\,\|\underset{\sim}{v}\|$$

so $\|\underset{\sim}{u}_{par}\|=\dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{v}\|}=\underset{\sim}{u}\cdot\underset{\sim}{\hat{v}}$

Then, $\underset{\sim}{u}_{par}=\|\underset{\sim}{u}_{par}\|\underset{\sim}{\hat{v}}$

$$=(\underset{\sim}{u}\cdot\underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}}.\ ✓$$

$\underset{\sim}{u}_{perp}$: $\underset{\sim}{u}_{perp}=(\|\underset{\sim}{u}\|\sin\theta)\underset{\sim}{\hat{u}}_{perp}$

$$=(\|\underset{\sim}{u}\|\cancel{\sin}\theta)\left[\underset{\sim}{\hat{v}}\times\left(\frac{\underset{\sim}{\hat{u}}\times\underset{\sim}{\hat{v}}}{\cancel{\sin\theta}}\right)\right]$$

$$=\underset{\sim}{\hat{v}}\times(\underset{\sim}{u}\times\underset{\sim}{\hat{v}})\ ✓$$

$\underset{\sim}{u}_{par}$: $\underset{\sim}{u}\cdot\underset{\sim}{v}=\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|\cos\theta=\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|\cos[(\theta-\frac{\pi}{2})+\frac{\pi}{2}]$

$$=\|\underset{\sim}{u}\|\|\underset{\sim}{v}\|[-\sin(\theta-\tfrac{\pi}{2})]$$

$$=-\|\underset{\sim}{u}_{par}\|\,\|\underset{\sim}{v}\|$$

so $\|\underset{\sim}{u}_{par}\|=-\dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{v}\|}=-\underset{\sim}{u}\cdot\underset{\sim}{\hat{v}}$

Then, $\underset{\sim}{u}_{par}=\|\underset{\sim}{u}_{par}\|(-\underset{\sim}{\hat{v}})$

$=(\underset{\sim}{u}\cdot\underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}}$ again. ✓

$\underset{\sim}{u}_{perp}$: $\underset{\sim}{u}_{perp}=(\|\underset{\sim}{u}\|\cos(\theta-\frac{\pi}{2}))\underset{\sim}{\hat{u}}_{perp}$

$$=\|\underset{\sim}{u}\|\sin\theta\ \underset{\sim}{\hat{u}}_{perp}$$

$$=\|\underset{\sim}{u}\|\cancel{\sin}\theta\left[\underset{\sim}{\hat{v}}\times\left(\frac{\underset{\sim}{\hat{u}}\times\underset{\sim}{\hat{v}}}{\cancel{\sin\theta}}\right)\right]$$

$$=\underset{\sim}{\hat{v}}\times(\underset{\sim}{u}\times\underset{\sim}{\hat{v}})\ ✓$$

9. (a) $(\underset{\sim}{A}\times\underset{\sim}{B})\cdot(\underset{\sim}{C}\times\underset{\sim}{D})=\underset{\sim}{A}\cdot\underset{\sim}{B}\times(\underset{\sim}{C}\times\underset{\sim}{D})$ per (7)

$=\underset{\sim}{A}\cdot[(\underset{\sim}{B}\cdot\underset{\sim}{D})\underset{\sim}{C}-(\underset{\sim}{B}\cdot\underset{\sim}{C})\underset{\sim}{D}]$ per (9)

$=(\underset{\sim}{A}\cdot\underset{\sim}{C})(\underset{\sim}{B}\cdot\underset{\sim}{D})-(\underset{\sim}{A}\cdot\underset{\sim}{D})(\underset{\sim}{B}\cdot\underset{\sim}{D})$ ✓

(b) $(\underset{\sim}{A}\times\underset{\sim}{B})\times(\underset{\sim}{C}\times\underset{\sim}{D})=(\underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{D})\underset{\sim}{C}-(\underset{\sim}{A}\times\underset{\sim}{B}\cdot\underset{\sim}{C})\underset{\sim}{D}$ per (9), where "$\underset{\sim}{u}$" is $\underset{\sim}{A}\times\underset{\sim}{B}$, "$\underset{\sim}{v}$" is $\underset{\sim}{C}$, and "$\underset{\sim}{w}$" is $\underset{\sim}{D}$

$=(\underset{\sim}{A}\cdot\underset{\sim}{B}\times\underset{\sim}{D})\underset{\sim}{C}-(\underset{\sim}{A}\cdot\underset{\sim}{B}\times\underset{\sim}{C})D$ per (7).

11. First, suppose $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}=0$. This will of course be satisfied if at least one of $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w}$ is $\underset{\sim}{0}$, in which case the set is LD by Theorem 9.8.3. If $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w}$ are all nonzero then $\underset{\sim}{v}\times\underset{\sim}{w}$ is perpendicular to the $\underset{\sim}{v},\underset{\sim}{w}$ plane (if $\underset{\sim}{v},\underset{\sim}{w}$ do not determine a plane then they must be collinear and hence LD), and $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}=0$ implies that $\underset{\sim}{u}$ lies in the $\underset{\sim}{v},\underset{\sim}{w}$ plane so $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w}$ must be LD. Conversely, suppose $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w}$ are LD, such that we can express

$$\underset{\sim}{u}=\alpha\underset{\sim}{v}+\beta\underset{\sim}{w}.$$

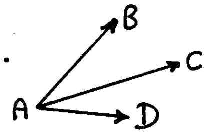
Then $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}=\alpha\cancel{\underset{\sim}{v}\cdot\underset{\sim}{v}}\times\underset{\sim}{w}+\beta\cancel{\underset{\sim}{w}\cdot\underset{\sim}{v}}\times\underset{\sim}{w}=0$. Or, if

$$\underset{\sim}{v}=\alpha\underset{\sim}{u}+\beta\underset{\sim}{w}$$

then $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}=-\underset{\sim}{u}\cdot\underset{\sim}{w}\times\underset{\sim}{v}=-\underset{\sim}{u}\times\underset{\sim}{w}\cdot\underset{\sim}{v}=-\alpha\,\underset{\sim}{u}\cancel{\times\underset{\sim}{w}}\cdot\underset{\sim}{u}-\beta\,\underset{\sim}{u}\cancel{\times\underset{\sim}{w}}\cdot\underset{\sim}{w}=0$. Or, if

$$\underset{\sim}{w}=\alpha\underset{\sim}{u}+\beta\underset{\sim}{v}$$

then $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w} = \underset{\sim}{u}\times\underset{\sim}{v}\cdot(\alpha\underset{\sim}{u}+\beta\underset{\sim}{v}) = 0+0=0$, so linear independence of $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w}$ implies that $\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}=0$. ✓

12. 

The points A,B,C,D lie in a plane if and only if the vectors $\underset{\sim}{AB}$, $\underset{\sim}{AC}$, $\underset{\sim}{AD}$ do, i.e., if they are coplanar and hence LD. Thus, A,B,C,D lie in a plane if and only if $\underset{\sim}{AB}\cdot\underset{\sim}{AC}\times\underset{\sim}{AD}=0$.
↑ i.e., $\underset{\sim}{B}-\underset{\sim}{A}$, ...

(b) $\underset{\sim}{A}=(2,1,-1)$, $\underset{\sim}{B}=(1,3,0)$, $\underset{\sim}{C}=(5,0,9)$, $\underset{\sim}{D}=(0,0,4)$, so

$$\underset{\sim}{AB}\cdot\underset{\sim}{AC}\times\underset{\sim}{AD} = \begin{vmatrix} -1 & 2 & 1 \\ 3 & -1 & 10 \\ -2 & -1 & 5 \end{vmatrix} = (-1)(5)-(2)(35)+(1)(-5) = -80 \neq 0 \text{ so A,B,C,D } \underline{\text{not}} \text{ coplanar.}$$

(c) $\underset{\sim}{A}=(4,0,0)$, $\underset{\sim}{B}=(0,1,0)$, $\underset{\sim}{C}=(1,2,-4)$, $\underset{\sim}{D}=(0,0,1)$, so

$$\underset{\sim}{AB}\cdot\underset{\sim}{AC}\times\underset{\sim}{AD} = \begin{vmatrix} -4 & 1 & 0 \\ -3 & 2 & -4 \\ -4 & 0 & 1 \end{vmatrix} = (-4)(2)-(1)(-19)+0 = 11 \neq 0 \text{ so A,B,C,D } \underline{\text{not}} \text{ coplanar.}$$

(e) $\underset{\sim}{A}=(1,0,1)$, $\underset{\sim}{B}=(2,1,3)$, $\underset{\sim}{C}=(1,-1,0)$, $\underset{\sim}{D}=(3,-1,2)$, so

$$\underset{\sim}{AB}\cdot\underset{\sim}{AC}\times\underset{\sim}{AD} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = (1)(-2)-(1)(2)+(2)(2) = 0 \text{ so A,B,C,D } \underline{\text{are}} \text{ coplanar.}$$

13. Let $\underset{\sim}{u}=(1,0,0)$, $\underset{\sim}{v}=(1,1,0)$, $\underset{\sim}{w}=(1,1,1)$, say. Then

$$\begin{aligned} \underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}) &= (\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v}-(\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w} \quad ① \\ &= \underset{\sim}{v}-\underset{\sim}{w} = (0,0,-1) \end{aligned}$$

and $(\underset{\sim}{u}\times\underset{\sim}{v})\times\underset{\sim}{w} = -\underset{\sim}{w}\times(\underset{\sim}{u}\times\underset{\sim}{v}) = -(\underset{\sim}{w}\cdot\underset{\sim}{v})\underset{\sim}{u}+(\underset{\sim}{w}\cdot\underset{\sim}{u})\underset{\sim}{v}$ ②

$$= -2\underset{\sim}{u}+\underset{\sim}{v} = (-1,1,0) \neq \underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}).$$

To make up an example where equality holds, note from ① and ② that we need to have

$$\cancel{(\underset{\sim}{u}\cdot\underset{\sim}{w})\underset{\sim}{v}}-(\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w} = -(\underset{\sim}{w}\cdot\underset{\sim}{v})\underset{\sim}{u}+\cancel{(\underset{\sim}{w}\cdot\underset{\sim}{u})\underset{\sim}{v}}$$

or,

$$(\underset{\sim}{u}\cdot\underset{\sim}{v})\underset{\sim}{w} = (\underset{\sim}{w}\cdot\underset{\sim}{v})\underset{\sim}{u}. \quad ③$$

Thus, we need $\underset{\sim}{w}$ collinear with $\underset{\sim}{u}$. Let $\underset{\sim}{u}=(1,0,0)$ (with no real loss of generality since we can orient the coordinate system and scale the axes so as to ensure this). Then $\underset{\sim}{w}=(\alpha,0,0)$, $\underset{\sim}{v}=(\beta,\gamma,\delta)$, and ③ gives

$$(\alpha\beta,0,0)=(\alpha\beta,0,0),$$

which is satisfied for any $\alpha,\beta,\gamma,\delta$. For instance,

$$\underset{\sim}{u}=(1,0,0),\ \underset{\sim}{v}=(5,9,-3),\ \underset{\sim}{w}=(4,0,0).$$

Section 14.5

1. (b) $\underset{\sim}{u}=6\hat{\imath}-e^{t}\hat{\jmath}$, $\underset{\sim}{u}'=-e^{t}\hat{\jmath}$, $u''=-e^{t}\hat{\jmath}$, $\|\underset{\sim}{u}''\|=e^{t}$

(c) $\underset{\sim}{u}=t^2\hat{\imath}-4\hat{\jmath}+3\cos 2t\,\hat{k}$, $\underset{\sim}{u}'=2t\hat{\imath}-6\sin 2t\,\hat{k}$, $\underset{\sim}{u}''=2\hat{\imath}-12\cos 2t\,\hat{k}$,
$\|u''\|=\sqrt{4+144\cos^2 2t}$

(e) $\underset{\sim}{u}=e^{-t}(\cos 2t\,\hat{\imath}+\sin 2t\,\hat{\jmath})$, $\underset{\sim}{u}'=e^{-t}[(-\overset{\cos 2t}{c}-2\overset{\sin 2t}{s})\hat{\imath}+(-s+2c)\hat{\jmath}]$,

$$\underset{\sim}{u}'' = e^{-t}[(3s-c)\hat{\imath}-(s+3c)\hat{\jmath}],\ \|\underset{\sim}{u}''\| = e^{t}\sqrt{(3s-c)^2+(s+3c)^2}$$
$$= e^{-t}\sqrt{9s^2-\cancel{6sc}+c^2+s^2+\cancel{6sc}+9c^2} = \sqrt{10}\,e^{-t}$$

2. (c) Want to verify (7e): $(\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w})' = \underset{\sim}{u}'\cdot\underset{\sim}{v}\times\underset{\sim}{w} + \underset{\sim}{u}\cdot\underset{\sim}{v}'\times\underset{\sim}{w} + \underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}'$, for $\underset{\sim}{u} = t^2\hat{\imath}-t\hat{\jmath}-3t\hat{k}$, $\underset{\sim}{v} = 2\cos t\,\hat{\imath}-t\hat{k}$, $\underset{\sim}{w} = \sin t\,\hat{\jmath}$.

$$\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w} = \begin{vmatrix} t^2 & -t & -3t \\ 2c & 0 & -t \\ 0 & s & 0 \end{vmatrix} = -s(-t^3+6tc) = s(t^3-6tc) \quad (c,s \equiv \cos t, \sin t)$$

so $(\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w})' = c(t^3-6tc)+s(3t^2-6c+6ts)$

$$\left.\begin{aligned} \underset{\sim}{u}'\cdot\underset{\sim}{v}\times\underset{\sim}{w} &= \begin{vmatrix} 2t & -1 & -3 \\ 2c & 0 & -t \\ 0 & s & 0 \end{vmatrix} = -s(-2t^2+6c) \\ \underset{\sim}{u}\cdot\underset{\sim}{v}'\times\underset{\sim}{w} &= \begin{vmatrix} t^2 & -t & -3t \\ -2s & 0 & -1 \\ 0 & s & 0 \end{vmatrix} = -s(-t^2-6st) \\ \underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}' &= \begin{vmatrix} t^2 & -t & -3t \\ 2c & 0 & -t \\ 0 & c & 0 \end{vmatrix} = -c(-t^3+6ct) \end{aligned}\right\} \text{Their sum is } 3st^2-6sc+6s^2t+ct^3-6c^2t$$
$$= c(t^3-6tc)+s(3t^2-6c+6st) \quad ✓$$

Actually, it is easy to prove (7e) in general:

$$(\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w})' = \frac{d}{dt}\begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ v_1(t) & v_2(t) & v_3(t) \\ w_1(t) & w_2(t) & w_3(t) \end{vmatrix},$$ then use (16.3) from Exercise 16 in Sec. 10.4, then use (6) in Sec. 14.4.

3. (a) Want to verify (7c): $(\underset{\sim}{u}\cdot\underset{\sim}{v})' = \underset{\sim}{u}'\cdot\underset{\sim}{v}+\underset{\sim}{u}\cdot\underset{\sim}{v}'$

$\underset{\sim}{u} = 3t\hat{\imath}+t^4\hat{k}$, $\underset{\sim}{v} = \hat{\imath}-4t\hat{\jmath}$.

$(\underset{\sim}{u}\cdot\underset{\sim}{v})' = (3t)' = 3$, $\quad \underset{\sim}{u}'\cdot\underset{\sim}{v}+\underset{\sim}{u}\cdot\underset{\sim}{v}' = 3+0 = 3$ ✓

(b) Want to verify (7d): $(\underset{\sim}{u}\times\underset{\sim}{v})' = \underset{\sim}{u}'\times\underset{\sim}{v}+\underset{\sim}{u}\times\underset{\sim}{v}'$

$$(\underset{\sim}{u}\times\underset{\sim}{v})' = \frac{d}{dt}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t & 0 & t^4 \\ 1 & -4t & 0 \end{vmatrix} = \frac{d}{dt}(4t^5\hat{\imath}+t^4\hat{\jmath}-12t^2\hat{k}) = 20t^4\hat{\imath}+4t^3\hat{\jmath}-24t\hat{k}$$

$$\underset{\sim}{u}'\times\underset{\sim}{v}+\underset{\sim}{u}\times\underset{\sim}{v}' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 0 & 4t^3 \\ 1 & -4t & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t & 0 & t^4 \\ 0 & -4 & 0 \end{vmatrix} = 16t^4\hat{\imath}+4t^3\hat{\jmath}-12t\hat{k}+4t^4\hat{\imath}-12t\hat{k}$$
$$= 20t^4\hat{\imath}+4t^3\hat{\jmath}-24t\hat{k} \quad ✓$$

4. (a)
$$(f\underset{\sim}{u})' = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)\underset{\sim}{u}(t+\Delta t)-f(t)\underset{\sim}{u}(t)}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{f(t+\Delta t)\underset{\sim}{u}(t)-f(t)\underset{\sim}{u}(t)}{\Delta t} + \lim_{\Delta t\to 0}\frac{f(t+\Delta t)\underset{\sim}{u}(t+\Delta t)-f(t+\Delta t)\underset{\sim}{u}(t)}{\Delta t}$$
$$= \left(\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\right)\underset{\sim}{u}(t) + \left(\lim_{\Delta t\to 0} f(t+\Delta t)\right)\left(\lim_{\Delta t\to 0}\frac{\underset{\sim}{u}(t+\Delta t)-\underset{\sim}{u}(t)}{\Delta t}\right)$$
$$= f'(t)\underset{\sim}{u}(t)+f(t)\underset{\sim}{u}'(t) \quad ✓$$

5. (a)
$$(\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w})' = \underset{\sim}{u}'\cdot\underset{\sim}{v}\times\underset{\sim}{w}+\underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{w})' \quad \text{per (7c)}$$
$$= \underset{\sim}{u}'\cdot\underset{\sim}{v}\times\underset{\sim}{w}+\underset{\sim}{u}\cdot(\underset{\sim}{v}'\times\underset{\sim}{w}+\underset{\sim}{v}\times\underset{\sim}{w}') \quad \text{per (7d)}$$
$$= \underset{\sim}{u}'\cdot\underset{\sim}{v}\times\underset{\sim}{w}+\underset{\sim}{u}\cdot\underset{\sim}{v}'\times\underset{\sim}{w}+\underset{\sim}{u}\cdot\underset{\sim}{v}\times\underset{\sim}{w}' \quad ✓$$

(b) $[\underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w})]' = \underset{\sim}{u}'\times(\underset{\sim}{v}\times\underset{\sim}{w}) + \underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w})'$ per (7d)

$= \underset{\sim}{u}'\times(\underset{\sim}{v}\times\underset{\sim}{w}) + \underset{\sim}{u}\times(\underset{\sim}{v}'\times\underset{\sim}{w} + \underset{\sim}{v}\times\underset{\sim}{w}')$ per (7d)

$= \underset{\sim}{u}'\times(\underset{\sim}{v}+\underset{\sim}{w}) + \underset{\sim}{u}\times(\underset{\sim}{v}'\times\underset{\sim}{w}) + \underset{\sim}{u}\times(\underset{\sim}{v}\times\underset{\sim}{w}')$

6. $\frac{d}{d\tau}\underset{\sim}{u}(f(\tau)) = \frac{d}{d\tau}[u_1(f(\tau))\hat{\imath} + u_2(f(\tau))\hat{\jmath} + u_3(f(\tau))\hat{k}]$

$= \frac{d}{d\tau}u_1(f(\tau))\hat{\imath} + \frac{d}{d\tau}u_2(f(\tau))\hat{\jmath} + \frac{d}{d\tau}u_3(f(\tau))\hat{k}$

$= u_1'(f)\frac{df}{d\tau}\hat{\imath} + u_2'(f)\frac{df}{d\tau}\hat{\jmath} + u_3'(f)\frac{df}{d\tau}\hat{k}$

$= (u_1'\hat{\imath} + u_2'\hat{\jmath} + u_3'\hat{k})f'$

$= \frac{d\underset{\sim}{u}}{df}\frac{df}{d\tau}$

7. (b) $(\underset{\sim}{u}\times\underset{\sim}{v})'' = (\underset{\sim}{u}'\times\underset{\sim}{v} + \underset{\sim}{u}\times\underset{\sim}{v}')' = (\underset{\sim}{u}'\times\underset{\sim}{v})' + (\underset{\sim}{u}\times\underset{\sim}{v}')'$

$= \underset{\sim}{u}''\times\underset{\sim}{v} + \underset{\sim}{u}'\times\underset{\sim}{v}' + \underset{\sim}{u}'\times\underset{\sim}{v}' + \underset{\sim}{u}\times\underset{\sim}{v}''$

$= \underset{\sim}{u}''\times\underset{\sim}{v} + 2\underset{\sim}{u}'\times\underset{\sim}{v}' + \underset{\sim}{u}\times\underset{\sim}{v}''$

8. (a) $\|\underset{\sim}{u}(\tau)\| = (\underset{\sim}{u}\cdot\underset{\sim}{u})^{1/2}$ so $\|\underset{\sim}{u}(\tau)\|' = \frac{1}{2}(\underset{\sim}{u}'\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{u}')/(\underset{\sim}{u}\cdot\underset{\sim}{u})^{1/2}$

$= \frac{1}{2}\,\frac{2\underset{\sim}{u}'\cdot\underset{\sim}{u}}{\sqrt{\underset{\sim}{u}\cdot\underset{\sim}{u}}} = \frac{\underset{\sim}{u}'\cdot\underset{\sim}{u}}{\|\underset{\sim}{u}\|}$

(b) If $\|\underset{\sim}{u}(\tau)\| =$ constant $\neq 0$ then (8.1) becomes $0 = \underset{\sim}{u}\cdot\underset{\sim}{u}'$ so $\underset{\sim}{u}'$ is either perpendicular to $\underset{\sim}{u}$ or it is $\underset{\sim}{0}$.

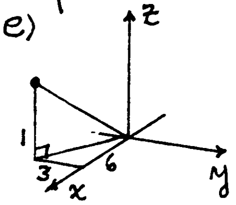
Section 14.6

1. (b) $r = \sqrt{2^2+(-1)^2} = \sqrt{5}$, $\tan^{-1}\frac{1}{2} = 26.57°$, so $\theta = 360-26.57 = 333.43° = 5.819$ rad, $z = 0$

(c) $r = \sqrt{(-3)^2+2^2} = \sqrt{13}$, $\tan^{-1}\frac{2}{3} = 33.69°$, so $\theta = 180-33.69 = 146.31° = 2.554$ rad, $z = 1$

2. (b) $\rho = 3$, $\phi = 90° = \pi/2$ rad, $\theta = 90° = \pi/2$ rad

(e)

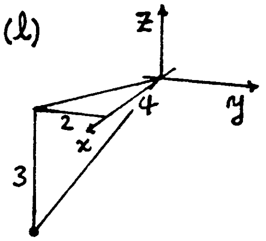


$\tan^{-1}\frac{1}{\sqrt{3^2+6^2}} = \tan^{-1}\frac{1}{\sqrt{45}} = 8.48°$ so $\phi = 90-8.48 = 81.52° = 1.423$ rad

$\tan^{-1}\frac{3}{6} = 26.57°$ so $\theta = 360-26.57 = 333.43° = 5.819$ rad

and $\rho = \sqrt{6^2+(-3)^2+1^2} = \sqrt{46}$

(l)



$\tan^{-1}\frac{3}{\sqrt{2^2+4^2}} = 33.85°$ so $\phi = 90+33.85° = 123.85° = 2.162$ rad

$\tan^{-1}\frac{2}{4} = 26.57°$ so $\theta = 360-26.57 = 333.43° = 5.819$ rad

$\rho = \sqrt{4^2+(-2)^2+(-3)^2} = \sqrt{29}$

3. (b) Write (3.1) in the matrix form $\begin{pmatrix}\hat{e}_\rho\\ \hat{e}_\phi\\ \hat{e}_\theta\end{pmatrix} = \begin{pmatrix}\sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi\\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi\\ -\sin\theta & \cos\theta & 0\end{pmatrix}\begin{pmatrix}\hat{\imath}\\ \hat{\jmath}\\ \hat{k}\end{pmatrix}$,

and call the 3x3 matrix $\underset{\sim}{A}$. Observe that $\underset{\sim}{A}$ is an orthogonal matrix since if we call its columns $\underset{\sim}{c}_1, \underset{\sim}{c}_2, \underset{\sim}{c}_3$ then $\underset{\sim}{c}_1 \cdot \underset{\sim}{c}_1 = (\sin^2\phi + \cos^2\phi)\cos^2\theta + \sin^2\theta = 1$, $\underset{\sim}{c}_1 \cdot \underset{\sim}{c}_2 = \sin^2\phi \sin\phi \cos\theta + \cos^2\phi \sin\theta \cos\theta - \sin\theta\cos\theta = 0$, $\underset{\sim}{c}_1 \cdot \underset{\sim}{c}_3 = \sin\phi\cos\theta\cos\phi - \cos\phi\cos\theta\sin\phi = 0$, $\underset{\sim}{c}_2 \cdot \underset{\sim}{c}_2 = (\sin^2\phi + \cos^2\phi)\sin^2\theta + \cos^2\theta = 1$, $\underset{\sim}{c}_2 \cdot \underset{\sim}{c}_3 = \sin\phi\sin\theta\cos\phi - \cos\phi\sin\theta\sin\phi = 0$, $\underset{\sim}{c}_3 \cdot \underset{\sim}{c}_3 = \cos^2\phi + \sin^2\phi = 1$. Since $\underset{\sim}{A}$ is orthogonal, $\underset{\sim}{A}^{-1}$ is simply $\underset{\sim}{A}^T$. Thus,

$$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\phi \\ \hat{e}_\theta \end{pmatrix},$$

which gives (3.2).

4. (b) $\underset{\sim}{A} = t\hat{e}_r$, $r = 4, \theta = 2t$, so $\underset{\sim}{A}'(t) = \hat{e}_r + t\left(\cancelto{\hat{e}_\theta}{\frac{d\hat{e}_r}{d\theta}} \cancelto{2}{\frac{d\theta}{dt}} + \cancelto{0}{\frac{d\hat{e}_r}{dr}} \frac{dr}{dt}\right) = \hat{e}_r + 2t\,\hat{e}_\theta$

(c) $\underset{\sim}{A} = r\underset{\sim}{\hat{e}}_r$, $r = 6 + \sin t, \theta = \cos t$, so $\underset{\sim}{A}'(t) = \frac{dr}{dt}\hat{e}_r + r\left(\cancelto{0}{\frac{d\hat{e}_r}{dr}} \frac{dr}{dt} + \cancelto{\hat{e}_\theta}{\frac{d\hat{e}_r}{d\theta}} \frac{d\theta}{dt}\right) = \cos t\,\hat{e}_r$
$- (6 + \sin t)\sin t\,\hat{e}_\theta$

(e) $\underset{\sim}{A} = \underset{\sim}{\hat{e}}_\phi$, $\rho = 1 + t, \phi = t^2, \theta = \sin 2t$, so $\underset{\sim}{A}'(t) = \cancelto{0}{\frac{\partial\hat{e}_\phi}{\partial\rho}} \frac{d\rho}{dt} + \frac{\partial\hat{e}_\phi}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial\hat{e}_\phi}{\partial\theta}\frac{d\theta}{dt}$
$= (2t)(-\hat{e}_\rho) + (2\cos 2t)\cos t^2\,\hat{e}_\theta$
$= -2t\,\hat{e}_\rho + 2\cos 2t\cos t^2\,\hat{e}_\theta$

(f) $\underset{\sim}{A} = \underset{\sim}{\hat{e}}_\rho + t\underset{\sim}{\hat{e}}_\theta$, $\rho = 1, \phi = t, \theta = t^2$ so
$\underset{\sim}{A}'(t) = \left(\cancel{\frac{d\hat{e}_\rho}{d\rho}} \frac{d\rho}{dt} + \frac{d\hat{e}_\rho}{d\phi}\frac{d\phi}{dt} + \frac{d\hat{e}_\rho}{d\theta}\frac{d\theta}{dt}\right) + \hat{e}_\theta + t\left(\cancelto{0}{\frac{d\hat{e}_\theta}{\partial\rho}} \frac{\partial\rho}{\partial t} + \cancelto{0}{\frac{\partial\hat{e}_\theta}{\partial\phi}} \frac{d\phi}{dt} + \frac{\partial\hat{e}_\theta}{\partial\theta}\frac{d\theta}{dt}\right)$

$= (\hat{e}_\phi)(1) + (\sin t\,\hat{e}_\theta)(2t) + \hat{e}_\theta + t(-\sin t\,\hat{e}_\rho - \cos t\,\hat{e}_\phi)(2t)$
$= -t\sin t\,\hat{e}_\rho + (1 - 2t^2\cos t)\hat{e}_\phi + (1 + 2t\sin t)\hat{e}_\theta$

5. (b) $\underset{\sim}{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = 2t\hat{\imath} + 5t\hat{\jmath} + 3t\hat{k}$
$\underset{\sim}{v} = \dot{\underset{\sim}{R}} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k} = 2(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) + 5(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) + 3\,\hat{e}_z$ per (21)
but $\cos\theta = x/r = 2t/\sqrt{(2t)^2 + (5t)^2} = 2/\sqrt{29}$, $\sin\theta = y/r = 5t/\sqrt{29}t = 5/\sqrt{29}$, so
$\underset{\sim}{v} = \left(2\frac{2}{\sqrt{29}} + 5\frac{5}{\sqrt{29}}\right)\hat{e}_r + \left(-2\cancel{\frac{5}{\sqrt{29}}} + 5\cancel{\frac{2}{\sqrt{29}}}\right)\hat{e}_\theta + 3\,\hat{e}_z = \sqrt{29}\,\hat{e}_r + 3\,\hat{e}_z$

$\underset{\sim}{a} = \frac{d}{dt}(2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}) = \underset{\sim}{0}$.

(c) $\underset{\sim}{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = (t^2 - t)\hat{\imath} + t^2\hat{\jmath} - 3t\hat{k}$
$\underset{\sim}{v} = \dot{\underset{\sim}{R}} = (2t - 1)\hat{\imath} + 2t\hat{\jmath} - 3\hat{k}$
$= (2t - 1)(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) + 2t(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) - 3\hat{e}_z$
but $\cos\theta = x/r = (t^2 - t)/\sqrt{(t^2 - t)^2 + (t^2)^2} = (t^2 - t)/\sqrt{2t^4 - 2t^3 + t^2} = (t - 1)/\sqrt{2t^2 - 2t + 1}$
and $\sin\theta = y/r = t^2/\sqrt{\;''\;} = t/\sqrt{2t^2 - 2t + 1}$, so
$\underset{\sim}{v} = \frac{4t^2 - 3t + 1}{\sqrt{2t^2 - 2t + 1}}\hat{e}_r - \frac{t}{\sqrt{2t^2 - 2t + 1}}\underset{\sim}{\hat{e}}_\theta - 3\underset{\sim}{\hat{e}}_z$

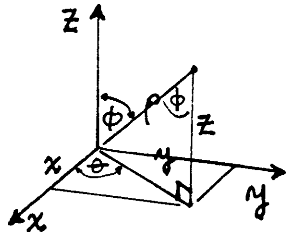
$\underset{\sim}{a} = 2\hat{\imath} + 2\hat{\jmath} = 2(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) + 2(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$
$= \frac{4t - 2}{\sqrt{2t^2 - 2t + 1}}\hat{e}_r - \frac{2}{\sqrt{2t^2 - 2t + 1}}\hat{e}_\theta$

(d) This one is easy because we can see that $r = 3, \theta = t, z = 6t$
Then (23) gives $\underset{\sim}{v} = 0\hat{e}_r + (3)(1)\hat{e}_\theta + 6\hat{e}_z = 3\hat{e}_\theta + 6\hat{e}_z$
and (24) gives $\underset{\sim}{a} = (0 - 3(1)^2)\hat{e}_r + ((3)(0) + 2(0)(1))\hat{e}_\theta + 0\hat{e}_z = -3\hat{e}_r$

(e) $\underset{\sim}{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \hat{\imath} + 4t\hat{\jmath} + 2\hat{k}$
$\underset{\sim}{v} = \dot{\underset{\sim}{R}} = 4\hat{\jmath} = 4(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$
but $\cos\theta = x/r = 1/\sqrt{16t^2+1}$ and $\sin\theta = y/r = 4t/\sqrt{16t^2+1}$, so
$$\underset{\sim}{v} = \frac{16t}{\sqrt{16t^2+1}}\hat{e}_r + \frac{4}{\sqrt{16t^2+1}}\underset{\sim}{\hat{e}_\theta}$$
$\underset{\sim}{a} = \ddot{\underset{\sim}{R}} = 4\dot{\hat{\underset{\sim}{\jmath}}} = \underset{\sim}{0}$ because $\hat{\jmath}$ is a constant vector.

(f) Easy because we can see that $r=1, \theta = t$. Then (23) and (24) give
$\underset{\sim}{v} = \underset{\sim}{\hat{e}_\theta} + \cos t\,\hat{e}_z$, $\underset{\sim}{a} = -\sin t\,\hat{e}_z$

6. (b) $\underset{\sim}{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $(x = 2t, y = 5t, z = 3t)$
$\underset{\sim}{v} = \dot{\underset{\sim}{R}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$ and $\underset{\sim}{a} = \dot{\underset{\sim}{v}} = \underset{\sim}{0}$.



Now express $\hat{\imath}, \hat{\jmath}, \hat{k}$, here, in terms of $t, \hat{e}_\rho, \hat{e}_\phi, \hat{e}_\theta$ using (3.2).
We will need $\sin\phi, \cos\phi, \sin\theta, \cos\theta$.
$\sin\phi = \sqrt{x^2+y^2}/\rho = \sqrt{x^2+y^2}/\sqrt{x^2+y^2+z^2} = \sqrt{29/38}$
$\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{9/38}$
$\sin\theta = y/\sqrt{x^2+y^2} = 5/\sqrt{29}$
$\cos\theta = \sqrt{1-\sin^2\theta} = 2/\sqrt{29}$

so, with these results and (3.2),
$$\underset{\sim}{v} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$$
$$= 2\left(\sqrt{\tfrac{29}{38}}\tfrac{2}{\sqrt{29}}\hat{e}_\rho + \sqrt{\tfrac{9}{38}}\tfrac{2}{\sqrt{29}}\hat{e}_\phi - \tfrac{5}{\sqrt{29}}\hat{e}_\theta\right) + 5\left(\sqrt{\tfrac{29}{38}}\tfrac{5}{\sqrt{29}}\hat{e}_\rho + \sqrt{\tfrac{9}{38}}\tfrac{5}{\sqrt{29}}\hat{e}_\phi + \tfrac{2}{\sqrt{29}}\hat{e}_\theta\right)$$
$$+ 3\left(\sqrt{\tfrac{9}{38}}\hat{e}_\rho - \sqrt{\tfrac{29}{38}}\hat{e}_\phi\right) = \sqrt{38}\,\hat{e}_\rho.$$
Actually, we could have obtained these results ($\underset{\sim}{v} = \sqrt{38}\,\hat{e}_\rho$, $\underset{\sim}{a} = \underset{\sim}{0}$) more simply by noticing that $x = 2t, y = 5t, z = 3t$ gives a constant-velocity motion purely in the ρ-direction. Since the speed is $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = \sqrt{38}$, then $\underset{\sim}{v} = \sqrt{38}\,\hat{e}_\rho$ and $\underset{\sim}{a} = \underset{\sim}{0}$.

(d) $\underset{\sim}{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath} + 6t\,\hat{k}$
so $\underset{\sim}{v} = \dot{\underset{\sim}{R}} = -3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 6\hat{k}$
Now express $\hat{\imath}, \hat{\jmath}, \hat{k}$ in terms of $t, \hat{e}_\rho, \hat{e}_\phi, \hat{e}_\theta$ using (3.2). We'll need $\sin\phi, \cos\phi, \sin\theta, \cos\theta$.
$\sin\phi = \sqrt{x^2+y^2}/\rho = \sqrt{x^2+y^2}/\sqrt{x^2+y^2+z^2} = 3/\sqrt{9+36t^2} = 1/\sqrt{1+4t^2}$
$\cos\phi = \sqrt{1-\sin^2\phi} = 2t/\sqrt{1+4t^2}$
$\sin\theta = \sin t$ and $\cos\theta = \cos t$
so (3.2) gives $\hat{\imath} = \frac{\cos t}{\sqrt{1+4t^2}}\hat{e}_\rho + \frac{2t\cos t}{\sqrt{1+4t^2}}\hat{e}_\phi - \sin t\,\hat{e}_\theta$,

$$\hat{j} = \frac{\sin t}{\sqrt{1+4t^2}}\,\hat{e}_\rho + \frac{2t\sin t}{\sqrt{1+4t^2}}\,\hat{e}_\phi + \cos t\,\hat{e}_\theta,$$

$$\hat{k} = \frac{2t}{\sqrt{1+4t^2}}\,\hat{e}_\rho - \frac{1}{\sqrt{1+4t^2}}\,\hat{e}_\phi$$

so $\underset{\sim}{v} = \frac{12t}{\sqrt{1+4t^2}}\,\hat{e}_\rho - \frac{6}{\sqrt{1+4t^2}}\,\hat{e}_\phi + 3\hat{e}_\theta$

$$\underset{\sim}{a} = \ddot{\underset{\sim}{R}} = -3\cos t\,\hat{\imath} - 3\sin t\,\hat{j} = -\frac{3}{\sqrt{1+4t^2}}\,\hat{e}_\rho - \frac{6t}{\sqrt{1+4t^2}}\,\hat{e}_\phi$$

(e) $\underset{\sim}{R} = \hat{\imath} + 4t\,\hat{j} + 2\hat{k}$

$\underset{\sim}{v} = \dot{\underset{\sim}{R}} = 4\hat{j}$. Now use (3.2) to express $\hat{j}$ in terms of $t, \hat{e}_\rho, \hat{e}_\phi, \hat{e}_\theta$. We'll need

$\sin\phi = \sqrt{x^2+y^2}/\sqrt{x^2+y^2+z^2} = \sqrt{1+16t^2}/\sqrt{5+16t^2}$

$\cos\phi = \sqrt{1-\sin^2\phi} = 2/\sqrt{5+16t^2}$

$\sin\theta = y/\sqrt{x^2+y^2} = 4t/\sqrt{1+16t^2}$

$\cos\theta = x/\sqrt{x^2+y^2} = 1/\sqrt{1+16t^2}$.

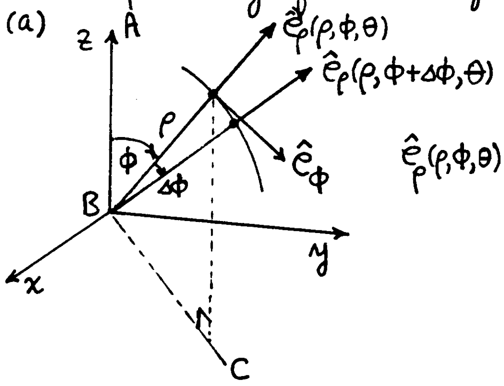
Then (3.2) gives

$$\underset{\sim}{v} = 4\hat{j} = \frac{16t}{\sqrt{5+16t^2}}\,\underset{\sim}{\hat{e}}_\rho + \frac{32t}{\sqrt{(5+16t^2)(1+16t^2)}}\,\underset{\sim}{\hat{e}}_\phi + \frac{4}{\sqrt{1+16t^2}}\,\underset{\sim}{\hat{e}}_\theta,$$

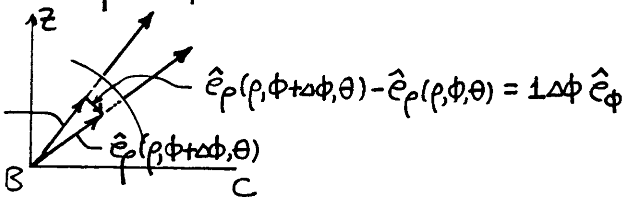
$$\underset{\sim}{a} = \frac{d}{dt}(4\hat{j}) = \underset{\sim}{0}.$$

7. These are tricky, if only because of the 3-dim. sketches needed, and they reveal the superiority of the "omega method."

(a)



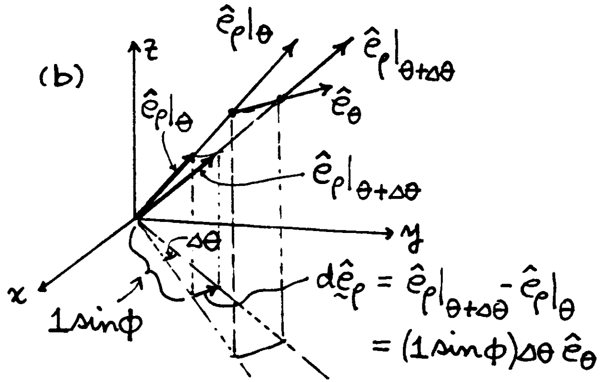
View of ABC plane:



$$\frac{\partial \hat{e}_\rho}{\partial\phi} = \lim_{\Delta\phi\to 0} \frac{\hat{e}_\rho(\rho,\phi+\Delta\phi,\theta) - \hat{e}_\rho(\rho,\phi,\theta)}{\Delta\phi}$$

$$= \lim_{\Delta\phi\to 0} \frac{1\Delta\phi}{\Delta\phi}\,\hat{e}_\phi = \hat{e}_\phi$$

(b)

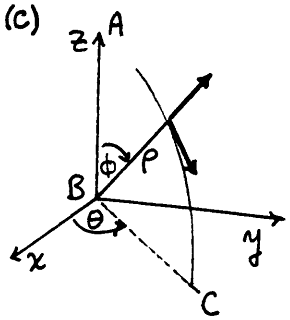


$$\frac{\partial \hat{e}_\rho}{\partial\theta} = \lim_{\Delta\theta\to 0} \frac{\hat{e}_\rho(\rho,\phi,\theta+\Delta\theta) - \hat{e}_\rho(\rho,\phi,\theta)}{\Delta\theta}$$

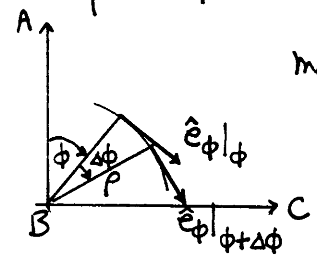
$$= \lim_{\Delta\theta\to 0} \frac{1\sin\phi\,\cancel{\Delta\theta}}{\cancel{\Delta\theta}}\,\hat{e}_\theta = \sin\phi\,\hat{e}_\theta$$

Partial checks: When $\phi=0$, $\hat{e}_\rho$ doesn't vary with θ so $\partial\hat{e}_\rho/\partial\theta$ should $= \underset{\sim}{0}$. ✓ When $\phi=\pi/2$, then $\hat{e}_\rho$ is the same as the plane polar, so we should have $\partial\hat{e}_\rho/\partial\theta = \hat{e}_\theta$, and we do (because $\sin\phi = \sin\pi/2 = 1$). ✓

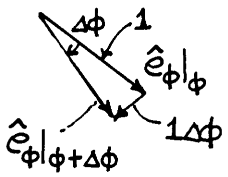
(c)



View of ABC plane:

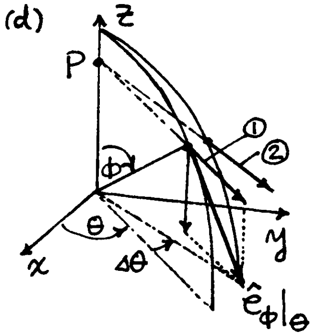


Move $\hat{e}_\phi|_\phi$ and $\hat{e}_\phi|_{\phi+\Delta\phi}$ to be tail-to-tail:



$$\frac{\partial\hat{e}_\phi}{\partial\phi} = \lim_{\Delta\phi\to 0} \frac{\hat{e}_\phi|_{\phi+\Delta\phi} - \hat{e}_\phi|_\phi}{\Delta\phi} = \lim_{\Delta\phi\to 0}\frac{(1\cancel{\Delta\phi})(-\hat{e}_\rho)}{\cancel{\Delta\phi}} = -\hat{e}_\rho$$

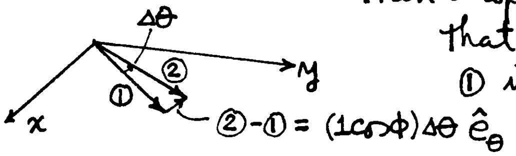
(d)

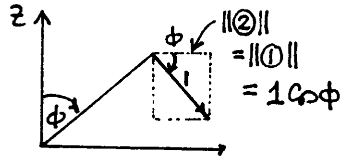


Break $\hat{e}_\phi$ into a component parallel to the x,y plane plus a component perpendicular to it. The latter does not vary as θ is varied, but the former does. Thus,

$$\frac{\partial\hat{e}_\phi}{\partial\theta} = \lim_{\Delta\theta\to 0}\frac{②-①}{\Delta\theta},$$

where the vectors "①" and "②" are shown at the left. To evaluate ②-①, slide them back until they meet at P, then drop them down to the x,y plane. Also, note that the length of ② and ① is (see figure at right)



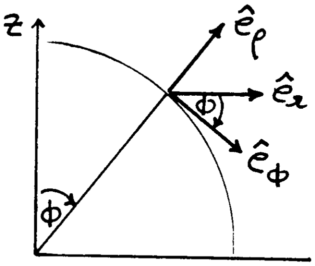


$$\text{so}\quad \frac{\partial\hat{e}_\phi}{\partial\theta} = \lim_{\Delta\theta\to 0}\frac{1\cos\phi\,\cancel{\Delta\theta}}{\cancel{\Delta\theta}}\,\hat{e}_\theta = \cos\phi\,\hat{e}_\theta$$

(e) This one is hard. However, if we realize that $\hat{e}_\theta$ in spherical coordinates is the same as $\hat{e}_\theta$ in plane polar (or cylindrical) coordinates, then we already know from Fig. 3 and equations (15) and (16) that $\partial\hat{e}_\theta/\partial\theta = -\hat{e}_r$, so all we need to do is to express $\hat{e}_r$ in terms of spherical coordinates. From the figure at the right we see that

$$\hat{e}_r = \cos\phi\,\hat{e}_\phi + \sin\phi\,\hat{e}_\rho$$

$$\text{so}\quad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r = -\cos\phi\,\hat{e}_\phi - \sin\phi\,\hat{e}_\rho$$



NOTE: Alternatively, though not using the difference-quotient method, we can do this one as follows: $\hat{e}_\theta = \hat{e}_\rho \times \hat{e}_\phi$

$$\text{so}\quad \partial\hat{e}_\theta/\partial\theta = \partial\hat{e}_\rho/\partial\theta \times \hat{e}_\phi + \hat{e}_\rho \times \partial\hat{e}_\phi/\partial\theta$$

$$= (\sin\phi\,\hat{e}_\theta)\times\hat{e}_\phi + \hat{e}_\rho \times (\cos\phi\,\hat{e}_\theta) = \text{etc.}$$

8. $\underset{\sim}{v} = \dot\rho\,\hat e_\rho + \rho\dot\phi\,\hat e_\phi + \rho\dot\theta\sin\phi\,\hat e_\theta$

$$\underset{\sim}{a} = \dot{\underset{\sim}{v}} = \ddot\rho\,\hat e_\rho + \dot\rho\left(\frac{\partial\hat e_\rho}{\partial\phi}\dot\phi + \frac{\partial\hat e_\rho}{\partial\theta}\dot\theta\right) + (\dot\rho\dot\phi + \rho\ddot\phi)\hat e_\phi + \rho\dot\phi\left(\frac{\partial\hat e_\phi}{\partial\phi}\dot\phi + \frac{\partial\hat e_\phi}{\partial\theta}\dot\theta\right)$$
$$+ (\dot\rho\dot\theta\sin\phi + \rho\ddot\theta\sin\phi + \rho\dot\theta\dot\phi\cos\phi)\hat e_\theta + \rho\dot\theta\sin\phi\left(\frac{\partial\hat e_\theta}{\partial\phi}\dot\phi + \frac{\partial\hat e_\theta}{\partial\theta}\dot\theta\right),$$

then use (28).

9. (a) $\underset{\sim}{R}(t) = x\hat\imath + y\hat\jmath + z\hat k = s\sin\frac{\pi}{4}\cos\Omega t\,\hat\imath + s\sin\frac{\pi}{4}\sin\Omega t\,\hat\jmath + s\cos\frac{\pi}{4}\hat k$

$$= s(\cos\Omega t\,\hat\imath + \sin\Omega t\,\hat\jmath + \hat k)/\sqrt2$$

$$\underset{\sim}{v}(t) = \dot R = \frac{\dot s}{\sqrt2}(\cos\Omega t\,\hat\imath + \sin\Omega t\,\hat\jmath + \hat k) + \frac{s}{\sqrt2}(-\Omega\sin\Omega t\,\hat\imath + \Omega\cos\Omega t\,\hat\jmath)$$
$$= \frac{1}{\sqrt2}(V\cos\Omega t - \Omega Vt\sin\Omega t)\hat\imath + \frac{1}{\sqrt2}(V\sin\Omega t + \Omega Vt\cos\Omega t)\hat\jmath + \frac{V}{\sqrt2}\hat k$$

$$\underset{\sim}{a}(t) = \dot{\underset{\sim}{v}} = -\frac{\Omega}{\sqrt2}(V\sin\Omega t + V\sin\Omega t + \Omega Vt\cos\Omega t)\hat\imath + \frac{\Omega}{\sqrt2}(V\cos\Omega t + V\cos\Omega t - \Omega Vt\sin\Omega t)\hat\jmath$$
$$= -\frac{\Omega V}{\sqrt2}(2\sin\Omega t + \Omega t\cos\Omega t)\hat\imath + \frac{\Omega V}{\sqrt2}(2\cos\Omega t - \Omega t\sin\Omega t)\hat\jmath$$

(b) $r = s/\sqrt2 = Vt/\sqrt2$, $\theta = \Omega t$, $z = s/\sqrt2 = Vt/\sqrt2$, so (23) and (24) give

$$\underset{\sim}{v}(t) = \frac{V}{\sqrt2}\hat e_r + \frac{V\Omega t}{\sqrt2}\hat e_\theta + \frac{V}{\sqrt2}\hat e_z \quad\text{and}\quad \underset{\sim}{a}(t) = \left(0 - \frac{Vt}{\sqrt2}\Omega^2\right)\hat e_r + \left(0 + 2\frac{V}{\sqrt2}\Omega\right)\hat e_\theta + 0\hat e_z$$
$$= -\frac{V}{\sqrt2}\Omega^2 t\,\hat e_r + \sqrt2 V\Omega\,\hat e_\theta$$

(c) $\rho = s = Vt$, $\theta = \Omega t$, $\phi = \pi/4$ so (30), (31) give

$$\underset{\sim}{v}(t) = V\hat e_\rho + \frac{V}{\sqrt2}\Omega t\,\hat e_\theta\,,$$
$$\underset{\sim}{a}(t) = \left(0 - 0 - Vt\Omega^2\tfrac12\right)\hat e_\rho + \left(0 + 0 - \frac{Vt\Omega^2}{2}\right)\hat e_\phi + \left(0 + 2V\Omega\tfrac{1}{\sqrt2} + 0\right)\hat e_\theta$$
$$= -\frac{V}{2}\Omega^2 t\,\hat e_\rho - \frac{V}{2}\Omega^2 t\,\hat e_\phi + \sqrt2 V\Omega\,\hat e_\theta$$
$$= -\frac{V\Omega^2 t}{2}(\hat e_\rho + \hat e_\phi) + \sqrt2 V\Omega\,\hat e_\theta$$

10. $\rho = \text{constant}$, $\phi = \Omega_1 t$, $\theta = \Omega_2 t$, so (30), (31) give

$$\underset{\sim}{v}(t) = \rho\Omega_1\,\hat e_\phi + \rho\Omega_2\sin\Omega_1 t\,\hat e_\theta\,.$$

at the equator $\phi = \Omega_1 t = \pi/2$ so $\underset{\sim}{v}|_{\text{equator}} = \rho\Omega\,\hat e_\phi + \rho\Omega_2\,\hat e_\theta$.

also,

$$\underset{\sim}{a}(t) = (0 - \rho\Omega_1^2 - \rho\Omega_2^2\sin^2\phi)\hat e_\rho + (0 + 0 - \rho\Omega_2^2\sin\phi\cos\phi)\hat e_\phi + (0 + 0 + 2\rho\Omega_1\Omega_2\cos\phi)\hat e_\theta$$

so $\underset{\sim}{a}|_{\text{equator}} = -\rho(\Omega_1^2 + \Omega_2^2)\hat e_\rho$.

Thus,
$$\|\underset{\sim}{v}|_{\text{equator}}\| = \rho\sqrt{\Omega_1^2 + \Omega_2^2}$$
$$\|\underset{\sim}{a}|_{\text{equator}}\| = \rho(\Omega_1^2 + \Omega_2^2)$$

11. Let it suffice to consider $\hat e_\rho$. $\|\hat e_\rho\| = 1$, so $\hat e_\rho \cdot \hat e_\rho = 1$. A differential of this gives $2\hat e_\rho \cdot d\hat e_\rho$, or, $\hat e_\rho \cdot d\hat e_\rho = 0 = \hat e_\rho \cdot \left(\frac{\partial\hat e_\rho}{\partial\rho}d\rho + \frac{\partial\hat e_\rho}{\partial\phi}d\phi + \frac{\partial\hat e_\rho}{\partial\theta}d\theta\right)$. Since the $d\rho, d\phi$, $d\theta$ increments are arbitrary it follows that

$$\hat e_\rho \cdot \partial\hat e_\rho/\partial\rho = 0,\quad \hat e_\rho \cdot \partial\hat e_\rho/\partial\phi = 0,\quad \hat e_\rho \cdot \partial\hat e_\rho/\partial\theta = 0\,.$$

Thus, the ρ components of $\partial\hat e_\rho/\partial\rho$, $\partial\hat e_\rho/\partial\phi$, $\partial\hat e_\rho/\partial\theta$ are zero. In words, if they were not, then that would correspond to a rate of change of $\hat e_\rho$ along the direction of $\hat e_\rho$; i.e., it would have a nonzero rate of change of its length whereas $\hat{e}_\rho$ is a unit vector of fixed length.

12. Consider the existence of a vector $\underset{\sim}{\Omega}$ such that $\underset{\sim}{\Omega} \times \underset{\sim}{u} = \underset{\sim}{v}$; i.e.,

$$\begin{aligned} \hat{\imath}: &\quad u_3\Omega_2 - u_2\Omega_3 = v_1 \\ \hat{\jmath}: &\quad -u_3\Omega_1 + u_1\Omega_3 = v_2 \\ \hat{k}: &\quad u_2\Omega_1 - u_1\Omega_2 = v_3 \end{aligned}$$

Let us try to solve by Gauss elimination:

$$\begin{pmatrix} 0 & u_3 & -u_2 & v_1 \\ -u_3 & 0 & u_1 & v_2 \\ u_2 & -u_1 & 0 & v_3 \end{pmatrix} \rightarrow \begin{pmatrix} u_2 & -u_1 & 0 & v_3 \\ -u_3 & 0 & u_1 & v_2 \\ 0 & u_3 & -u_2 & v_1 \end{pmatrix} \rightarrow \begin{pmatrix} u_2 & -u_1 & 0 & v_3 \\ 0 & -u_3u_1 & u_2u_1 & u_3v_3+u_2v_2 \\ 0 & u_3 & -u_2 & v_1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} u_2 & -u_1 & 0 & v_3 \\ 0 & -u_3u_1 & u_2u_1 & u_3v_3+u_2v_2 \\ 0 & 0 & 0 & u_3v_3+u_2v_2+u_1v_1 \end{pmatrix},$$

which does have a (nonunique) solution if $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$, provided that $u_3v_3+u_2v_2+u_1v_1 = = \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, as assumed. Further, we see that there is a solution even if any one or any two components of $\underset{\sim}{u}$ are zero. For instance, suppose $u_1 = 0$ and u_2, u_3 are not 0. Then the original system becomes

$$\begin{pmatrix} 0 & u_3 & -u_2 & v_1 \\ -u_3 & 0 & 0 & v_2 \\ u_2 & 0 & 0 & v_3 \end{pmatrix} \rightarrow \begin{pmatrix} u_2 & 0 & 0 & v_3 \\ 0 & u_3 & -u_2 & v_1 \\ 0 & 0 & 0 & u_3v_3+u_2v_2 \end{pmatrix}$$

which is consistent if, again, $u_3v_3+u_2v_2 = u_3v_3+u_2v_2+\overset{0}{\cancel{u_1}}v_1 = 0$.

15. (a) On the one hand, $\frac{d\hat{e}_\phi}{dt} = \underset{\sim}{\Omega} \times \hat{e}_\phi = [\dot{\phi}\hat{e}_\theta + \dot{\theta}(\cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi)] \times \hat{e}_\phi$
$= -\dot{\phi}\,\hat{e}_\rho + \dot{\theta}\cos\phi\,\hat{e}_\theta$

On the other hand, $\frac{d\hat{e}_\phi}{dt}[\rho(t),\phi(t),\theta(t)] = \frac{\partial \hat{e}_\phi}{\partial \rho}\dot{\rho} + \frac{\partial \hat{e}_\phi}{\partial \phi}\dot{\phi} + \frac{\partial \hat{e}_\phi}{\partial \theta}\dot{\theta}$.

Since $\dot{\rho}, \dot{\phi}, \dot{\theta}$ are arbitrary, it follows from these that

$\dot{\rho}$ terms: $\partial\hat{e}_\phi/\partial\rho = \underset{\sim}{0}$

$\dot{\phi}$ terms: $\partial\hat{e}_\phi/\partial\phi = -\hat{e}_\rho$

$\dot{\theta}$ terms: $\partial\hat{e}_\phi/\partial\theta = \cos\phi\,\hat{e}_\theta$

(b) On the one hand, $\frac{d\hat{e}_\theta}{dt} = \underset{\sim}{\Omega} \times \hat{e}_\theta = [\dot{\phi}\hat{e}_\theta + \dot{\theta}(\cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi)] \times \hat{e}_\theta$
$= -\dot{\theta}\cos\phi\,\hat{e}_\phi - \dot{\theta}\sin\phi\,\hat{e}_\rho$

On the other hand, $\frac{d\hat{e}_\theta}{dt}[\rho(t),\phi(t),\theta(t)] = \frac{\partial \hat{e}_\theta}{\partial \rho}\dot{\rho} + \frac{\partial \hat{e}_\theta}{\partial \phi}\dot{\phi} + \frac{\partial \hat{e}_\theta}{\partial \theta}\dot{\theta}$.

Since $\dot{\rho}, \dot{\phi}, \dot{\theta}$ are arbitrary, it follows from these that

$\dot{\rho}$ terms: $\partial\hat{e}_\theta/\partial\rho = \underset{\sim}{0}$

$\dot{\phi}$ terms: $\partial\hat{e}_\theta/\partial\phi = \underset{\sim}{0}$

$\dot{\theta}$ terms: $\partial\hat{e}_\theta/\partial\theta = -\sin\phi\,\hat{e}_\rho - \cos\phi\,\hat{e}_\phi$

CHAPTER 15

Section 15.2

1. (b) $4x-3y+5z=1$

$x+y-2z=6$. Gauss elimination gives $x=(19+\alpha)/7$, $y=(23+13\alpha)/7$, $z=\alpha$. These are parametric equations of the line, the parameter being α, for $-\infty<\alpha<\infty$.

(c) $2x+y+z=3$

$x+4z=5$. Gauss elimination gives $x=5-4\alpha$, $y=-7+7\alpha$, $z=\alpha$. These are parametric equations of the line, the parameter being α, for $-\infty<\alpha<\infty$.

(g) $4x-y+z=3$

$x^2+z^2=1$. The latter suggests letting $x=\cos\tau$, $z=\sin\tau$. Then the first equation gives $y=4\cos\tau+\sin\tau-3$. These are parametric equations of the curve, for $0\le\tau<2\pi$, say.

(h) $x-y+2z=1$

$x^2+4y^2=4$. The latter suggests letting $x=2\cos\tau$, $y=\sin\tau$. Then the first equation gives $z=(1+\sin\tau-2\cos\tau)/2$, for $0\le\tau<2\pi$.

(i) $2x-y-3z=5$

$z=x^2+y+1$. These are linear equations on y and z. Solving $\begin{matrix} y+3z=2x-5 \\ -y+z=x^2+1 \end{matrix}$ so $z=(x^2+2x-4)/4$, $y=(-3x^2+2x-8)/4$ so, letting x be τ, a set of parametric equations for the curve is

$$x=\tau,\quad y=(-3\tau^2+2\tau-8)/4,\quad z=(\tau^2+2\tau-4)/4 \quad \text{for } 0<\tau<2,$$

where the τ limits follow from the endpoints $(0,-2,-1)$ and $(2,-4,1)$.

(j) $x-y^2+z=0$

$x-y+2z=0$. Solve these for x and z in terms of y, and let $y=\tau$. That gives $x=2y^2-y$ and $z=y-y^2$, so $x=2\tau^2-\tau$, $y=\tau$, $z=\tau-\tau^2$ for $1\le\tau\le2$.

(k) $x=y^2+z^2$

$z=xy$. Putting the 2nd into the 1st gives $x=y^2+x^2y^2$ so $y=\pm\sqrt{\dfrac{x}{1+x^2}}$ Of the two signs, choose the minus because we seek the curve between $(0,0,0)$ and $(1,-1/\sqrt{2},-1/\sqrt{2})$. Letting $x=\tau$, we have

$$x=\tau,\quad y=-\sqrt{\frac{\tau}{1+\tau^2}},\quad z=-\tau\sqrt{\frac{\tau}{1+\tau^2}} \quad \text{for } 0\le\tau\le1.$$

2. We can (but don't have to) use (7).

(b) $x=1+2\tau$, $y=2+5\tau$, $z=3-7\tau$ for $0\le\tau\le1$

(c) $x=8-8\tau$, $y=1+27\tau$, $z=7+5\tau$ for $0\le\tau\le1$

(d) $x=1-7\tau$, $y=0$, $z=0$ for $0\le\tau\le1$

3. Use (9), $s(\tau)=\int_{\tau_0}^{\tau}\sqrt{\underset{\sim}{R}'(t)\cdot\underset{\sim}{R}'(t)}\,dt$.

(b) $\underset{\sim}{R}(\tau)=\cos\tau\,\hat{\imath}+4\tau\,\hat{\jmath}+\sin\tau\,\hat{k}$, so $s(\tau)=\int_0^{\tau}\sqrt{\sin^2 t+16+\cos^2 t}\,dt=\sqrt{17}\,\tau$.

(c) $\underset{\sim}{R}(\tau) = \tau\hat{\imath} - 3\tau\hat{\jmath} + 5(\tau+4)\hat{k}$, so $s(\tau) = \int_0^\tau \sqrt{1+9+25}\, dt = \sqrt{35}\,\tau$

(d) $\underset{\sim}{R}(\tau) = \cos\tau(\hat{\imath}+\hat{\jmath}) - \sqrt{2}\sin\tau\,\hat{k}$, so $s(\tau) = \int_0^\tau \sqrt{\sin^2 t + \sin^2 t + 2\cos^2 t}\, dt = \sqrt{2}\,\tau$

(e) $\underset{\sim}{R}(\tau) = \hat{\imath} - 7\tau^2\hat{\jmath} + 3\hat{k}$, so $s(\tau) = \int_0^\tau \sqrt{(-14t)^2}\, dt = \int_0^\tau 14t\,dt = 7\tau^2$

(f) $\underset{\sim}{R}(\tau) = \sin\tau\,\hat{\imath} + \cos\tau\,\hat{\jmath} - \cos\tau\,\hat{k}$, so $s(\tau) = \int_0^\tau \sqrt{\cos^2 t + 2\sin^2 t}\, dt = \int_0^\tau \sqrt{1+\sin^2 t}\, dt$

Though it looks simple, the integral

$$s(\tau) = \int_0^\tau \sqrt{1+\sin^2 t}\, dt$$

is nonelementary, and the int Maple command gives no response. It turns out that $s(\tau)$ can be evaluated in terms of nonelementary functions called elliptic integrals, which are tabulated functions. We proceed as follows:

$$\begin{aligned} s(\tau) = \int_0^\tau \sqrt{\cos^2 t + 2\sin^2 t}\, dt &= \int_0^\tau \sqrt{2-\cos^2 t}\, dt \quad (\text{now set } t = \tfrac{\pi}{2} - \phi) \\ &= \int_{\pi/2}^{\pi/2-\tau} \sqrt{2-\sin^2\phi}\,(-d\phi) \\ &= \sqrt{2}\int_{\pi/2-\tau}^{\pi/2} \sqrt{1-\tfrac{1}{2}\sin^2\phi}\, d\phi \\ &= \sqrt{2}\left\{ \int_0^{\pi/2} \sqrt{1-\tfrac{1}{2}\sin^2\phi}\, d\phi - \int_0^{\pi/2-\tau} \sqrt{1-\tfrac{1}{2}\sin^2\phi}\, d\phi \right\} \\ &= \sqrt{2}\left\{ E(\tfrac{1}{\sqrt{2}}) - E(\tfrac{1}{\sqrt{2}}, \tfrac{\pi}{2} - \tau) \right\}, \end{aligned}$$

where

$$E(k,\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\, d\phi \qquad (0 \le k \le 1)$$

is Legendre's normal form of the **elliptic integral of the second kind**, as a function of the **amplitude** ϕ and the **modulus** k. If $\phi = \pi/2$ then the integral is said to be **complete**. The complete elliptic integral of the second kind, $E(k,\pi/2)$, is denoted more simply as $E(k)$:

$$E(k,\pi/2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\, d\phi \equiv E(k).$$

Alternative to solving for $s(\tau)$ in terms of elliptic integrals, we could solve for it as an infinite series of elementary functions by using the Taylor series

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{2\cdot4}x^2 - \frac{1\cdot1\cdot3}{2\cdot4\cdot6}x^3 - \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}x^4 - \cdots$$

and writing

$$s(\tau) = \sqrt{2}\int_{\pi/2-\tau}^{\pi/2} (1 - \overset{``x"}{\boxed{\tfrac{1}{2}\sin^2\phi}})^{1/2}\, d\phi$$

$$= \sqrt{2}\left\{ \int_{\frac{\pi}{2}-\tau}^{\frac{\pi}{2}} 1\,d\phi - \frac{1}{2}\int_{\frac{\pi}{2}-\tau}^{\frac{\pi}{2}} \frac{1}{2}\sin^2\phi\, d\phi - \frac{1}{2\cdot4}\int_{\frac{\pi}{2}-\tau}^{\frac{\pi}{2}} (\tfrac{1}{4})\sin^4\phi\, d\phi - \cdots \right\},$$

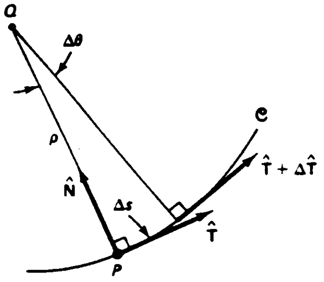
and using

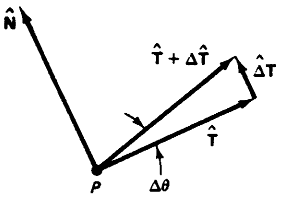
$$\int_a^b \sin^n\phi\, d\phi = -\frac{1}{n}\sin^{n-1}\phi\cos\phi\Big|_a^b + \frac{n-1}{n}\int_a^b \sin^{n-2}\phi\, d\phi.$$

4. (a) $\|d\underset{\sim}{R}\|$ is ds, so $d\underset{\sim}{R}/ds$ is normalized to unit length.

(b) Easy: if $\hat{\underset{\sim}{T}}(s)\cdot\hat{\underset{\sim}{T}}(s) = 1$ then d/ds gives $\hat{\underset{\sim}{T}}'\cdot\hat{\underset{\sim}{T}} + \hat{\underset{\sim}{T}}\cdot\hat{\underset{\sim}{T}}' = 0$ or $2\hat{\underset{\sim}{T}}\cdot\hat{\underset{\sim}{T}}' = 0$ or $\hat{\underset{\sim}{T}}\cdot\hat{\underset{\sim}{T}}' = 0$.

(c)





Denote $\underset{\sim}{\hat{T}}(s+\Delta s)$ as $\underset{\sim}{\hat{T}}+\Delta\underset{\sim}{\hat{T}}$. We see from the figure that

$$\Delta\underset{\sim}{\hat{T}} \sim (\|\underset{\sim}{\hat{T}}\|\Delta\theta)\underset{\sim}{\hat{N}} = \underset{\sim}{\hat{N}}\Delta\theta.$$

Alternatively, (4.3) gives

$$\Delta\underset{\sim}{\hat{T}} \sim \kappa\underset{\sim}{\hat{N}}\Delta s,$$

so $\underset{\sim}{\hat{N}}\Delta\theta \sim \kappa\underset{\sim}{\hat{N}}\Delta s$, so $\Delta\theta \sim \kappa\Delta s$. But $\Delta s \sim \rho\Delta\theta$, so it follows that $\kappa = 1/\rho$. ✓

(d) $\underset{\sim}{\hat{B}} \equiv \underset{\sim}{\hat{T}}\times\underset{\sim}{\hat{N}}$, so $\underset{\sim}{\hat{B}}' = \underset{\sim}{\hat{T}}'\times\underset{\sim}{\hat{N}} + \underset{\sim}{\hat{T}}\times\underset{\sim}{\hat{N}}' = \kappa\underset{\sim}{\hat{N}}\times\underset{\sim}{\hat{N}} + \underset{\sim}{\hat{T}}\times\underset{\sim}{\hat{N}}'$ per (4.3)

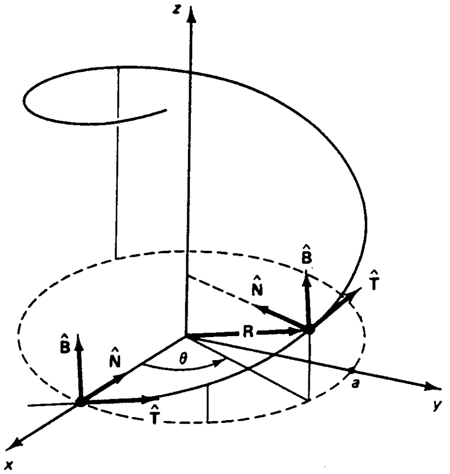
$= \underset{\sim}{0} + \underset{\sim}{\hat{T}}\times\underset{\sim}{\hat{N}}'$ ✓

(e) $\underset{\sim}{\hat{N}} \equiv \underset{\sim}{\hat{B}}\times\underset{\sim}{\hat{T}}$, so $\underset{\sim}{\hat{N}}' = \underset{\sim}{\hat{B}}'\times\underset{\sim}{\hat{T}} + \hat{B}\times\hat{T}' = \nu\underset{\sim}{\hat{N}}\times\underset{\sim}{\hat{T}} + \underset{\sim}{\hat{B}}\times\kappa\underset{\sim}{\hat{N}}$ per (4.6) and (4.3)

$= \nu(-\underset{\sim}{\hat{B}}) + \kappa(-\underset{\sim}{\hat{T}})$ since $\underset{\sim}{\hat{T}}, \hat{N}, \hat{B}$ is a right-handed ON set.

5. $\underset{\sim}{R} = a\cos\tau\,\hat{\imath} + a\sin\tau\,\hat{\jmath} + b\tau\,\hat{k} \quad (0 \le \tau < \infty)$

so
$$s(\tau) = \int_0^\tau \sqrt{\underset{\sim}{R}'(t)\cdot\underset{\sim}{R}'(t)}\,dt = \int_0^\tau (a^2\sin^2 t + a^2\cos^2 t + b^2)^{1/2}\,dt = \int_0^\tau (a^2+b^2)^{1/2}\,dt = \sqrt{a^2+b^2}\,\tau$$



Next,
$$\underset{\sim}{\hat{T}} = d\underset{\sim}{R}/ds = \frac{d\underset{\sim}{R}}{d\tau}\frac{d\tau}{ds} = (-a\sin\tau\,\hat{\imath} + a\cos\tau\,\hat{\jmath} + b\hat{k})\frac{1}{\sqrt{a^2+b^2}} = c(-a\sin\tau\,\hat{\imath} + a\cos\tau\,\hat{\jmath} + b\hat{k})$$

where $c \equiv 1/\sqrt{a^2+b^2}$. Then
$$\frac{d^2\underset{\sim}{R}}{ds^2} = \frac{d}{d\tau}\left(\frac{d\underset{\sim}{R}}{ds}\right)\frac{d\tau}{ds} = (-a\cos\tau\,\hat{\imath} - a\sin\tau\,\hat{\jmath})c^2$$

so
$$\kappa = \frac{1}{\rho} = \|d^2\underset{\sim}{R}/ds^2\| = \sqrt{a^2\cos^2\tau + a^2\sin^2\tau}\;c^2 = ac^2 = a/(a^2+b^2)$$

$\underset{\sim}{\hat{N}} = \underset{\sim}{R}''/\|\underset{\sim}{R}''\|$ since, from (4.2) and (4.3), $\underset{\sim}{\hat{N}}$ is a normalized version of $\underset{\sim}{R}''$

$= -\cos\tau\,\hat{\imath} - \sin\tau\,\hat{\jmath}$.

Then
$$\underset{\sim}{\hat{B}} = \underset{\sim}{\hat{T}}\times\underset{\sim}{\hat{N}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -ac\sin\tau & +ac\cos\tau & bc \\ -\cos\tau & -\sin\tau & 0 \end{vmatrix} = c(b\sin\tau\,\hat{\imath} - b\cos\tau\,\hat{\jmath} + a\hat{k})$$

$\rho = 1/\kappa = (a^2+b^2)/a$.

From (4.6) we see that the torsion ν is the scale factor between $d\underset{\sim}{\hat{B}}/ds$ and $\hat{N}$.

Since
$$\frac{d\underset{\sim}{\hat{B}}}{ds} = \frac{d\hat{B}}{d\tau}\frac{d\tau}{ds} = c(bc\cos\tau\,\hat{\imath} + b\sin\tau\,\hat{\jmath})\frac{1}{\sqrt{a^2+b^2}} = c^2b(\cos\tau\,\hat{\imath} + \sin\tau\,\hat{\jmath})$$

we see, from our expressions for $d\underset{\sim}{\hat{B}}/ds$ and $\underset{\sim}{\hat{N}}$ and from (4.6), that
$$\nu = -b/(a^2+b^2).$$

Note that the curvature $\kappa(s)$, the radius of curvature $\rho(s)$, and the torsion $\nu(s)$ are all constant (i.e., they don't vary with s) for a helix.

6. $\hat{\utilde{T}}' = \kappa \hat{\utilde{N}}$ (4.8a)

$\hat{\utilde{N}}' = -\kappa \hat{\utilde{T}} - \nu \hat{\utilde{B}}$ (4.8b)

$\hat{\utilde{B}}' = \nu \hat{\utilde{N}}$ (4.8c)

(4.8a) and (4.8c) give $\nu \hat{\utilde{T}}' = \kappa \hat{\utilde{B}}'$. ①

Putting d^2/ds^2 of (4.8a) into d/ds of (4.8b) gives

$$\hat{\utilde{T}}''' = \kappa \hat{\utilde{N}}'' = \kappa(-\kappa \hat{\utilde{T}}' - \nu \hat{\utilde{B}}'). \qquad ②$$

Then, putting ① into ② gives

$$\hat{\utilde{T}}''' = -\kappa^2 \hat{\utilde{T}}' - \nu^2 \hat{\utilde{T}}',$$

or, $\hat{\utilde{T}}''' + \omega^2 \hat{\utilde{T}}' = \utilde{0}$ where $\omega^2 = \kappa^2 + \nu^2$.

Thus,

$$\hat{\utilde{T}} = \utilde{A} \sin\omega s + \utilde{B} \cos\omega s + \utilde{C}.$$ ($\utilde{A}, \utilde{B}, \utilde{C}$ are arbitrary constant vectors)

Then

$$\utilde{R} = \int \hat{T}\, ds = -\utilde{A}\frac{\cos\omega s}{\omega} + \utilde{B}\frac{\sin\omega s}{\omega} + \utilde{C}s + \utilde{D},$$

or,

$$\utilde{R} = \utilde{E}\cos\omega s + \utilde{F}\sin\omega s + \utilde{G}s + \utilde{H},$$

where $\utilde{E}, \utilde{F}, \utilde{G}, \utilde{H}$ are vector constants of integration. This establishes (6.2).

For the helix in Exercise 5, recall that

$$\utilde{R} = a\cos\tau\, \hat{\imath} + a\sin\tau\, \hat{\jmath} + b\tau\, \hat{k}$$

but $\tau = s/\sqrt{a^2+b^2}$. also,

$$\kappa = a/(a^2+b^2), \quad \nu = -b/(a^2+b^2)$$

so $\omega^2 = \kappa^2 + \nu^2 = 1/(a^2+b^2)$.

Thus, $\tau = \omega s$,

so $\utilde{R} = \underbrace{(a\hat{\imath})}_{\utilde{E}}\cos\omega s + \underbrace{(a\hat{\jmath})}_{\utilde{F}}\sin\omega s + \underbrace{(b\omega\hat{k})}_{\utilde{G}}s + \underbrace{\utilde{0}}_{\utilde{H}}$ ✓

7.(a) $\utilde{v} = \frac{d\utilde{R}}{dt} = \frac{d\utilde{R}}{ds}\frac{ds}{dt} = \hat{\utilde{T}} v = v\hat{\utilde{T}}$

$$\utilde{a} = \frac{d\utilde{v}}{dt} = \frac{dv}{dt}\hat{\utilde{T}} + v\frac{d\hat{T}}{ds}\frac{ds}{dt} = \frac{dv}{dt}\hat{\utilde{T}} + \kappa v^2 \hat{\utilde{N}}.$$

(b) $\utilde{j} = \frac{d^2v}{dt^2}\hat{\utilde{T}} + \frac{dv}{dt}\frac{d\hat{T}}{ds}\frac{ds}{dt} + 2\kappa v\frac{dv}{dt}\hat{\utilde{N}} + \kappa v^2 \frac{d\hat{\utilde{N}}}{d\utilde{s}}\frac{ds}{dt}$

$= \frac{d^2v}{dt^2}\hat{\utilde{T}} + \frac{dv}{dt}\kappa\hat{\utilde{N}}v + 2\kappa v\frac{dv}{dt}\hat{\utilde{N}} + \kappa v^2(-\kappa\hat{\utilde{T}} - \nu\hat{\utilde{B}})v$, which gives (7.2). ✓

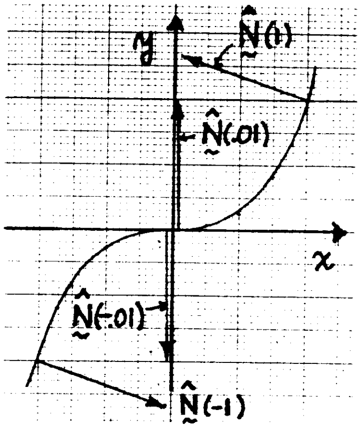
8. We can use x as τ. Then $\utilde{R} = x\hat{\imath} + y\hat{\jmath} = x\hat{\imath} + x^3\hat{\jmath}$.

$ds = \sqrt{\utilde{R}'(x)\cdot\utilde{R}'(x)}\,dx = \sqrt{1+9x^4}\,dx$, so $ds/dx = \sqrt{1+9x^4}$.

$\frac{d\utilde{R}}{ds} = \frac{d\utilde{R}}{dx}\frac{dx}{ds} = (\hat{\imath} + 3x^2\hat{\jmath})(1+9x^4)^{-1/2}$,

$\frac{d^2\utilde{R}}{ds^2} = \frac{d}{dx}(\ ''\ )\frac{dx}{ds} =$ etc. $= (-18x^3\hat{\imath} + 6x\hat{\jmath})/(1+9x^4)^2$

so $\hat{\utilde{N}} = \utilde{R}''(s)/\|\utilde{R}''(s)\| = (-3x^3\hat{\imath} + x\hat{\jmath})/\sqrt{9x^6+x^2} = (-3x^3\hat{\imath} + x\hat{\jmath})/(|x|\sqrt{1+9x^4})$.



Thus, $\underset{\sim}{\hat{N}}(1) = -0.949\hat{\imath} + 0.316\hat{\jmath}$

$\underset{\sim}{\hat{N}}(.01) = -0.030\hat{\imath} + 0.9996\hat{\jmath}$

$\underset{\sim}{\hat{N}}(-.01) = +0.030\hat{\imath} - 0.9996\hat{\jmath}$

$\underset{\sim}{\hat{N}}(-1) = 0.949\hat{\imath} - 0.316\hat{\jmath}$,

as sketched. Observe that $\underset{\sim}{\hat{N}}(x)$ is discontinuous at $x=0$, where it suffers a reversal of direction.

9. $\underset{\sim}{R}(x) = x\hat{\imath} + y(x)\hat{\jmath}$, $d\underset{\sim}{R} = dx\,\hat{\imath} + y'dx\,\hat{\jmath}$, $ds = \sqrt{1+y'^2}\,dx$

$$\frac{d\underset{\sim}{R}}{ds} = \frac{d\underset{\sim}{R}}{dx}\frac{dx}{ds} = (\hat{\imath}+y'\hat{\jmath})(1+y'^2)^{-1/2}$$

$$\frac{d^2\underset{\sim}{R}}{ds^2} = \frac{d}{dx}\left[(\hat{\imath}+y'\hat{\jmath})(1+y'^2)^{-1/2}\right](1+y'^2)^{-1/2} = y''\hat{\jmath}(\ \ )^{-1} + (\hat{\imath}+y'\hat{\jmath})(-\tfrac{1}{2})(2y'y'')(\ \ )^{-2}$$

$$= \frac{y''(1+y'^2)\hat{\jmath} - y'y''(\hat{\imath}+y'\hat{\jmath})}{(1+y'^2)^2}$$

$$\kappa = \frac{1}{\rho} = \|\underset{\sim}{R}''\| = |y''|\sqrt{1+y'^2}/(1+y'^2)^2$$

$$= |y''|/(1+y'^2)^{3/2} \quad ✓$$

10. Suppose the curve is a plane curve. Then there exist constants a, b, c, d such that

$$ax(\tau)+by(\tau)+cz(\tau)=d,$$

repeated differentiation of which (with respect to τ) gives

$$ax'+by'+cz'=0$$
$$ax''+by''+cz''=0$$
$$ax'''+by'''+cz'''=0.$$

Since a, b, c are not all zero, it must (Thm 10.5.5) be true that

$$\det = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

Conversely, if that determinant $= 0$ then there exist constants a, b, c such that $ax'+by'+cz'=0$ (Thm. 10.5.5), integration of which gives $ax+by+cz=d$.

11. (b) $\det = \begin{vmatrix} 2\tau & 2\tau & -3 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$, hence it <u>is</u> a plane curve.

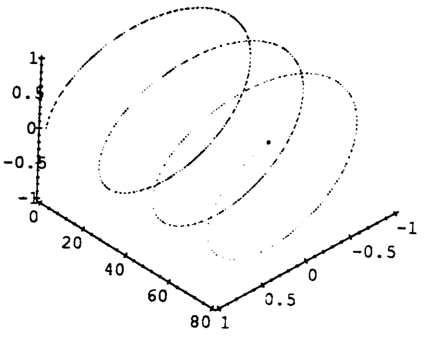
(c) $\det = \begin{vmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$, hence it <u>is</u> a plane curve.

(d) $\det = \begin{vmatrix} 0 & \cos\tau & -\sin\tau \\ 0 & -\sin\tau & -\cos\tau \\ 0 & -\cos\tau & \sin\tau \end{vmatrix} = 0$, hence it <u>is</u> a plane curve.

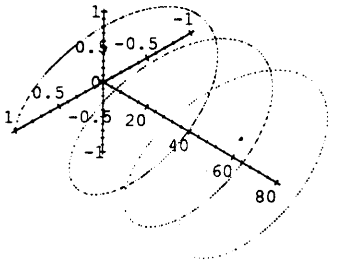
12.(b) I used $0 \le t \le 20$ instead of $0 \le t \le 5$. The maple commands

```
> with(plots):
> spacecurve([cos(t),4*t,sin(t)],t=0..20,numpoints=1000,axes=FRAMED)
  ;
```

and the same, but with `axes=NORMAL` instead gave



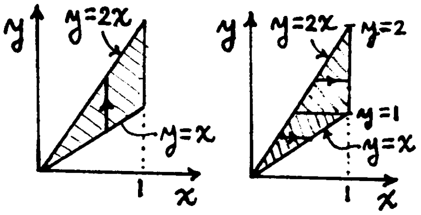
and



, respectively.

Section 15.3

1. (b) $\int_0^1\int_x^{2x} x^3y\,dy\,dx = \int_0^1 x^3\frac{y^2}{2}\Big|_x^{2x}dx = \int_0^1 x^3\left(\frac{4x^2}{2}-\frac{x^2}{2}\right)dx = \frac{3}{2}\int_0^1 x^5dx = \frac{1}{4}$



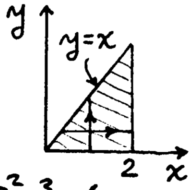
Reversing the order,

$$\int_0^1\int_{y/2}^{y} x^3y\,dx\,dy + \int_1^2\int_{y/2}^{1} x^3y\,dx\,dy$$

$$= \frac{15}{64}\int_0^1 y^5dy + \frac{1}{4}\int_1^2\left(y-\frac{y^5}{16}\right)dy = \frac{1}{4}\ ✓$$

(c) $\int_0^2\int_y^2 xe^{x+y}dx\,dy = \int_0^2 e^ye^x(x-1)\Big|_y^2 dy$

$= \int_0^2 [e^{y+2}-(y-1)e^{2y}]dy = \frac{3}{4}e^4-e^2-\frac{3}{4}$

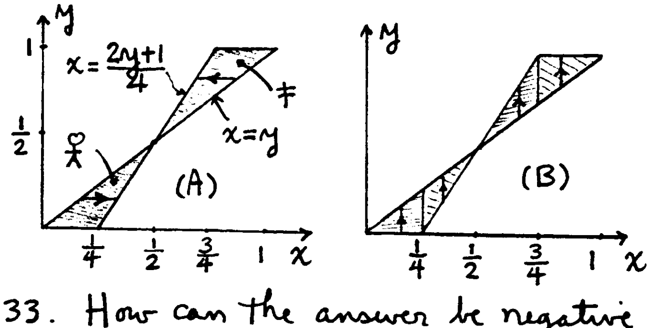


or, $\int_0^2\int_0^x xe^{x+y}dy\,dx = \int_0^2 xe^{x+y}\Big|_0^x dx = \int_0^2 x(e^{2x}-e^x)dx = \frac{3}{4}e^4-e^2-\frac{3}{4}$ ✓

(e) $\int_0^1\int_y^{(2y+1)/4}(x+y)^8\,dx\,dy$

$= \frac{1}{9}\int_0^1\left[\left(\frac{2y+1}{4}+y\right)^9-(2y)^9\right]dy$

$= \frac{1}{9}\left\{\int_{1/4}^{7/4}u^9\frac{2du}{3} - 2^9\int_0^1 y^9dy\right\}$



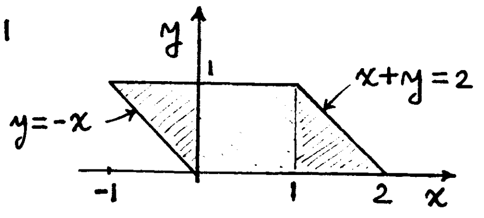
$= \frac{1}{135}\left[\left(\frac{7}{4}\right)^{10}-\left(\frac{1}{4}\right)^{10}-768\right] \approx -3.693411933$. How can the answer be negative when the integrand $(x+y)^8 > 0$? From Fig.(A) we see that the integral over ♀ will be positive (because the integrand and the dx's and the dy's are all positive), but the integral over ‡ will be negative because the dx's are negative, the dy's are positive and $(x+y)^8$ is positive.

Reversing the order we need to break the integral into several parts, as seen from Fig. (B):

$$\int_0^{1/4}\int_0^{x}(x+y)^8\,dy\,dx+\int_{1/4}^{1/2}\int_{(4x-1)/2}^{x}(x+y)^8\,dy\,dx+\int_{3/4}^{1/2}\int_{x}^{(4x-1)/2}(x+y)^8\,dy\,dx+\int_{1}^{3/4}\int_{x}^{1}(x+y)^8\,dy\,dx$$

which gives the same result, -3.693411933. ✓

(f) $\int_0^1\int_{-y}^{2-y}x\,dx\,dy=\int_0^1\frac{(2-y)^2-y^2}{2}\,dy=(2y-y^2)\big|_0^1=1$



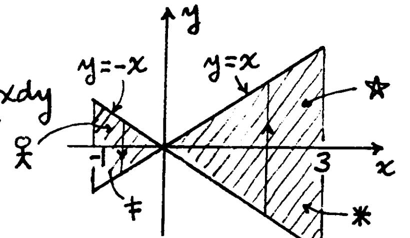
or, $\int_{-1}^{0}\int_{-x}^{1}x\,dy\,dx+\int_0^1\int_0^1 x\,dy\,dx+\int_1^2\int_0^{2-x}x\,dy\,dx$

$=\int_{-1}^{0}(x+x^2)\,dx+\int_0^1 x\,dx+\int_1^2(2x-x^2)\,dx=1$ ✓

(g) $\int_{-1}^{3}\int_{-x}^{x}x^2\,dy\,dx=\int_{-1}^{3}x^3y\big|_{-x}^{x}\,dx=\int_{-1}^{3}2x^3\,dx=40$

or, $\underbrace{\int_1^0\int_{-1}^{-y}x^2\,dx\,dy}_{\text{♀}}+\underbrace{\int_0^{-1}\int_{-1}^{y}x^2\,dx\,dy}_{\ddagger}+\underbrace{\int_0^3\int_y^3 x^2\,dx\,dy}_{\star}+\underbrace{\int_{-3}^{0}\int_{-y}^{3}x^2\,dx\,dy}_{*}$

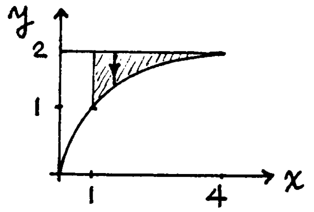
$=40$ ✓



(h) $\int_1^4\int_2^{\sqrt{x}}x^2\,dy\,dx=\int_1^4-x^2(2-x^{1/2})\,dx$

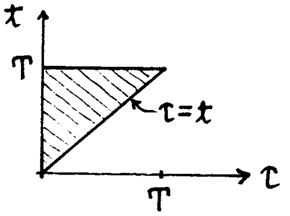
$=-\frac{2}{3}x^3+\frac{2}{7}x^{7/2}\big|_1^4=-\frac{40}{7}$

or, $\int_2^1\int_1^{y^2}x^2\,dx\,dy=\int_2^1\left(\frac{y^6}{3}-\frac{1}{3}\right)dy=-\frac{40}{7}$ ✓



2. $\int_0^T\int_0^t x(\tau)\,d\tau\,dt=\int_0^T\int_\tau^T x(\tau)\,dt\,d\tau$

$=\int_0^T x(\tau)(T-\tau)\,d\tau.$



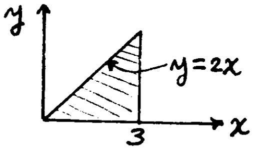
Or, we could have used integration by parts, with $u=\int_0^t x(\tau)\,d\tau$, $dv=dt$. Then we obtain

$\left[t\int_0^t x(\tau)\,d\tau\right]\Big|_0^T-\int_0^T t\,x(t)\,dt$

$=T\int_0^T x(\tau)\,d\tau-\int_0^T\tau x(\tau)\,d\tau=\int_0^T x(\tau)(T-\tau)\,d\tau.$

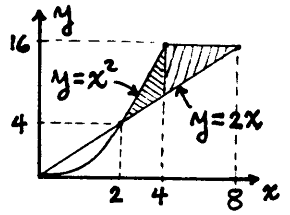
3. (a) $\int_0^6\int_{y/2}^{3}\frac{1}{x}e^{y/x}\,dx\,dy=\int_0^3\int_0^{2x}\frac{1}{x}e^{y/x}\,dy\,dx$

$=\int_0^3 e^{y/x}\big|_0^{2x}\,dx=\int_0^3(e^2-1)\,dx=3(e^2-1).$



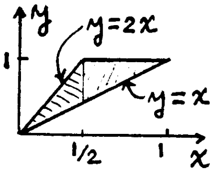
(b) $\int_4^{16}\int_{y/2}^{\sqrt{y}}\frac{1}{x}e^{y/x}\,dx\,dy=\int_4^2\int_{2x}^{x^2}\frac{1}{x}e^{y/x}\,dy\,dx+\int_8^4\int_{2x}^{16}\frac{1}{x}e^{y/x}\,dy\,dx$

$=\int_4^2(e^x-e^2)\,dx+\int_8^4(e^{16/x}-e^2)\,dx=7e^2-e^4-\int_4^8 e^{16/x}\,dx.$



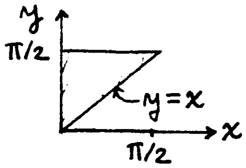
(c) $\int_0^1\int_{y/2}^{y} \sin(x^2y)\,dx\,dy = \int_0^{1/2}\int_x^{2x} \sin(x^2y)\,dy\,dx + \int_{1/2}^{1}\int_x^{1} \sin(x^2y)\,dy\,dx$

$= \int_0^{1/2} \frac{\cos x^3 - \cos 2x^3}{x^3}\,dx + \int_{1/2}^{1} \frac{\cos x^3 - \cos x^2}{x^2}\,dx$



(d) $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\,dy\,dx = \int_0^{\pi/2}\int_0^{y} \frac{\sin y}{y}\,dx\,dy$

$= \int_0^{\pi/2} \frac{\sin y}{\cancel{y}}\cancel{y}\,dy = -\cos y\Big|_0^{\pi/2} = 1$



4. (c) $M = \sigma A = \sigma\int_0^1\int_{x/2}^{\sqrt{x}} dy\,dx + \sigma\int_1^{4/3}\int_{x/2}^{2-x} dy\,dx = \sigma\int_0^1\left[\sqrt{x} - \frac{x}{2}\right]dx + \sigma\int_1^{4/3}(2-x-\frac{x}{2})\,dx = \frac{\sigma}{2}$

$x_c = \frac{\sigma}{\sigma/2}\int_0^1\int_{x/2}^{\sqrt{x}} x\,dy\,dx + \frac{\sigma}{\sigma/2}\int_1^{4/3}\int_{x/2}^{2-x} x\,dy\,dx = \frac{88}{135}$

$y_c = \frac{\sigma}{\sigma/2}\int_0^1\int_{x/2}^{\sqrt{x}} \sigma y\,dy\,dx + \frac{\sigma}{\sigma/2}\int_1^{4/3}\int_{x/2}^{2-x} \sigma y\,dy\,dx = \frac{29}{54}$

(d) $M = \sigma A = \sigma(8 - 2 - \int_0^2 x^2\,dx) = \frac{10\sigma}{3}$

$x_c = \frac{\sigma}{10\sigma/3}\int_0^4\int_{y/4}^{\sqrt{y}} x\,dx\,dy = 1, \quad y_c = \frac{\sigma}{10\sigma/3}\int_0^4\int_{y/4}^{\sqrt{y}} y\,dx\,dy = \frac{56}{25}$

(g) $M = \int_0^1\int_{x/2}^{\sqrt{x}} x\,dy\,dx + \int_1^{4/3}\int_{x/2}^{2-x} x\,dy\,dx = \frac{44}{135}$

$x_c = \frac{135}{44}\int_0^1\int_{x/2}^{\sqrt{x}} x^2\,dy\,dx + \frac{135}{44}\int_1^{4/3}\int_{x/2}^{2-x} x^2\,dy\,dx = \frac{2995}{3696}$

$y_c = \frac{135}{44}\int_0^1\int_{x/2}^{\sqrt{x}} xy\,dy\,dx + \frac{135}{44}\int_1^{4/3}\int_{x/2}^{2-x} xy\,dy\,dx = \frac{655}{1056}$

5. (d) $I_x = \sigma\int_0^2\int_x^{(x+6)/2} y^2\,dy\,dx = \frac{\sigma}{3}\int_0^2\left[\frac{(x+6)^3}{8} - x^3\right]dx = \frac{167}{6}\sigma$

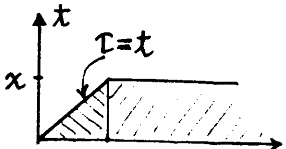
$I_y = \sigma\int_0^2\int_x^{(x+6)/2} x^2\,dy\,dx = \sigma\int_0^2 x^2\left(\frac{x+6}{2} - x\right)dx = 6\sigma$

(g) $I_x = \int_0^a\int_0^{bx/a} (1+x)y^2\,dy\,dx = \frac{ab^3}{60}(5+4a)$

$I_y = \int_0^a\int_0^{bx/a} (1+x)x^2\,dy\,dx = \int_0^a (x^2+x^3)\frac{bx}{a}\,dx = \frac{a^3b}{20}(5+4a)$

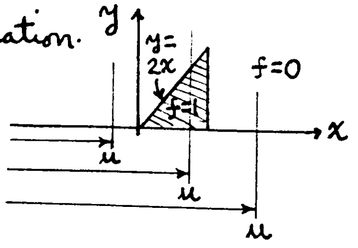
6. $\int_0^x \mathrm{Ei}(t)\,dt = \int_0^x\int_t^\infty \frac{e^{-\tau}}{\tau}\,d\tau\,dt = \int_0^x\int_0^\tau \frac{e^{-\tau}}{\tau}\,dt\,d\tau + \int_x^\infty\int_0^x \frac{e^{-\tau}}{\tau}\,dt\,d\tau$

$= \int_0^x e^{-\tau}\,d\tau + x\int_x^\infty \frac{e^{-\tau}}{\tau}\,d\tau = 1 - e^{-x} + x\,\mathrm{Ei}(x)$



7. If $u \le 0$, $\iint = 0$ since $f = 0$ everywhere in domain of integration.
If $0 < u < 1$, $\int_{-\infty}^{\infty}\int_{-\infty}^{u} f\,dx\,dy = \int_0^u\int_0^{2x} 1\,dy\,dx = u^2$.

And if $u \ge 1$, $\int_{-\infty}^{\infty}\int_{-\infty}^{u} f\,dx\,dy = \int_0^1\int_0^{2x} 1\,dy\,dx = 1$.



8. The Maple commands

```
A:= int(x* exp(2/x), x=1.. sqrt(2));
evalf( exp(2)/2 + (sqrt(2)-1)* exp(sqrt(2)) - A);
```

gives the result

2.740067688

9. (a) The Maple command

```
int( int((x+y)^8, x=y..(2*y+1)/4), y=0..1);
```

gives $-\dfrac{2178463}{589824}$

and evalf("); gives −3.693411933

10. (b) $\int_{-1}^{1}\int_0^z\int_2^3 z^2\sin(yz)\,dx\,dy\,dz = \int_{-1}^{1}\int_0^z z^2\sin(yz)\,dy\,dz = \int_{-1}^{1} -z\cos(yz)\Big|_0^z\,dz$
$= \int_{-1}^{1}(z - z\cos z^2)\,dz = 0$

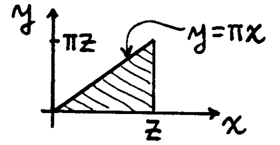
(c) $\int_0^{\pi}\int_0^{2y}\int_0^{y+z}\cos(x+y)\,dx\,dz\,dy = \int_0^{\pi}\int_0^{2y}\sin(x+y)\Big|_0^{y+z}dz\,dy$
$= \int_0^{\pi}\int_0^{2y}[\sin(2y+z)-\sin y]\,dz\,dy = \int_0^{\pi}(\cos 2y - \cos 4y - 2y\sin y)\,dy = -2\pi$

(d) $\int_{-1}^{3}\int_0^{z^2}\int_{\pi y}^{0}\sin\frac{x}{y}\,dx\,dy\,dz = \int_{-1}^{3}\int_0^{z^2}(-2y)\,dy\,dz = \int_{-1}^{3} -z^4\,dz = -\frac{244}{5}$

(e) $\int_0^2\int_0^{4z}\int_0^{2-z} e^{x+y}\,dx\,dy\,dz = \int_0^2\int_0^{4z}(e^{2-z+y} - e^y)\,dy\,dz$
$= \int_0^2(e^{3z+2} - e^{4z} - e^{2-z} + 1)\,dz = \frac{e^8}{12} - \frac{4}{3}e^2 + \frac{13}{4}$

(f) $\int_0^1\int_0^1\int_0^y e^{xyz}y^2z^3\,dx\,dy\,dz = \int_0^1\int_0^1 z^2y\,e^{xyz}\Big|_0^y\,dy\,dz = \int_0^1\left(\frac{z}{2}e^z - \frac{z^2}{2} - \frac{z}{2}\right)dz = \frac{1}{12}$

11. $I = \int_0^1\int_0^{\pi z}\int_{y/\pi}^{z}\sin\frac{y}{x}\,dx\,dy\,dz = \int_0^1\int_0^z\int_0^{\pi x}\sin\frac{y}{x}\,dy\,dx\,dz$
$= \int_0^1\int_0^z 2x\,dx\,dz = \int_0^1 z^2\,dz = 1/3$



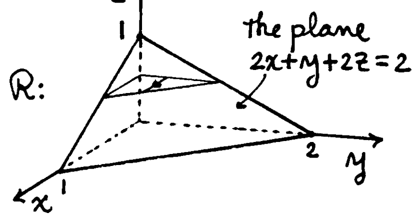
12. The 3-dim. version of the formulas given in Exercise 4 are

$$x_c = \frac{1}{M}\iiint_R x\sigma(x,y,z)\,dV,\quad y_c = \frac{1}{M}\iiint_R y\sigma(x,y,z)\,dV,\quad z_c = \frac{1}{M}\iiint_R z\sigma(x,y,z)\,dV$$

where the mass is $M = \iiint_R \sigma(x,y,z)\,dV$.

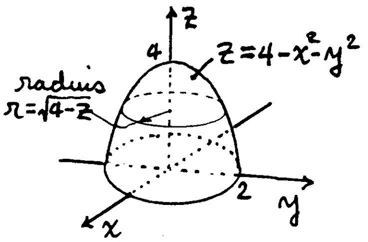
(a) $M = \sigma\int_0^1\int_0^{2-2z}\int_0^{(2-y-2z)/2}dx\,dy\,dz = \frac{\sigma}{3}$

$x_c = \frac{\sigma}{\sigma/3}\int_0^1\int_0^{2-2z}\int_0^{(2-y-2z)/2}x\,dx\,dy\,dz = \frac{1}{4}$

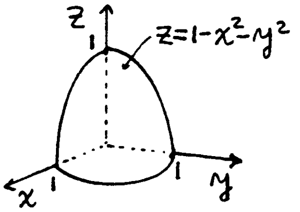


(c) $M = \sigma \iiint dV = \sigma \int_0^4 \pi r^2 dz = \pi\sigma \int_0^4 (4-z)\,dz = 8\pi\sigma$

$z_c = \frac{\sigma}{8\pi\sigma} \int_0^4 z(\pi r^2 dz) = \frac{1}{8} \int_0^4 z(4-z)\,dz = \frac{4}{3}$



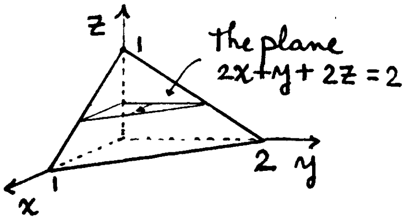
(d)



$M = \sigma \iiint dV = \sigma \int_0^1 \frac{\pi r^2}{4} dz$
$= \frac{\pi\sigma}{4} \int_0^1 (1-z)\,dz = \frac{\pi\sigma}{8}$
$z_c = \frac{\sigma}{\pi\sigma/8} \int_0^1 z\, \frac{\pi}{4}(1-z)\,dz = \frac{1}{3}$

(e) $M = \frac{\pi}{4} \int_0^1 z(1-z)\,dz = \pi/24$
and $z_c = \frac{24}{\pi} \int_0^1 z^2 \frac{\pi(1-z)}{4}\,dz = \frac{1}{6}$

13. (a) R and σ as in Exercise 12(a): $\sigma =$ constant, R:



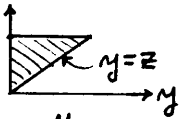
$\mathcal{I}_x = \sigma \int_0^1 \int_0^{2-2z} \int_0^{(2-y-2z)/2} (y^2+z^2)\,dx\,dy\,dz$

$= \sigma \int_0^1 \int_0^{2-2z} (y^2+z^2)\left(\frac{2-y-2z}{2}\right) dy\,dz$
$= (\sigma/6) \int_0^1 [4(z-1)^4 + 6(z^2-2z^3+z^4)]\,dz = \frac{\sigma}{6}$

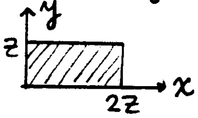
(b) $\mathcal{I}_x = \sigma \int_0^c \int_0^b \int_0^a (y^2+z^2)\,dx\,dy\,dz = \sigma a \int_0^c \int_0^b (y^2+z^2)\,dy\,dz = \sigma a \int_0^c \left(\frac{b^3}{3} + bz^2\right) dz$
$= \sigma abc\,(b^2+c^2)/3$, or, $M(b^2+c^2)/3$

(c) $\mathcal{I}_x = \frac{\sigma abc}{3}(b^2+c^2) - \frac{\sigma}{3}\frac{a}{2}\frac{b}{2}\frac{c}{2}\left(\frac{b^2}{4}+\frac{c^2}{4}\right) = \frac{31\sigma}{96} abc\,(b^2+c^2)$, or, $\frac{31M}{96}(b^2+c^2)$

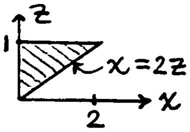
14. (a) $\int_0^1 \int_0^z \int_0^z f\,dx\,dy\,dz = \int_0^1 \int_y^1 \int_0^z f\,dx\,dz\,dy$



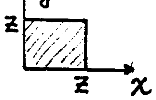
(b) $\int_0^1 \int_0^z \int_0^{2z} f\,dx\,dy\,dz = \int_0^1 \int_0^{2z} \int_0^z f\,dy\,dx\,dz$



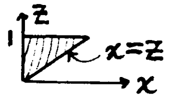
$= \int_0^2 \int_{x/2}^1 \int_0^z f\,dy\,dz\,dx$



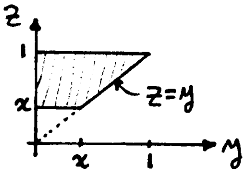
(c) $\int_0^1 \int_0^z \int_0^z f\,dx\,dy\,dz = \int_0^1 \int_0^z \int_0^z f\,dy\,dx\,dz$



$= \int_0^1 \int_x^1 \int_0^z f\,dy\,dz\,dx$

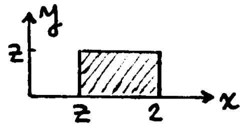


$= \int_0^1 \int_0^x \int_x^1 f\,dz\,dy\,dx + \int_0^1 \int_x^1 \int_y^1 f\,dz\,dy\,dx$

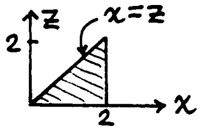


NOTE: As a partial check we could set $f=1$, say. Then original integral gives 1/3 and the final two integrals give $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$. ✓

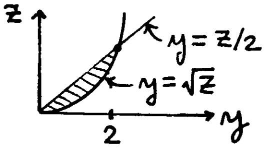
(d) $\int_0^2\int_0^z\int_z^2 f\,dx\,dy\,dz = \int_0^2\int_z^2\int_0^z f\,dy\,dx\,dz$



$= \int_0^2\int_0^x\int_0^z f\,dy\,dz\,dx$



(e) $\int_0^4\int_{z/2}^{\sqrt{z}}\int_0^{y+z} f\,dx\,dy\,dz = \int_0^2\int_{y^2}^{2y}\int_0^{y+z} f\,dx\,dz\,dy$



(f) $\int_4^6\int_1^3\int_0^2 f\,dx\,dy\,dz = \int_0^2\int_1^3\int_4^6 f\,dz\,dy\,dx$

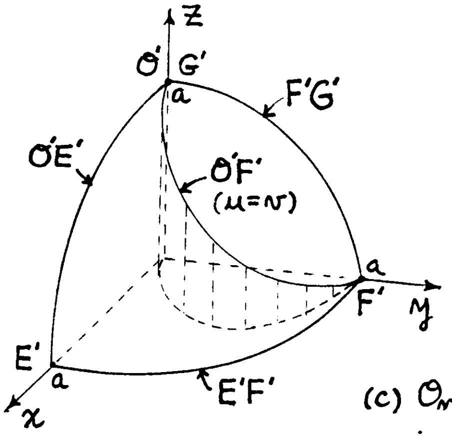
15. (e) The Maple command

```
int(int(int(exp(x+y), x=0..2-z), y=0..4*z), z=0..2);
```

gives $\frac{e^8}{12}-\frac{4}{3}e^2+\frac{13}{4}$, as obtained in 10(e).

Section 15.4

1.



(a) On OE, $u=0$ and $0 \le v \le \pi/2$,

so $\left.\begin{aligned} x&=a\sin v\\ y&=0\\ z&=a\cos v\end{aligned}\right\}$ give a circular arc in x,z plane, of radius a; $v=0$ is at north pole and $v=\pi/2$ is at equator

(b) On EF, $v=\pi/2$ and $0\le u\le \pi/2$,

so $\left.\begin{aligned} x&=a\cos u\\ y&=a\sin u\\ z&=0\end{aligned}\right\}$ give the circular arc in x,y plane

(c) On FG, $u=\pi/2$ and $0\le v\le\pi/2$, so $\begin{aligned} x&=0\\ y&=a\sin v\\ z&=a\cos v\end{aligned}$, which give a circular arc in y,z plane

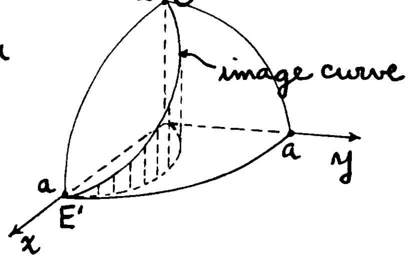
(d) The entire line OG maps into the single point at the north pole.

(e) On $v=u$, $\left.\begin{aligned} x&=a\sin u\cos u\\ y&=a\sin u\sin u\end{aligned}\right\}$ $z=a\cos u$ Eliminate u between these two equations: $\sin u=\sqrt{y/a}$, $\cos u=\sqrt{1-\sin^2 u}=\sqrt{1-y/a}$, so $x=a\sqrt{y/a}\sqrt{1-y/a}=\sqrt{y(a-y)}$, or, $x^2+y^2-ay=0$, or, $x^2+(y-\frac{a}{2})^2=(\frac{a}{2})^2$,

which is a circular cylinder of radius $a/2$, with the line $x=0, y=a/2$ as its central axis. The intersection of that cylinder with the spherical surface $x^2+y^2+z^2=a^2$ gives the desired image curve. This curve is sketched in the figure above.

(f) On $v=\frac{\pi}{2}-u$, $\begin{aligned} x&=a\sin(\tfrac{\pi}{2}-u)\cos u=a\cos^2 u\\ y&=a\sin(\ \text{"}\ )\sin u=a\cos u\sin u\\ z&=a\cos(\ \text{"}\ )\qquad\ \ =a\sin u\end{aligned}$

Eliminating u between the first two of these gives $y=a\sqrt{x/a}\sqrt{1-x/a}=\sqrt{x(a-x)}$ or $x^2-ax+y^2=0$, or, $(x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$, which is a circular



cylinder of radius a/2 with its centerline along $x=a/2, y=0$. The intersection of that cylinder with the spherical surface gives the desired image, as sketched at the bottom of the preceding page.

2. $x=u, y=u+v, z=0;\ 0<u<4, 0<v<4.$

(a)

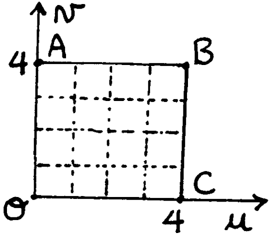
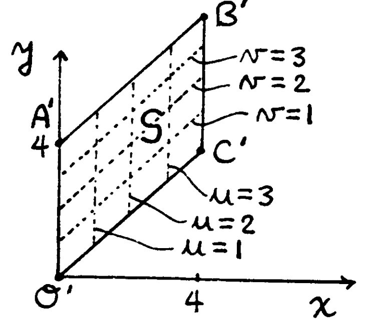


Image of OA is $x=0, 0<y<4$
Image of OC is $y=x, 0<x<4$
Image of AB is $y=x+4, 0<x<4$
Image of BC is $x=4, 4<y<8$,
so S is as shown at right:



(b) See the dashed lines, above.

(c) $\underset{\sim}{R} = u\hat{\imath}+(u+v)\hat{\jmath}+0\hat{k}$, $\underset{\sim}{R}_u = \hat{\imath}+\hat{\jmath}$, $\underset{\sim}{R}_v = \hat{\jmath}$

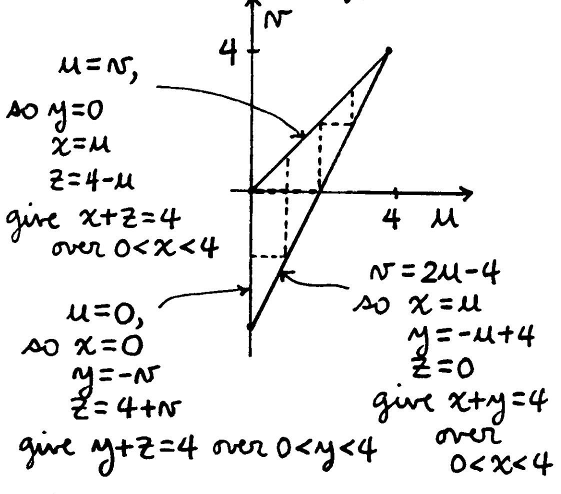
$\underset{\sim}{R}_u \times \underset{\sim}{R}_v = (\hat{\imath}+\hat{\jmath})\times\hat{\jmath} = \hat{k}$

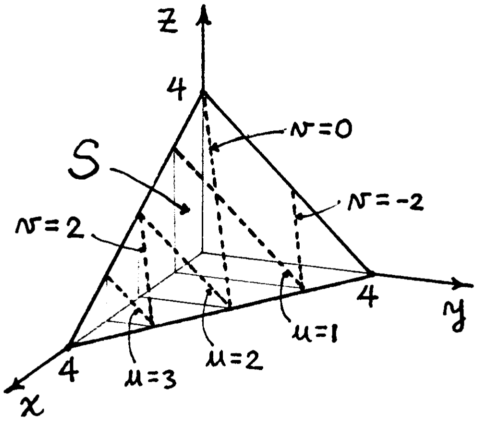
$\|\underset{\sim}{R}_u \times \underset{\sim}{R}_v\| = \|\hat{k}\| = 1$,

so (9) gives $\hat{\underset{\sim}{n}} = \hat{k}$ (which, of course, could have been seen by inspection in this case). (10) is satisfied everywhere.

3. $x=u, y=u-v, z=4-2u+v$

(a) $u=x$ and $v=x-y$, so $z=4-2x+(x-y)$, or, $x+y+z=4$, a plane.





(b) Image of $u=1$ is $x=1$
$y=1-v$
$z=2+v$, which is a straight line, over $\overset{(1,3,0)}{-2}<v<\overset{(1,0,3)}{1}$

Similarly for $u=2,3$, as shown, above, by the dashed lines.

Image of $v=-2$ is $x=u$
$y=u+2$
$z=2-2u$, which is a straight line, over $\overset{(0,2,2)}{0}<u<\overset{(1,3,0)}{1}$

Image of $v=0$ is $x=u$
$y=u$
$z=4-2u$, which is a straight line, over $\overset{(0,0,4)}{0}<u<\overset{(2,2,0)}{2}$

Image of $v=2$ is $\begin{array}{l} x=u \\ y=u-2 \\ z=6-2u, \end{array}$ which is a straight line, over $\overset{\curvearrowright(2,0,2)}{2} < u < \overset{\curvearrowright(3,1,0)}{3}$ as shown, above, by the dashed lines.

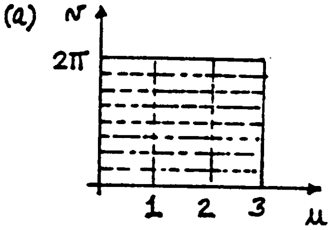
(c) $\underset{\sim}{R} = u\hat{\imath}+(u-v)\hat{\jmath}+(4-2u+v)\hat{k}$, $\underset{\sim}{R}_u=\hat{\imath}+\hat{\jmath}-2\hat{k}$, $\underset{\sim}{R}_v=-\hat{\jmath}+\hat{k}$

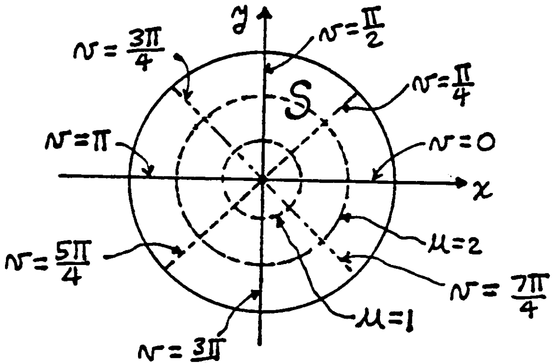
$\underset{\sim}{R}_u\times\underset{\sim}{R}_v=-\hat{\imath}-\hat{\jmath}-\hat{k}$,

so $\underset{\sim}{\hat{n}}=-(\hat{\imath}+\hat{\jmath}+\hat{k})/\sqrt{3}$.

4. $x=u\cos v$, $y=u\sin v$, $z=0$ (these are the familiar "polar" coordinates, $0\le u<3$, $0\le v<2\pi$ $\quad u=r$, $v=\theta$.)

(a)





(b) See dashed lines →

(c) $\underset{\sim}{R}=u\cos v\,\hat{\imath}+u\sin v\,\hat{\jmath}$

$\underset{\sim}{R}_u=\cos v\,\hat{\imath}+\sin v\,\hat{\jmath}$

$\underset{\sim}{R}_v=-u\sin v\,\hat{\imath}+u\cos v\,\hat{\jmath}$

$\underset{\sim}{R}_u\times\underset{\sim}{R}_v=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos v & \sin v & 0\\ -u\sin v & u\cos v & 0\end{vmatrix}=u\underset{\sim}{\hat{k}}$, so $\underset{\sim}{\hat{n}}=\frac{u\underset{\sim}{\hat{k}}}{u}=\underset{\sim}{\hat{k}}$,

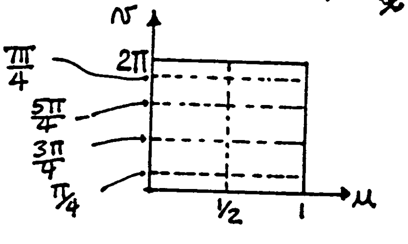
for $u\neq0$. For $u=0$, $\underset{\sim}{R}_u\times\underset{\sim}{R}_v=\underset{\sim}{0}$ and $\hat{n}=\frac{0\underset{\sim}{\hat{k}}}{0}$ is undefined. However, inspection of S (which is flat, in x,y plane) reveals that $\underset{\sim}{\hat{n}}=\underset{\sim}{\hat{k}}$ at $u=0$ as well.

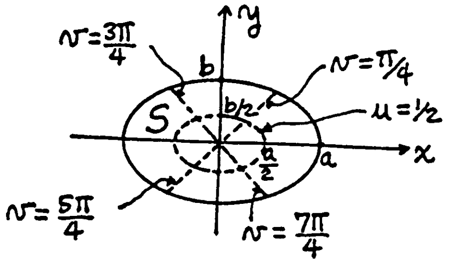
5. $x=au\cos v$, $y=bu\sin v$, $z=0$

$0\le u<1$, $0\le v<2\pi$

(a) $(\frac{x}{a})^2+(\frac{y}{b})^2=u^2$, so constant u curves are ellipses.

Also, $\frac{y}{x}=\frac{b}{a}\tan v$, so const. v curves are radial lines.





(b) See the dashed curves →

(c) $\underset{\sim}{R}=au\cos v\,\hat{\imath}+bu\sin v\,\hat{\jmath}+0\underset{\sim}{\hat{k}}$

$\underset{\sim}{R}_u=a\cos v\,\hat{\imath}+b\sin v\,\hat{\jmath}$

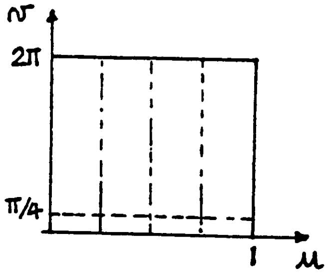
$\underset{\sim}{R}_v=-au\sin v\,\hat{\imath}+bu\cos v\,\hat{\jmath}$, $\underset{\sim}{R}_u\times\underset{\sim}{R}_v=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a\cos v & b\sin v & 0\\ -au\sin v & bu\cos v & 0\end{vmatrix}=abu\underset{\sim}{\hat{k}}$

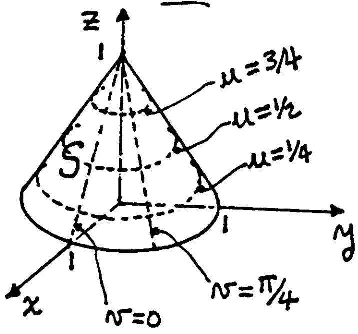
so $\hat{n} = \frac{abu\hat{k}}{abu} = \hat{k}$ for $u \neq 0$. For $u=0$, $\underset{\sim}{R}_u \times \underset{\sim}{R}_v = \underset{\sim}{0}$ and $\underset{\sim}{\hat{n}} = 0\underset{\sim}{\hat{k}}/0$ is undefined. Nevertheless, inspection of S (which is flat and lies in the x, y plane) reveals that $\underset{\sim}{\hat{n}} = \underset{\sim}{\hat{k}}$ at $u=0$ as well.

6. $x = (1-u)\cos v$, $y = (1-u)\sin v$, $z = u$

$0 < u \leqq 1$, $0 \leq v < 2\pi$

(a) $x^2 + y^2 = (\underbrace{1-z})^2$ radius varies linearly with z, so S is a cone.





(b) See the dashed lines ⟶

(c) $\underset{\sim}{R} = (1-u)\cos v\,\underset{\sim}{\hat{i}} + (1-u)\sin v\,\underset{\sim}{\hat{j}} + u\underset{\sim}{\hat{k}}$

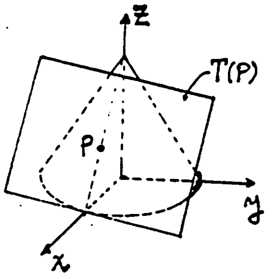
$\underset{\sim}{R}_u = -\cos v\,\underset{\sim}{\hat{i}} - \sin v\,\underset{\sim}{\hat{j}} + \underset{\sim}{\hat{k}}$

$\underset{\sim}{R}_v = -(1-u)\sin v\,\underset{\sim}{\hat{i}} + (1-u)\cos v\,\underset{\sim}{\hat{j}}$

$$\underset{\sim}{R}_u \times \underset{\sim}{R}_v = \begin{vmatrix} \underset{\sim}{\hat{i}} & \underset{\sim}{\hat{j}} & \underset{\sim}{\hat{k}} \\ -c & -s & 1 \\ -(1-u)s & (1-u)c & 0 \end{vmatrix} = -(1-u)c\,\underset{\sim}{\hat{i}} - (1-u)s\,\underset{\sim}{\hat{j}} - (1-u)\underset{\sim}{\hat{k}}$$

$= \underset{\sim}{0}$ only at $u=1$, i.e., at the apex of the cone. Clearly, a unique normal does <u>not</u> exist at that point.

away from that point, $$\underset{\sim}{\hat{n}} = \frac{-(1-u)c\,\underset{\sim}{\hat{i}} - (1-u)s\,\underset{\sim}{\hat{j}} - (1-u)\underset{\sim}{\hat{k}}}{|1-u|\sqrt{c^2+s^2+1}}$$
$$= -(\cos v\,\underset{\sim}{\hat{i}} + \sin v\,\underset{\sim}{\hat{j}} + \underset{\sim}{\hat{k}})/\sqrt{2}.$$

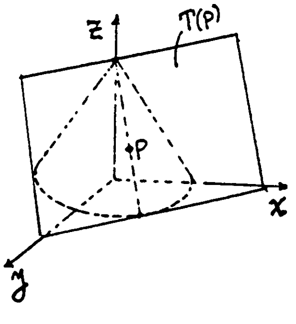


(d) $(\underset{\sim}{R} - \underset{\sim}{R}_P) \cdot \underset{\sim}{\hat{n}} = 0$

at $u = 1/2$, $v = 0$ we have $\underset{\sim}{R}_P = \frac{1}{2}\underset{\sim}{\hat{i}} + \frac{1}{2}\underset{\sim}{\hat{k}}$,

and $\underset{\sim}{\hat{n}} = -\frac{\hat{i} + \hat{k}}{\sqrt{2}}$,

so $(x - \frac{1}{2})(\frac{-1}{\sqrt{2}}) + (z - \frac{1}{2})(-\frac{1}{\sqrt{2}}) = 0$ or, $x + z = 1$.



(e) $x + z = 1$ again, as in (d).

(f) at $u = 1/2$, $v = \pi/4$ we have $\underset{\sim}{R}_P = \frac{1}{2\sqrt{2}}\underset{\sim}{\hat{i}} + \frac{1}{2\sqrt{2}}\hat{j} + \frac{1}{2}\underset{\sim}{\hat{k}}$

and $\underset{\sim}{\hat{n}} = -(\frac{1}{\sqrt{2}}\underset{\sim}{\hat{i}} + \frac{1}{\sqrt{2}}\hat{j} + \underset{\sim}{\hat{k}})/\sqrt{2}$

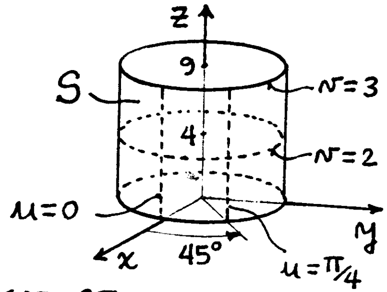
so $(x - \frac{1}{2\sqrt{2}})(-\frac{1}{2}) + (y - \frac{1}{2\sqrt{2}})(-\frac{1}{2}) + (z - \frac{1}{2})(-\frac{1}{\sqrt{2}}) = 0$

or, $x + y + \sqrt{2}z = \sqrt{2}$

7. $x=a\cos u,\ y=b\sin u,\ z=v^2;\ 0\le u<2\pi, 0<v<3$

(a) $(\frac{x}{a})^2+(\frac{y}{b})^2=1,\ 0<z<9$ so S is an elliptic cylinder, as shown at the right.

(b) See the figure.



8. (a) $x=u, y=v;\ -\infty<u<\infty, -\infty<v<\infty$

Another: $x=u\cos v, y=u\sin v;\ 0\le u<\infty,\ 0\le v<2\pi$

Another: $x=u, y=u^2+v;\ -\infty<u<\infty, -\infty<v<\infty$

(b) $x=u, z=v;\ -\infty<u<\infty, -\infty<v<\infty$

Another: $x=u\cos v, z=u\sin v;\ 0\le u<\infty, 0\le v<2\pi$

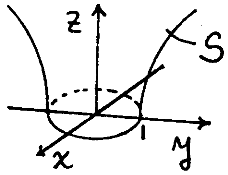
(d) $x=u, y=v, z=6-3u+2v;\ -\infty<u<\infty, -\infty<v<\infty$

another: $y=u, z=v, x=(6+2u-v)/3;\ -\infty<u<\infty, -\infty<v<\infty$

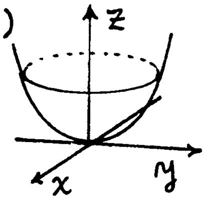
(e) $x=u, y=v, z=\sqrt{u^2+v^2-1};\ u^2+v^2\ge 1$

Another: $x=u\cos v, y=u\sin v, z=\sqrt{u^2-1};$

$1\le u<\infty,\ 0\le v<2\pi$



(f)



$x=u, y=v, z=u^2+v^2;\ -\infty<u<\infty, -\infty<v<\infty$

Another: $x=u\cos v, y=u\sin v, z=u^2;\ 0\le u<\infty, 0\le v<2\pi$

Another: $x=u, y=v^3, z=u^2+v^6;\ -\infty<u<\infty, -\infty<v<\infty$

(g) $x=u, y=v, z=\sqrt{u^2+v^2};\ -\infty<u<\infty, -\infty<v<\infty$

Another: $x=u\cos v, y=u\sin v, z=u;\ 0\le u<\infty,\ 0\le v<2\pi$

Another: $x=u\sin v, y=u\cos v, z=u;\ 0\le u<\infty,\ 8\pi<v\le 10\pi$

9. (a) From Fig. 4 we know that the u,v coordinate curves on S will be orthogonal if $\underset{\sim}{R}_u\cdot\underset{\sim}{R}_v=0$ at each point. In this example the surface is

$$x+2y-4z=5$$

so let us parametrize it by setting

$$z=u \text{ and } y=f(u,v)$$

$$\text{so } x=5+4u-2f(u,v),$$

where $f(u,v)$ is as yet unspecified to give us the flexibility to have $\underset{\sim}{R}_u\cdot\underset{\sim}{R}_v=0$.

Well, $\underset{\sim}{R}(u,v)=x(u,v)\hat{\underset{\sim}{\imath}}+y(u,v)\hat{\jmath}+z(u,v)\hat{k}$

$=(5+4u-2f)\hat{\imath}+f\hat{\jmath}+u\hat{k}$

so $\underset{\sim}{R}_u=(4-2f_u)\hat{\imath}+f_u\hat{\jmath}+\hat{k}$, $\underset{\sim}{R}_v=-2f_v\hat{\imath}+f_v\hat{\jmath}$

and $\underset{\sim}{R}_u\cdot\underset{\sim}{R}_v=(-8+4f_u+f_u)f_v=0$.

We could satisfy by having $f_v=0$ but then x,y,z would be functions only of u and the parametrization would give a curve, not a surface. Instead, let $-8+4f_u+f_u=5f_u-8=0$, $f_u=8/5$. Let

$f(u,v)=\frac{8}{5}u+v,$

say. Then $x = 5 + \frac{4}{5}u - 2v$
$y = \frac{8}{5}u + v$
$z = u$

is a suitable parametrization of S, namely, such that the u, v coordinate curves are orthogonal. (This answer is by no means unique.)

(b) Same strategy as explained in (a). For instance, let

$$x = u,\ y = v + f(u),\ z = 9 - 2u + v + f(u).$$

Then $\underset{\sim}{R} = u\hat{i} + (v+f)\hat{j} + (9-2u+v+f)\hat{k}$

$\underset{\sim}{R}_u \cdot \underset{\sim}{R}_v = (1)(0) + (f')(1) + (-2+f')(1) = 0$

$f' = 1,\ f(u) = u$

so $x = u,\ y = u + v,\ z = 9 + v - u$ (Of course $-\infty < u < \infty$ and $-\infty < v < \infty$.)

10. Surely the plane $z = 0$ has a normal at <u>every</u> x, y point. Yet, with

$$\underset{\sim}{R}(u,v) = u^3\hat{i} + v^3\hat{j} + 0\hat{k}$$

we have $\underset{\sim}{R}_u \times \underset{\sim}{R}_v = 3u^2\hat{i} \times 3v^2\hat{j} = 9u^2v^2\hat{k} = \underset{\sim}{0}$ at the origin.

11. (a) Following the method used in Example 3, begin by parametrizing S as

$$x = u,\ y = v,\ z = u^2 + v^2,$$

say. Then $\underset{\sim}{R} = u\hat{i} + v\hat{j} + (u^2+v^2)\hat{k}$,

$$\underset{\sim}{n} = \underset{\sim}{R}_u \times \underset{\sim}{R}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\hat{i} - 2v\hat{j} + \hat{k},$$

$$\hat{\underset{\sim}{n}} = \frac{-2u\hat{i} - 2v\hat{j} + \hat{k}}{\sqrt{4u^2+4v^2+1}},\ \pm \text{that, of course.}$$

At $P = (2,1,5)$, $u = 2$ and $v = 1$, so $\boxed{\hat{\underset{\sim}{n}} = (-4\hat{i} - 2\hat{j} + \hat{k})/\sqrt{21}}$. Hence, $a = -4/\sqrt{21}$, $b = -2/\sqrt{21}$, $c = 1/\sqrt{21}$, $x_P = 2$, $y_P = 1$, $z_P = 5$, so (12b) gives

$$-\frac{4}{\sqrt{21}}(x-2) - \frac{2}{\sqrt{21}}(y-1) + \frac{1}{\sqrt{21}}(z-5) = 0,$$

or,

$$\boxed{4x + 2y - z = 5.}$$

Alternatively, we could proceed more directly using the ideas given in the closure section: $f(x,y,z) = x^2 + y^2 - z = 0$

so (25) gives $\hat{\underset{\sim}{n}} = \left.\frac{2x\hat{i} + 2y\hat{j} - \hat{k}}{\sqrt{4x^2+4y^2+1}}\right|_{(2,1,5)} = \frac{4\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{21}}$

and (24) gives $4(x-2) + 2(y-1) - (z-5) = 0$, as above.

(b) $f = x^2 - 4y^2 - z$, $P = (1,2,-3)$

$\underset{\sim}{n} = 2x\hat{i} - 8y\hat{j} - \hat{k}\,|_P = 2\hat{i} - 16\hat{j} - \hat{k}$, $\hat{\underset{\sim}{n}} = (2\hat{i} - 16\hat{j} - \hat{k})/\sqrt{261}$

Tangent plane: $2(x-1) - 16(y-2) - (z+3) = 0$

(c) $f = x^4 + y^4 + z^4 - 18$, $P = (1,-1,2)$

$\underset{\sim}{n} = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}\,|_P = 4\hat{i} - 4\hat{j} + 32\hat{k}$, $\hat{\underset{\sim}{n}} = (\hat{i} - \hat{j} + 8\hat{k})/\sqrt{66}$

Tangent plane: $(x-1) - (y+1) + 8(z-2) = 0$

(d) $f = x - yz$, $P = (3,-1,-3)$

$\underset{\sim}{n} = \hat{\imath} - z\hat{\jmath} - y\hat{k}\,\big|_P = \hat{\imath} + 3\hat{\jmath} + \hat{k}$, $\underset{\sim}{\hat{n}} = (\hat{\imath} + 3\hat{\jmath} + \hat{k})/\sqrt{11}$

Tangent plane: $(x-3) + 3(y+1) + (z+3) = 0$

(e) $f = x^2 - y + z^4$, $P = (1,2,1)$

$\underset{\sim}{n} = 2x\hat{\imath} - \hat{\jmath} + 4z^3\hat{k}\,\big|_P = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$, $\underset{\sim}{\hat{n}} = (2\hat{\imath} - \hat{\jmath} + 4\hat{k})/\sqrt{21}$

Tangent plane: $2(x-1) - (y-2) + 4(z-1) = 0$

(g) $\underset{\sim}{R}(u,v) = u\hat{\imath} + (u+v^2)\hat{\jmath} + (v+1)\hat{k}$, $P: u=3, v=2$ so $x=3, y=7, z=3$

$$\underset{\sim}{n} = \underset{\sim}{R}_u \times \underset{\sim}{R}_v\big|_P = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 2v & 1 \end{vmatrix}_P = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = \hat{\imath} - \hat{\jmath} + 4\hat{k} \text{ so } \underset{\sim}{\hat{n}} = (\hat{\imath} - \hat{\jmath} + 4\hat{k})/\sqrt{18}$$

Tangent plane: $(x-3) - (y-7) + 4(z-3) = 0$

(h) $\underset{\sim}{R} = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u^2\hat{k}$, $P: u=2, v=\pi/6$ so $x=\sqrt{3}, y=1, z=4$

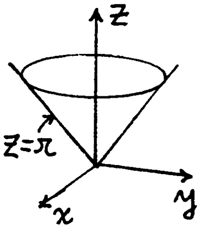
$$\underset{\sim}{n} = \underset{\sim}{R}_u \times \underset{\sim}{R}_v\big|_P = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}_P = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sqrt{3}/2 & 1/2 & 4 \\ -1 & \sqrt{3} & 0 \end{vmatrix} = -4\sqrt{3}\,\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$$

so $\underset{\sim}{\hat{n}} = (-2\sqrt{3}\,\hat{\imath} - 2\hat{\jmath} + \hat{k})/\sqrt{17}$

Tangent plane: $-2\sqrt{3}(x-\sqrt{3}) - 2(y-1) + (z-4) = 0$

(i) $\underset{\sim}{R} = u\sin v\,\hat{\imath} + u\cos v\,\hat{\jmath} + u\hat{k}$, $P: u=0, v=\pi$ so $x=y=z=0$

$$\underset{\sim}{n} = \underset{\sim}{R}_u \times \underset{\sim}{R}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin v & \cos v & 1 \\ u\cos v & -u\sin v & 0 \end{vmatrix}_P = \underset{\sim}{0}$$



so a unique normal line and tangent plane do not exist at P. Geometrically, the reason is that the surface S is a cone with its apex at P. (That is, $x^2+y^2=z^2$, so $z = \sqrt{x^2+y^2} = r$.)

(j) $\underset{\sim}{R} = ve^u\,\hat{\imath} + (u-v)\hat{\jmath} + v^2\hat{k}$, $P: u=2, v=1$ so $x=e^2, y=1, z=1$

$$\underset{\sim}{n} = \underset{\sim}{R}_u \times \underset{\sim}{R}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ ve^u & 1 & 0 \\ e^u & -1 & 2v \end{vmatrix}_P = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^2 & 1 & 0 \\ e^2 & -1 & 2 \end{vmatrix} = 2\hat{\imath} - 2e^2\hat{\jmath} - 2e^2\hat{k}$$

so $\underset{\sim}{\hat{n}} = (\hat{\imath} - e^2\hat{\jmath} - e^2\hat{k})/\sqrt{2e^4+1}$

Tangent plane: $(x-e^2) - e^2(y-1) - e^2(z-1) = 0$

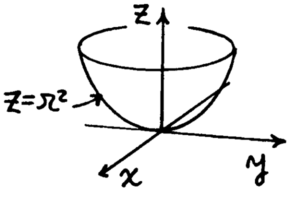
Section 15.5

1\. (a) $S: x^2+y^2=1$, $0<z<1+y$. Use $x=\cos v$, $y=\sin v$, $z=u$

(5) gives $dA = du\,dv$ so $A = \int_0^{2\pi}\int_0^{1+\sin v} du\,dv = 2\pi$.

(b) $S: x^2+y^2=1$, $0<z<1-y^2$. Use $x=\cos v$, $y=\sin v$, $z=u$

(5) gives $dA = du\,dv$ so $A = \int_0^{2\pi}\int_0^{1-\sin^2 v} du\,dv = \pi$



2\. (a) $x^2+y^2=z$, $0<z<h$. Use $x=u\cos v$, $y=u\sin v$, $z=u^2$

Then $E = \cos^2 v + \sin^2 v + 4u^2 = 1+4u^2$

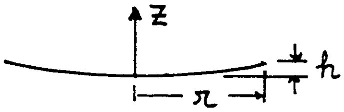
$F = (\cancel{\cos v)(-u}\sin v) + (\cancel{\sin v)(u}\cos v) + 0 = 0$

$G = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2$ so $dA = u\sqrt{1+4u^2}\,du\,dv$, so $A = \int_0^{2\pi}\int_0^{\sqrt{h}} u\sqrt{1+4u^2}\,du\,dv = \frac{2\pi}{12}\left[(1+4h)^{3/2}-1\right]$

(b) A Taylor series of the latter, about $h=0$, gives

$$A = \frac{\pi}{6}\left[1+\tfrac{3}{2}(4h)+\cdots -1\right] = \frac{\pi}{6}(6h+\cdots) \sim \pi h \text{ as } h\to 0.$$

This result makes sense since as $h\to 0$ S approaches a flat "dish" of radius $\sqrt{x^2+y^2} = \sqrt{z} = \sqrt{h}$ and the area $\sim$ "πr^2" $= \pi(\sqrt{h})^2 = \pi h$.



3. S: $x^2+y^2-z^2=1$, $0<z<h$.

(a) Let $x = u\cos v$, $y = u\sin v$, $z = \sqrt{u^2-1}$.

Then $E = \cos^2 v + \sin^2 v + \left[(\tfrac{1}{2})(2u)/\sqrt{u^2-1}\,\right]^2 = \frac{2u^2-1}{u^2-1}$, $F=0$, $G=u^2$

so

$$A = \int_0^{2\pi}\int_1^{\sqrt{1+h^2}} u\sqrt{\frac{2u^2-1}{u^2-1}}\,du\,dv = 2\pi\int_1^{\sqrt{1+h^2}} u\sqrt{\frac{2u^2-1}{u^2-1}}\,du \qquad (u^2=t)$$

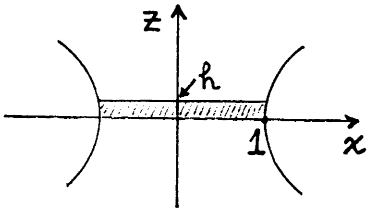
$$= \pi\int_1^{1+h^2}\sqrt{\frac{2t-1}{t-1}}\,dt$$

and now using integral tables or computer software (such as the Maple int command) gives

$$A = \pi\left\{h\sqrt{1+2h^2} + \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\,h+\sqrt{1+2h^2}\,\right|\right\}$$

(b) As $h\to 0$, $A = \pi\left[h(1+\tfrac{1}{2}2h^2+\cdots)+\frac{1}{\sqrt{2}}(\sqrt{2}\,h+\cdots)\right] \sim 2\pi h$.

This result looks correct because as $h\to 0$ S approaches a short cylinder (i.e., a circular cylinder) of radius 1 and length h. Thus, its surface area is $\sim (2\pi 1)h = 2\pi h$. ✓



4. (a) S: $z^2-x^2-y^2=1$, $1<z<h$.

Let $x = u\cos v$, $y = u\sin v$, $z = \sqrt{1+u^2}$.

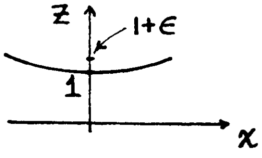
Then $E = \cos^2 v + \sin^2 v + \left[\tfrac{1}{2}2u/\sqrt{1+u^2}\,\right]^2 = \frac{2u^2+1}{u^2+1}$, $F=0$, $G=u^2$,

so $dA = u\sqrt{\frac{2u^2+1}{u^2+1}}\,du\,dv$ and

$$A = \int_0^{2\pi}\int_0^{\sqrt{h^2-1}} u\sqrt{\frac{2u^2+1}{u^2+1}}\,du\,dv \overset{u^2=t}{\underset{\downarrow}{=}} \pi\int_0^{h^2-1}\sqrt{\frac{2t+1}{t+1}}\,dt$$

$$= \pi\left\{h\sqrt{2h^2-1} - \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\,h+\sqrt{2h^2-1}\,\right| - 1 + \frac{1}{\sqrt{2}}\ln\left|1+\sqrt{2}\,\right|\right\}$$

(b) If $h = 1+\epsilon$, then as $\epsilon\to 0$ the hyperboloid becomes a flat dish of radius $u = \sqrt{z^2-1}$ $= \sqrt{(1+\epsilon)^2-1} = \sqrt{2\epsilon+\epsilon^2} \sim \sqrt{2\epsilon}$, and its area $\sim \pi u^2 \sim 2\pi\epsilon$. Let's see if we get that result:

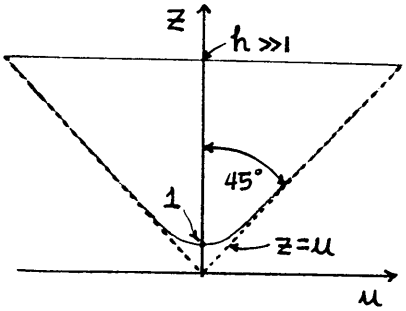


$$A = \pi\left\{(1+\epsilon)\sqrt{2(1+2\epsilon+\epsilon^2)-1} - \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}(1+\epsilon)+\sqrt{2(1+2\epsilon+\epsilon^2)-1}\,\right| - 1 + \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\right\}$$

and a Taylor expansion in ϵ, about $\epsilon = 0$, gives

$$A = 0 + 2\pi\epsilon + \cdots \sim 2\pi\epsilon. \checkmark$$

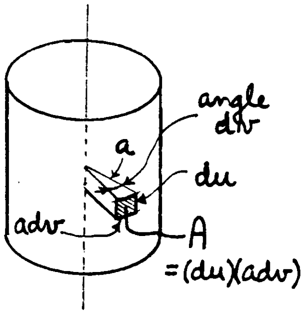
(c) This time the limit as h becomes large. Since $z = \sqrt{u^2+1}$ (where $u = \sqrt{x^2+y^2}$ is the radius), we see that large z corresponds to large u, and $z = \sqrt{u^2+1} \sim u$, which is a cone with vertex at the origin as shown (dotted) as the right. Thus, for large h we should have $A \sim$ area of a 45° cone, between $z=0$ and $z=h$, namely, $\sqrt{2}\pi h^2$. Let's see:



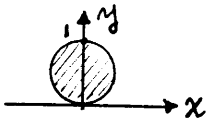
$$A = \pi\left\{ h\sqrt{2h^2-1} - \frac{1}{\sqrt{2}} \ln\left|\sqrt{2}h + \sqrt{2h^2-1}\right| - 1 + \frac{1}{\sqrt{2}} \ln|1+\sqrt{2}|\right\}.$$

$$\sim \pi\left\{\sqrt{2}h^2 - \frac{1}{\sqrt{2}}\ln(2\sqrt{2}h)\right\} \sim \sqrt{2}\pi h^2. \checkmark$$

5. S: $x^2+y^2=a^2$; $x = a\cos v$, $y = a\sin v$, $z = u$. Obtain $E=1, F=0, G=a^2$, so $dA = \sqrt{a^2}\,du\,dv$ $= a\,du\,dv$



6. (a)

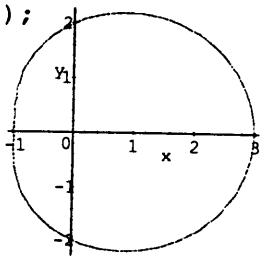


$$A = \int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \frac{1}{2}\sin^2\theta\,d\theta = \frac{\pi}{4}.$$

(In fact, $r = \sin\theta$ gives $r^2 = r\sin\theta$ or $x^2+y^2=y$ or $x^2+(y-\frac{1}{2})^2 = \frac{1}{4}$ which is a circle of radius 1/2, centered at (0,1/2).)

(b) Let us plot it first, using maple:

```
[ > with(plots):
[ > implicitplot(x^2+y^2=2*sqrt(x^2+y^2)+x,x=-4..4,y=-4..4);
```



$$A = \int_0^{2\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}(2+\cos\theta)^2\,d\theta$$

$$= \int_0^{2\pi}(2+2\cos\theta + \frac{1}{2}\cos^2\theta)\,d\theta = 4\pi + \frac{\pi}{2} = \frac{9\pi}{2}$$

(c) Let us first plot it, using Maple: First, write

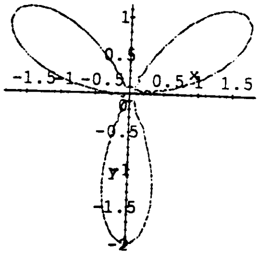
$$r = 2\sin 3\theta = 2(3\sin\theta - 4\sin^3\theta)$$

$$= 2\left(3\frac{r\sin\theta}{r} - 4\frac{r^3\sin^3\theta}{r^3}\right)$$

or,

$$r^4 = 6r^2y - 8y^3, \text{ or, } (x^2+y^2-6y)(x^2+y^2) = -8y^3$$

```
> implicitplot((x^2+y^2-6*y)*(x^2+y^2)=-8*y^3,x=-5..5,
  y=-5..5,numpoints=5000);
```



What are the θ limits for the leaf (actually, petal) that is in the first quadrant? We can see from $r = 2\sin 3\theta$ that $r \to 0$ as $\theta \to 0, \pm\pi/3, \pm 2\pi/3, \pm 3\pi/3, \ldots$, of which only $\theta =$

$0, \pm\pi/3, \pm 2\pi/3$, and π are not redundant. In particular, the first-quadrant petal lies between $\theta=0$ and $\theta=\pi/3$, so

$$A=\int_0^{\pi/3}\int_0^{2\sin 3\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\sin^2 3\theta\,d\theta \overset{3\theta=\phi}{=} 2\int_0^{\pi}\sin^2\phi\,\frac{d\phi}{3}=\frac{\pi}{3}$$

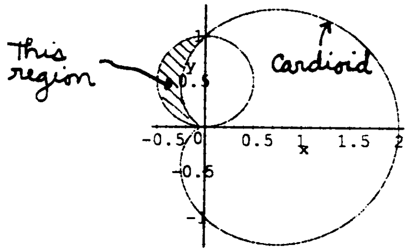
(d) Let's look at a graph first. Using Maple,

```
> with(plots):
> implicitplot({x^2+(y-1/2)^2=1/4,x^2+y^2=sqrt(x^2+y^2)+x},x=-2..2,y
  =-2..2,numpoints=5000);
```

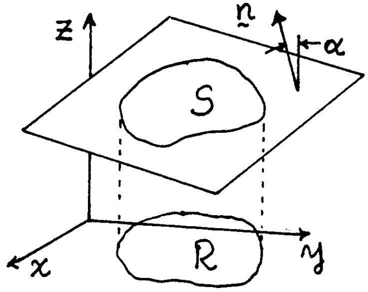
$$A=\int_{\pi/2}^{\pi}\int_{1+\cos\theta}^{\sin\theta} r\,dr\,d\theta$$

$$=\int_{\pi/2}^{\pi}\left(\frac{\sin^2\theta}{2}-\frac{1+2\cos\theta+\cos^2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left(\frac{\pi}{4}-\frac{\pi}{2}+2-\frac{\pi}{4}\right)=\frac{4-\pi}{4}$$



7.



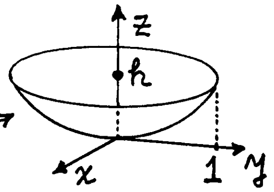
$z=(d-ax-by)/c$

$$\text{area of } S = \iint_R \sqrt{1+\frac{a^2}{c^2}+\frac{b^2}{c^2}}\,dx\,dy$$

$$=\frac{\sqrt{a^2+b^2+c^2}}{c}\times \text{area of } R.$$

Now, $\hat{n}=\pm\dfrac{\frac{a}{c}\hat{\imath}+\frac{b}{c}\hat{\jmath}+\hat{k}}{\sqrt{(a^2+b^2+c^2)/c^2}}$. Of these, chose the + since we want $\hat{n}$ to have a nonnegative z component if α is to be acute. Then $\hat{n}\cdot\hat{k}=\dfrac{c}{\sqrt{a^2+b^2+c^2}}=\cos\alpha$, so $\sqrt{a^2+b^2+c^2}/c=\sec\alpha$ and

$$\text{area of } S = (\sec\alpha)(\text{area of } R).$$



8. The shadow of the surface $z=h(1-x^2-y^2)$ $(0\le z\le h)$ on the x,y plane is the unit disk. also,

$$\sqrt{1+f_x^2+f_y^2}=\sqrt{1+(-2hx)^2+(-2hy)^2}=\sqrt{1+4h^2(x^2+y^2)}\text{ , so}$$

$$A=\iint_{\text{unit disk}}\sqrt{1+4h^2(x^2+y^2)}\,dx\,dy=\int_0^{2\pi}\int_0^1\sqrt{1+4h^2r^2}\,r\,dr\,d\theta=\frac{\pi}{6h^2}\left[(1+4h^2)^{3/2}-1\right].$$

Using the Taylor series $(1+x)^{3/2}=1+\frac{3}{2}x+\cdots$ we have

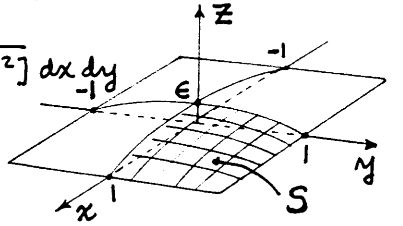
$$A=\frac{\pi}{6h^2}\left[1+\frac{3}{2}4h^2+\cdots-1\right]\sim\frac{\pi}{6h^2}\frac{3}{2}4h^2=\pi \text{ as } h\to 0,$$

which makes sense because as $h\to 0$ the surface flattens out to be the unit disk in the x,y plane, with area $=\pi(1)^2=\pi$.

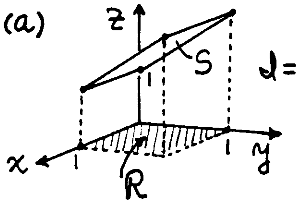
9. $A=\int_0^1\int_0^1\sqrt{1+z_x^2+z_y^2}\,dx\,dy=\int_0^1\int_0^1\sqrt{1+4\epsilon^2[x^2(1-y^2)^2+(1-x^2)^2y^2]}\,dx\,dy$

$$\approx\int_0^1\int_0^1\left\{1+\frac{1}{2}4\epsilon^2[x^2(1-y^2)^2+(1-x^2)^2y^2]\right\}dx\,dy$$

$$=1+\frac{32}{45}\epsilon^2.$$



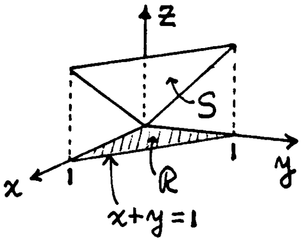
10. (a)



$\delta = \iint_S (1+x)\,dA = \iint_R (1+x)\sqrt{1+z_x^2+z_y^2}\,dx\,dy = \int_0^1\int_0^1 (1+x)\sqrt{2}\,dx\,dy$
$= 3/\sqrt{2}$

(b) $\delta = \iint_S (1+x)\,dA = \iint_R (1+x)\sqrt{1+z_x^2+z_y^2}\,dx\,dy$

$= \int_0^1\int_0^{1-y} \sqrt{3}(1+x)\,dx\,dy = \sqrt{3}\int_0^1 (\frac{3}{2}-2y+\frac{y^2}{2})\,dy = \frac{2}{\sqrt{3}}$



(d) $\delta = \iint_S (1+x)\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} (1+\cos\theta)\underbrace{dz\,d\theta}_{dA=(1d\theta)(dz)}$

$= \int_0^{2\pi}(1+\cos\theta)^2\,d\theta = \int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta = 2\pi+0+\pi = 3\pi$

(e) Use spherical coordinates, $x = 3\sin v\cos u$ $(\rho=3,\ v=\phi,\ u=\theta)$
$y = 3\sin v\sin u$
$z = 3\cos v$

$E = (-3\sin v\sin u)^2+(3\sin v\cos u)^2 = 9\sin^2 v$, $F=0$,
$G = (3\cos v\cos u)^2+(3\cos v\sin u)^2+(-3\sin v)^2 = 9$, so $dA = \sqrt{81\sin^2 v - 0^2}\,du\,dv$
so $\delta = \iint_S (1+x)\,dA = \int_0^{\pi}\int_{-\pi/2}^{\pi/2}(1+3\sin v\cos u)(9\sin v\,du\,dv) = 45\pi$

(f) Use spherical coordinates, $x = 2\sin v\cos u$ $(\rho=2,\ v=\phi,\ u=\theta)$
$y = 2\sin v\sin u$ so $dA = \sqrt{EG-F^2}\,du\,dv$
$z = 2\cos v$ $= 4\sin v\,du\,dv$

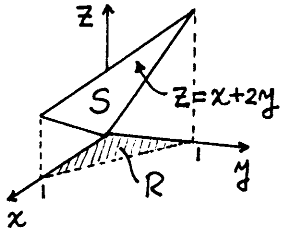
so $\delta = \iint_S (1+x)\,dA = \int_0^{\pi}\int_0^{2\pi}(1+2\sin v\cos u)(4\sin v\,du\,dv) = \int_0^{\pi} 8\pi\sin v\,dv = 16\pi$

(g) Let $x = u\cos v$, $y = u\sin v$, $z = u$, so $E=2$, $F=0$, $G=u^2$, $dA = \sqrt{2}\,u\,du\,dv$
so $\delta = \iint_S (1+x)\,dA = \int_0^{2\pi}\int_0^{h}(1+u\cos v)\sqrt{2}\,u\,du\,dv = \sqrt{2}\pi h^2$.

(h) Let $x = u\cos v$, $y = u\sin v$, $z = u^2$, so $E = 1+4u^2$, $F=0$, $G=u^2$, $dA = u\sqrt{1+4u^2}\,du\,dv$
so $\delta = \iint_S (1+x)\,dA = \int_0^{2\pi}\int_0^{\sqrt{h}}(1+u\cos v)\,u\sqrt{1+4u^2}\,du\,dv$ (Note: $\int_0^{2\pi}\cos v\,dv$ will $=0$)

$= 2\pi\int_0^{\sqrt{h}} u\sqrt{1+4u^2}\,du = 2\pi\int_1^{1+4h}\sqrt{t}\,\frac{dt}{8}$ (where $1+4u^2 = t$)
$= \frac{\pi}{6}[(1+4h)^{3/2}-1]$

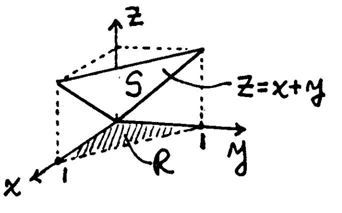
11. (a)



$M = \iint_S \sigma\,dA = \sigma_0\iint_R \sqrt{1+z_x^2+z_y^2}\,dx\,dy = \sigma_0\iint_R\sqrt{1+1+4}\,dx\,dy = \sigma_0\frac{\sqrt{6}}{2}$

$x_c = \frac{1}{M}\iint_S \sigma x\,dA = \frac{2}{\sigma_0\sqrt{6}}\iint_R x\sigma_0\sqrt{6}\,dx\,dy = 2\int_0^1\int_0^{1-y} x\,dx\,dy = \frac{1}{3}$

(b) $M = \iint_S \sigma\,dA = \iint_R (1+y)\sqrt{1+z_x^2+z_y^2}\,dx\,dy = \int_0^1\int_0^{1-y}(1+y)\sqrt{3}\,dx\,dy$
$= 2/\sqrt{3}$

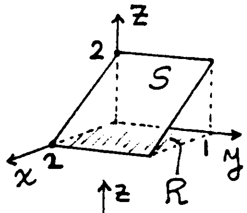


$$x_c = \frac{1}{M}\iint_S \sigma x\,dA = \frac{1}{2/\sqrt{3}}\iint_R (1+y)x\sqrt{3}\,dx\,dy = \frac{\sqrt{3}}{2}\sqrt{3}\int_0^1\int_0^{1-y} x(1+y)\,dx\,dy$$

$$= \frac{3}{2}\int_0^1 \frac{(1-y)^2}{2}(1+y)\,dy = \frac{3}{4}\left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right) = \frac{5}{16}$$

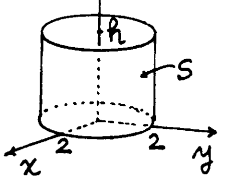
(c) $M = \iint_S \sigma\,dA = \iint_R (4-x)\sqrt{1+z_x^2+z_y^2}\,dx\,dy = \int_0^1\int_0^2 (4-x)\sqrt{2}\,dx\,dy = 6\sqrt{2}$

$x_c = \frac{1}{M}\iint_S x\sigma\,dA = \frac{1}{6\sqrt{2}}\int_0^1\int_0^2 x(4-x)\sqrt{2}\,dx\,dy = \frac{16\sqrt{2}}{3}/6\sqrt{2} = \frac{8}{9}$



(e) $M = \iint_S \sigma\,dA = \iint_S (4+x)\,dA = \int_0^h\int_0^{2\pi}(4+2\cos\theta)(2\,d\theta\,dz) = 16\pi h$

$x_c = \frac{1}{M}\iint_S x\sigma\,dA = \frac{1}{16\pi h}\int_0^h\int_0^{2\pi} 2\cos\theta\,(4+2\cos\theta)\,2\,d\theta\,dz = \frac{1}{2}$



(h) $M = \iint_S \sigma\,dA = \int_0^{2\pi}\int_0^{\pi}(4+3\sin\phi\cos\theta)(9\sin\phi\,d\phi\,d\theta) = 144\pi$. (Here we use spherical coords: so $x = 3\sin\phi\cos\theta$ and $dA = 9\sin\phi\,d\phi\,d\theta$.)

$x_c = \frac{1}{M}\iint_S x\sigma\,dA = \frac{1}{144\pi}\int_0^{2\pi}\int_0^{\pi}(3\sin\phi\cos\theta)(4+3\sin\phi\cos\theta)(9\sin\phi\,d\phi\,d\theta)$

so $x_c = \frac{3}{4}$

12. (a) $\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA = \iint_R (y^2+z^2)\sqrt{1+z_x^2+z_y^2}\,\sigma_0\,dx\,dy$

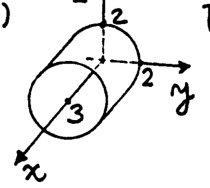
$$= \sigma_0\int_0^1\int_0^{1-y}\left[y^2+(x+2y)^2\right]\sqrt{1+1+4}\,dx\,dy = 4\sigma_0/\sqrt{6}$$

(b) $\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA = \iint_R (y^2+z^2)\sqrt{1+z_x^2+z_y^2}\,(1+y)\,dx\,dy$

$$= \int_0^1\int_0^{1-y}\left[y^2+(x+y)^2\right]\sqrt{1+1+1}\,(1+y)\,dx\,dy = \frac{29}{60}\sqrt{3}$$

(d) $\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA = \sigma_0\int_0^{2\pi}\int_0^{\pi}(c^2\sin^2\phi\sin^2\theta + c^2\cos^2\phi)(c^2\sin\phi\,d\phi\,d\theta) = \frac{8}{3}\pi\sigma_0 c^4$

(e)



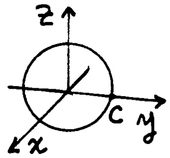
Use these cylindrical coordinates:

$y = 2\cos\theta$
$z = 2\sin\theta$
$x = x$
$dA = 2\,d\theta\,dx.$

Then,

$$\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA = \int_0^3\int_0^{2\pi}(4)\sigma_0\,2\,d\theta\,dx = 48\pi\sigma_0$$

(f)

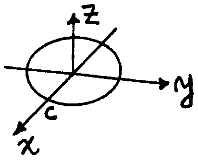


$y = r\cos\theta$
$z = r\sin\theta$
$dA = r\,dr\,d\theta$

Then,

$$\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA = \int_0^{2\pi}\int_0^c r^2\sigma_0\,r\,dr\,d\theta = \pi\sigma_0 c^4/2$$

(g)



$x = r\cos\theta$
$y = r\sin\theta$
$z = 0$
$dA = r\,dr\,d\theta$

Then, $\mathcal{I}_x = \iint_S (y^2+z^2)\sigma\,dA$

$$= \int_0^{2\pi}\int_0^c (r^2\sin^2\theta + 0)\,\sigma_0\,r\,dr\,d\theta = \pi\sigma_0\int_0^c r^3\,dr = \pi\sigma_0 c^4/4$$

13. (a) $I = \int_0^{\pi/4}\int_1^2 \frac{r\sin\theta}{r\cos\theta}\, r\,dr\,d\theta = \frac{3}{2}\int_0^{\pi/4} -\frac{d(\cos\theta)}{\cos\theta} = -\frac{3}{2}\ln(\cos\theta)\Big|_0^{\pi/4} = -\frac{3}{2}\ln\frac{1}{\sqrt{2}} = \frac{3}{4}\ln 2$

(b) $I = \int_0^{\pi/3}\int_0^2 \sin r^2\, r\,dr\,d\theta = \frac{\pi}{3}\int_0^2 \sin r^2\,\frac{d(r^2)}{2} = \frac{\pi}{6}(-\cos r^2)\Big|_0^2 = \frac{\pi}{6}(1-\cos 4)$

(c) $I = \int_{-\pi/3}^{\pi/3}\int_1^2 \frac{y(x^2+y^2)^2}{x}\,dx\,dy = \int_{-\pi/3}^{\pi/3}\int_1^2 \frac{r\sin\theta}{r\cos\theta}\, r^4\, r\,dr\,d\theta = 0$

(d) $I = \int_0^{\pi/2}\int_1^2 \frac{1}{r}\, r\,dr\,d\theta = \pi/2$

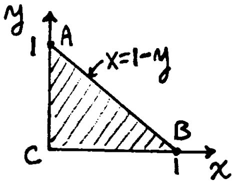
(f) $I = \int_0^{\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta = -\pi e^{-r^2}/2\,\Big|_0^1 = \frac{\pi}{2}(1-e^{-1})$

14. $I = \int_0^1\int_0^{1-y} e^{x/(x+2y)}\,dx\,dy$. Let $\begin{matrix} u = x \\ v = x+2y \end{matrix}$. Then $dA \to |J(u,v)|\,du\,dv = \left|\begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix}\right| du\,dv$. Observe that since u and v are given as functions of x and y, the required x_u, x_v, y_u, y_v derivatives are "reverse" derivatives. In the present example it is easy to solve $u=x, v=x+2y$ for x and y as $x=u, y=(v-u)/2$, so $x_u=1, x_v=0, y_u=-1/2, y_v=1/2$ and $J(u,v)=\left|\begin{matrix} 1 & 0 \\ -1/2 & 1/2 \end{matrix}\right| = 1/2$. Alternatively (e.g., if the change of variables were such that we could not solve for x, y explicitly as we did here), we can use the fact [see equation (47) in Sec. 13.6] $J(u,v)$ and $J(x,y)$ are numerical inverses of each other. Thus,

$$J(u,v) = 1/J(x,y) = 1\Big/\left|\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}\right| = 1\Big/\left|\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}\right| = 1/2 \text{ again. Thus,}$$

$$I = \int_0^1\int_0^{1-y} e^{x/(x+2y)}\,dx\,dy = \int_?^?\int_?^? e^{u/v}\,|J(u,v)|\,du\,dv = \int_?^?\int_?^? e^{u/v}\,\frac{1}{2}\,du\,dv.$$

Now we need the new limits:

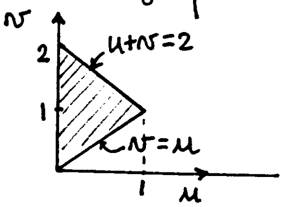


AB: $\begin{matrix} u = x = 1-y \\ v = x+2y = 1-y+2y = 1+y \end{matrix}$ } These are parametric equations for the image curve in the u,v plane, where y is the parameter and $0<y<1$. Or, $u+v=2$

CB: $\begin{matrix} u = x \\ v = x+2y = x \end{matrix}$ } Parametric equations for the image curve, where x is the parameter and $0<x<1$. Or, $u=v$

CA: $\begin{matrix} u = x \quad\;\; = 0 \\ v = x+2y = 2y \end{matrix}$ } Parametric equations for the image curve, where y is the parameter and $0<y<1$.

Thus, the image of the region S is as shown below:



$$\text{so } I = \frac{1}{2}\left[\int_0^1\int_0^v e^{u/v}\,du\,dv + \int_1^2\int_0^{2-v} e^{u/v}\,du\,dv\right]$$

$$= \frac{1}{2}\int_0^1 v e^{u/v}\Big|_0^v dv + \frac{1}{2}\int_1^2 v e^{u/v}\Big|_0^{2-v} dv$$

$$= \frac{1}{2}\int_0^1 v(e-1)\,dv + \frac{1}{2}\int_1^2 v\left(e^{\frac{2}{v}-1}-1\right)dv$$

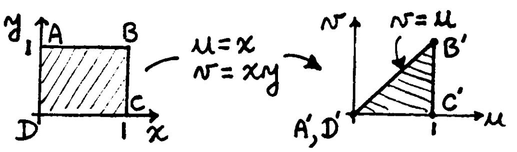
$$= \frac{e}{4} - 1 + \frac{1}{2e}\int_1^2 v e^{2/v}\,dv$$

Maple:

```
   A: = evalf(int(v* exp(2/v)/(2* exp(1)), v=1..2));
   evalf (exp(1)/4 -1+A);
```

gives the result 0.7662451690.

15.



since AB: $\left.\begin{matrix} u=x \\ v=x \end{matrix}\right\}$ gives $v=u$ over $0<u<1$

BC: $\left.\begin{matrix} u=1 \\ v=y \end{matrix}\right\}$ gives $u=1$ over $0<v<1$

CD: $\left.\begin{matrix} u=x \\ v=0 \end{matrix}\right\}$ gives $v=0$ over $0<u<1$

AD: $\left.\begin{matrix} u=0 \\ v=0 \end{matrix}\right\}$ gives the point $u=v=0$

$$J(u,v) = 1/J(x,y) = 1\Big/\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
$$= 1\Big/\begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = 1/x = 1/u,$$

so

$$\mathcal{I} = \int_0^1\int_0^1 \sin\frac{xy}{xy+1}\,dx\,dy = \int_0^1\int_v^1 \sin\frac{v}{v+1}\,\frac{1}{u}\,du\,dv$$

(Integrate on u first because integrand is an easy function of u, but a hard function of v.)

$$= -\int_0^1 \sin\frac{v}{v+1}\,\ln v\,dv,$$

which looks difficult – but at least we were able to reduce $\mathcal{I}$ from a double integral to a single integral. To complete, using Maple, use the command

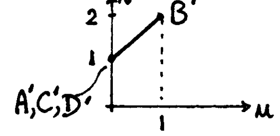
```
evalf(int(-sin(v/(v+1))*ln(v), v=0..1);
```

and obtain the result

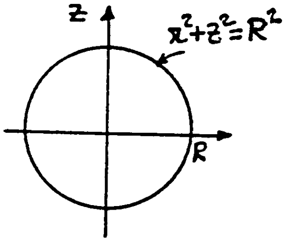
In fact, we could have used Maple to evaluate the original double integral, by the command

```
evalf(int(int(sin(x*y/(x*y+1)), x=0..1), y=0..1));
```

Could we have used the change of variables $u=xy, v=xy+1$ instead? Then $J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ y & x \end{vmatrix} = 0$ so $J(u,v) = 1/0 = \infty$. Further, the given unit square maps into a <u>line</u> – the straight line from $(0,1)$ to $(1,2)$. Thus we have the degenerate case of zero area and infinite integrand.



Section 15.6



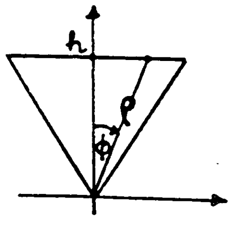
1. (a) $V = \int_{-R}^{R}\int_{0}^{2\pi}\int_{0}^{\sqrt{R^2-z^2}} r\,dr\,d\theta\,dz = \frac{2\pi}{2}\int_{-R}^{R}(R^2-z^2)\,dz = \frac{4\pi R^3}{3}$

(b) $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{R^3}{3}\cdot 2 = \frac{4\pi R^3}{3}$

2. (a) $V = \int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha} r\,dr\,d\theta\,dz = 2\pi\int_0^h \frac{z^2\tan^2\alpha}{2}\,dz = \frac{\pi h^3}{3}\tan^2\alpha$

(b) $V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{h/\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= 2\pi\int_0^{\alpha}\frac{1}{3}\frac{h^3}{\cos^3\phi}\sin\phi\,d\phi = -\frac{2\pi h^3}{3}\int_0^{\alpha}\frac{d(\cos\phi)}{\cos^3\phi}$$
$$= \frac{\pi h^3}{3}\left(\frac{1}{\cos^2\alpha}-1\right) = \frac{\pi h^3}{3}\tan^2\alpha$$



3. (a) $\hat{\underset{\sim}{i}}, \hat{\underset{\sim}{j}}$ components $= 0$ by symmetry, so

$$\underset{\sim}{F}(0,0,0) = G\sigma\hat{\underset{\sim}{k}}\iiint_V \frac{z\,dV}{(x^2+y^2+z^2)^{3/2}} = G\sigma\hat{\underset{\sim}{k}}\int_0^R\int_0^{2\pi}\int_0^{\sqrt{R^2-z^2}} \frac{z\,r\,dr\,d\theta\,dz}{(r^2+z^2)^{3/2}}$$

$$= G\sigma\hat{\underset{\sim}{k}}\, 2\pi\int_0^R z\left(-\frac{1}{R}+\frac{1}{z}\right)dz = \pi\sigma G R\hat{\underset{\sim}{k}}$$

(b) $$\underset{\sim}{F}(0,0,0) = G\sigma\hat{\underset{\sim}{k}}\iiint_V \frac{z\,dV}{(x^2+y^2+z^2)^{3/2}} = G\sigma\hat{\underset{\sim}{k}}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}\frac{(\cancel{\rho}\cos\phi)(\cancel{\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta)}{\cancel{\rho^3}}$$

$$= \hat{\underset{\sim}{k}}\,2\pi\sigma G R\int_0^{\pi/2}\sin\phi\,d(\sin\phi) = \pi\sigma G R\hat{\underset{\sim}{k}}$$

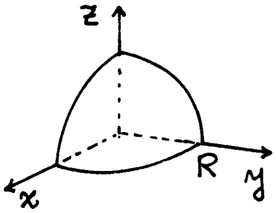
4. (a) $$M = \iiint\sigma\,dV = \sigma\int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha} r\,dr\,d\theta\,dz = 2\pi\sigma\int_0^h\frac{z^2\tan^2\alpha}{2}dz = \frac{\pi}{3}\sigma h^3\tan^2\alpha$$

$$z_c = \frac{1}{M}\iiint z\sigma\,dV = \frac{3}{\pi h^3\tan^2\alpha}\int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha} z r\,dr\,d\theta\,dz$$

$$= \frac{3}{\cancel{\pi}h^3\cancel{\tan^2\alpha}}\,2\cancel{\pi}\int_0^h\frac{z^3\cancel{\tan^2\alpha}}{2}dz = \frac{3}{4}h$$

(b) By symmetry, clearly, $x_c = y_c = z_c$

$M = \sigma\frac{1}{8}\frac{4}{3}\pi R^3 = \pi\sigma R^3/6$



$$x_c = \frac{6}{\pi\sigma R^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \overbrace{(\rho\sin\phi\cos\theta)}^{x}\sigma\overbrace{(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)}^{dV}$$

$$= \frac{6}{\pi R^3}\int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^R\rho^3 d\rho = \frac{6}{\pi R^3}\frac{\pi R^4}{16} = \frac{3}{8}R = y_c = z_c$$

(c) $$M = \sigma\int_a^R\int_0^{2\pi}\int_0^{\sqrt{R^2-z^2}} r\,dr\,d\theta\,dz = 2\pi\sigma\int_a^R\frac{R^2-z^2}{2}dz = \pi\sigma\left(\frac{2}{3}R^3 - aR^2 + \frac{a^3}{3}\right)$$

$$z_c = \frac{3}{\pi\cancel{\sigma}(2R^3-3aR^2+a^3)}\int_a^R\int_0^{2\pi}\int_0^{\sqrt{R^2-z^2}}\cancel{\sigma}z\,r\,dr\,d\theta\,dz$$

$$= \frac{3}{\cancel{\pi}(2R^3-3aR^2+a^3)}\cancel{2\pi}\int_a^R\frac{R^2z-z^3}{\cancel{2}}dz = \frac{3}{4}\frac{R^4-2R^2a^2+a^4}{2R^3-3aR^2+a^3}$$

Partial checks: As $a\to 0$, $z_c\to\frac{3}{8}R$, as in (b) ✓

As $a\to R$, L'Hospital's rule gives $z_c\to R$. ✓

(d) $$M = \sigma\int_0^{2\pi}\int_0^{\alpha}\int_0^{R}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \sigma\frac{R^3}{3}(1-\cos\alpha)2\pi$$

$$z_c = \frac{3}{2\pi\sigma R^3(1-\cos\alpha)}\int_0^{2\pi}\int_0^{\alpha}\int_0^{R}\sigma(\rho\cos\phi)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$$

$$= \frac{3\cancel{\sigma}}{\cancel{2\pi\sigma}R^3(1-\cos\alpha)}\frac{R^4}{4}\frac{\sin^2\alpha}{2}\cancel{2\pi} = \frac{3R\sin^2\alpha}{8(1-\cos\alpha)}$$

5. (a) $$I_x = \iiint_V\sigma(y^2+z^2)\,dV = \sigma\int_0^{2\pi}\int_0^{\pi}\int_a^b\rho^2(\sin^2\phi\sin^2\theta+\cos^2\phi)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$$

$$= \sigma\frac{b^5-a^5}{5}\int_0^{\pi}(\pi\sin^2\phi + 2\pi\cos^2\phi)\sin\phi\,d\phi$$

$$= \pi\sigma \frac{b^5-a^5}{5}\int_0^{\pi}(1+\cos^2\phi)\,d(-\cos\phi) = \frac{8}{3}\pi\sigma\frac{b^5-a^5}{5}$$

But $M = \frac{4}{3}\pi(b^3-a^3)\sigma$, so $I_x = \frac{8}{\cancel{3}}\frac{\cancel{3}}{4}\frac{M}{b^3-a^3}\frac{b^5-a^5}{5} = \frac{2}{5}M\frac{b^5-a^5}{b^3-a^3}$

(b) L'Hospital's rule gives $I_x \sim \frac{2}{5}M\left(\frac{-5a^4}{-3a^2}\right)\Big|_{a\to b}$ or, $I_x \sim \frac{2}{3}Mb^2$

(c) $$I_x = \iiint\limits_V \sigma(y^2+z^2)\,dV = \sigma\int_0^h\int_0^{2\pi}\int_a^b (r^2\sin^2\theta + z^2)\,r\,dr\,d\theta\,dz$$

$$= \sigma\int_0^h\int_a^b(\pi r^3 + 2\pi z^2 r)\,dr\,dz = \sigma\pi\left(\frac{b^4-a^4}{4}h + 2\frac{h^3}{3}\frac{b^2-a^2}{2}\right)$$

But $M = \sigma h\pi(b^2-a^2)$, so $I_x = \frac{M}{4}\frac{b^4-a^4}{b^2-a^2} + \frac{h^2}{3}M = \frac{M}{12}(3b^2+3a^2+4h^2)$

(d) $$I_z = \iiint\limits_V \sigma(x^2+y^2)\,dV = \sigma\int_0^h\int_0^{2\pi}\int_a^b r^2\,r\,dr\,d\theta\,dz = \sigma\frac{b^4-a^4}{4}2\pi h$$

$$= \underbrace{\sigma h\pi(b^2-a^2)}_{M}\frac{b^2+a^2}{2} = \frac{M}{2}(a^2+b^2)$$

(e) $$I_z = \iiint\limits_V \sigma(x^2+y^2)\,dV = \sigma\int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha} r^2\,r\,dr\,d\theta\,dz = \frac{\sigma}{4}2\pi\int_0^h z^4\tan^4\alpha\,dz = \frac{\pi\sigma}{2}\frac{h^5}{5}\tan^4\alpha$$

But $$M = \sigma\iiint dV = \sigma\int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha} r\,dr\,d\theta\,dz = \frac{\sigma}{2}2\pi\int_0^h z^2\tan^2\alpha\,dz = \frac{\pi\sigma}{3}h^3\tan^2\alpha$$

so $I_z = \frac{3M}{10}(h\tan\alpha)^2$

(f) $$I_x = \iiint\limits_V \sigma(y^2+z^2)\,dV = \sigma\int_0^h\int_0^{2\pi}\int_0^{z\tan\alpha}(r^2\sin^2\theta+z^2)\,r\,dr\,d\theta\,dz$$

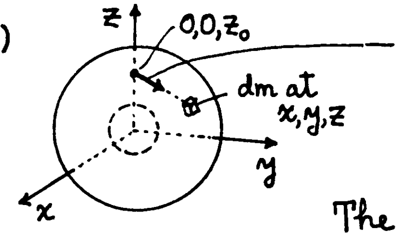
$$= \pi\sigma\int_0^h\int_0^{z\tan\alpha}(r^3+2rz^2)\,dr\,dz = \pi\sigma\int_0^h\left(\frac{z^4\tan^4\alpha}{4} + z^2\tan^2\alpha\,z^2\right)dz$$

$$= \pi\sigma\tan^2\alpha\left(\frac{\tan^2\alpha}{4}+1\right)\frac{h^5}{5} = \underbrace{\frac{\pi\sigma}{3}h^3\tan^2\alpha}_{M}\;\frac{3}{5}h^2\left(\frac{\tan^2\alpha+4}{4}\right) = \frac{3Mh^2}{20}(\tan^2\alpha+4)$$

6. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial(x,y,z)}{\partial(\theta,r,z)} = \begin{vmatrix} x_\theta & x_r & x_z \\ y_\theta & y_r & y_z \\ z_\theta & z_r & z_z \end{vmatrix} = \begin{vmatrix} -r\sin\theta & \cos\theta & 0 \\ r\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = -r$$

so (4) gives $dV = |-r|\,d\theta\,dr\,dz = r\,dr\,d\theta\,dz$ again.

7. (a)



$$d\underset{\sim}{F} = G\frac{1}{d^2}\underset{\sim}{\hat{e}}\,dm$$

$$= G\frac{1}{x^2+y^2+(z-z_0)^2}\frac{x\underset{\sim}{\hat{i}}+y\underset{\sim}{\hat{j}}+(z-z_0)\underset{\sim}{\hat{k}}}{\sqrt{x^2+y^2+(z-z_0)^2}}\sigma dV$$

The $\hat{i}, \hat{j}$ components integrate to 0 by symmetry, leaving

$$\underset{\sim}{F}(0,0,z_0) = \sigma G\hat{k} \iiint_V \frac{(z-z_0)dV}{[x^2+y^2+(z-z_0)^2]^{3/2}}$$

(b) $$\underset{\sim}{F}(0,0,z_0) = \sigma G\hat{k}\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{\rho\cos\phi - z_0}{(\underbrace{\rho^2+z_0^2-2z_0\rho\cos\phi}_{\text{Let this} = t})^{3/2}}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\sigma G\hat{k}\int_a^b\int_{(\rho-z_0)^2}^{(\rho+z_0)^2} \frac{\frac{\rho^2+z_0^2-t}{2z_0} - z_0}{t^{3/2}}\frac{dt}{2z_0\rho}\rho^2 d\rho$$

$$= \frac{2\pi\sigma G}{4z_0^2}\hat{k}\int_a^b\int_{(\rho-z_0)^2}^{(\rho+z_0)^2}[(\rho^2-z_0^2)t^{-3/2} - t^{-1/2}]\,dt\,\rho\, d\rho$$

$$= \frac{\pi\sigma G}{2z_0^2}\hat{k}\int_a^b -2\boxed{\left[\frac{\rho^2-z_0^2}{\sqrt{t}} + \sqrt{t}\,\right]\Bigg|_{t=(\rho-z_0)^2}^{t=(\rho+z_0)^2}}\xleftarrow{} ★\ \rho\, d\rho$$

If we trace $\sqrt{t}$ we see that it is a distance, so we must choose the positive root.
Thus, $$★ = \frac{\rho^2-z_0^2}{\rho+z_0} + \rho + z_0 - \frac{\rho^2-z_0^2}{|\rho-z_0|} - |\rho - z_0|$$

If $z_0 < a$ then $|\rho - z_0| = \rho - z_0$, and
$★ = \rho - z_0 + \rho + z_0 - (\rho+z_0) - (\rho - z_0) = 0$, and $\underset{\sim}{F}(0,0,z_0) = \underset{\sim}{0}$.
If $z_0 > b$ then $|\rho - z_0| = z_0 - \rho$ and
$★ = \rho - z_0 + \rho + z_0 + \rho + z_0 - z_0 + \rho = 4\rho$, so

$$\underset{\sim}{F}(0,0,z_0) = -\frac{4\pi\sigma G}{z_0^2}\hat{k}\int_a^b \frac{\rho^3}{3}d\rho = -\frac{4}{3}\pi(b^3-a^3)\sigma\frac{G}{z_0^2}\hat{k} = -\frac{MG}{z_0^2}\hat{k}$$

8. (a) It suffices to consider one point, say (0,0,0). Clearly, the x and y components of $\underset{\sim}{F}(0,0,0)$ will be 0 by symmetry, so begin with

$$\underset{\sim}{F}(0,0,0) = G\int_0^h\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\sigma\, dx\, dy\, dz$$

$$\overset{*}{=} \sigma G\hat{k}\int_0^h\int_0^{2\pi}\int_0^{\infty}\frac{z r\, dr\, d\theta\, dz}{(\underbrace{r^2+z^2}_{t})^{3/2}} \quad \text{(cylindrical coords.)}$$

$$= \frac{2\pi\sigma}{2}G\hat{k}\int_0^h z\, dz\int_{z^2}^{\infty}\frac{dt}{t^{3/2}} = \pi\sigma G\hat{k}\int_0^h 2\, dz = 2\pi\sigma G h\hat{k}$$

---

* The r and z integrands are of the same form, but it's easier to integrate on r first because of its infinite upper limit. Integrating on z first is trickier:

$$\underset{\sim}{F}(0,0,0) = \text{etc.} = 2\pi\sigma G\hat{k}\int_0^{\infty}\left(1 - \frac{r}{\sqrt{r^2+h^2}}\right)dr$$

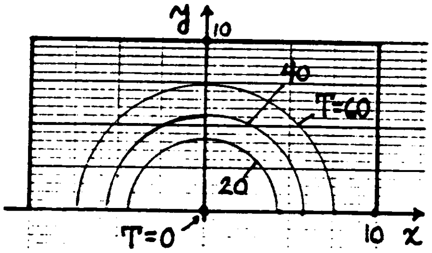
$$= \lim_{R\to\infty} 2\pi\sigma G\hat{k}(R - \underbrace{\boxed{\sqrt{R^2+h}}}_{\sim R + \frac{h^2}{2R}+\cdots} + h) \qquad = 2\pi\sigma G h\hat{k} \text{ again.}$$

(b) $\underset{\sim}{F}(0,0,0) = \lim_{h\to\infty} 2\pi\sigma G h \hat{k} =$ "$\infty \hat{k}$". Thus, we are fortunate that the earth was designed as a sphere rather than as a half-space.

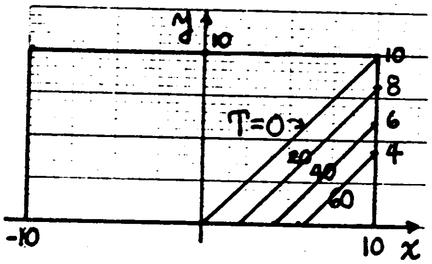
CHAPTER 16

Section 16.2

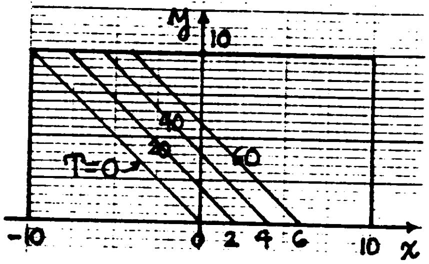
1. (a) $T=x^2+y^2$



(b) $T=10(x-y)$

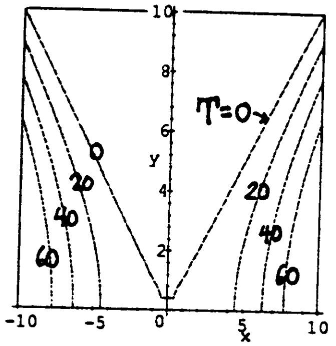


(c) $T=10(x+y)$

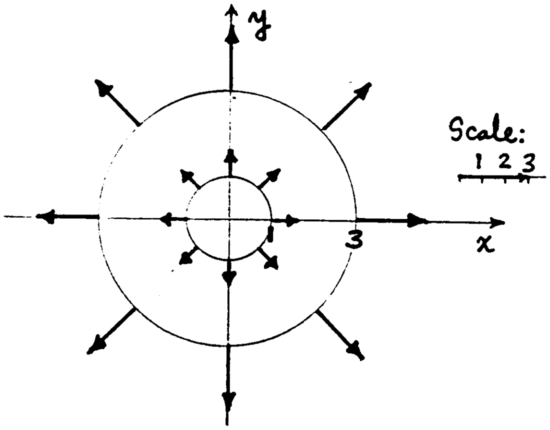


(d) $T=x^2-y^2$. Let's use Maple for this one. The contourplot command gives 8 contours by default and we can get any number using the option contours=c where c is the number of contours desired. For example, to generate Fig. 2b we used contours=18. But it doesn't necessarily give the specific contours that we desire. Thus, let us use the implicitplot command instead:

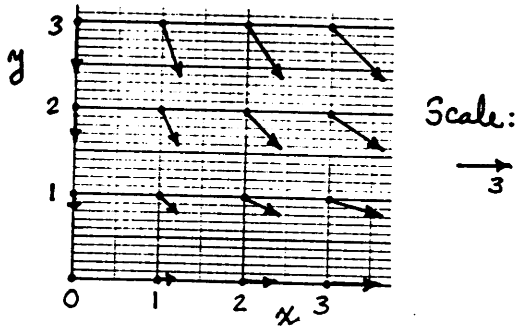
```
> with(plots):
> implicitplot({x^2-y^2=0,x^2-y^2=20,x^2-y^2=40,
  x^2-y^2=60},x=-10..10,y=0..10);
```



2.



3. $\underset{\sim}{v}(x,y) = x\hat{\imath}-y\hat{\jmath}$, as shown at the right:



4. (a) $\underset{\sim}{w}=y\hat{\imath}-x\hat{\jmath}+z\hat{k}$,

$\underset{\sim}{w}(2,3,-1)=3\hat{\imath}-2\hat{\jmath}-\hat{k}$, so we want $\frac{y}{3}=\frac{-x}{-2}=\frac{z}{-1}$. Setting $x=\tau$, say, we have the curve (straight line)

$$x=\tau,\ y=3\tau/2,\ z=-\tau/2$$

(b) $\underset{\sim}{w}=(y+2z)\hat{\imath}-x\hat{\jmath}+(x+y)\hat{k}$. $\underset{\sim}{w}(2,3,-1)=\hat{\imath}-2\hat{\jmath}+5\hat{k}$, so $\frac{y+2z}{1}=\frac{-x}{-2}=\frac{x+y}{5}$. Setting $x=\tau$, say, we have

$$x=\tau,\ y=3\tau/2,\ z=-\tau/2$$

(c) $\underset{\sim}{w} = x^2\hat{\imath} + y\hat{\jmath} + z\hat{k}$. $\underset{\sim}{w}(2,3,-1) = 4\hat{\imath} + 3\hat{\jmath} - \hat{k}$, so $\frac{x^2}{4} = \frac{y}{3} = \frac{z}{-1}$. Setting $x = \tau$, say, we have

$$x = \tau,\quad y = 3\tau^2/4,\quad z = -\tau^2/4$$

(d) $\underset{\sim}{w} = yz\hat{\imath} - \hat{\jmath} - (x+2)\hat{k}$. $\underset{\sim}{w}(2,3,-1) = -3\hat{\imath} - \hat{\jmath} - 4\hat{k}$, so $\frac{yz}{-3} = \frac{-1}{-1} = \frac{-(x+2)}{-4}$. Setting $x = \tau$, say, we have

$$x = \tau,\quad y = \cdots .$$

This doesnt work. We *must* have $x = \text{constant} = 2$. Then we can set $y = \tau$, say, and find $z = -3/\tau$:

$$x = 2,\quad y = \tau,\quad z = -3/\tau .$$

5. (a) $\underset{\sim}{v} = U\hat{\imath} + \frac{Ua^2}{(x^2+y^2)^2}\left[(y^2-x^2)\hat{\imath} - 2xy\hat{\jmath}\right]$.

On $y=0$, for $a \le |x| < \infty$, $\underset{\sim}{v}(x,0) = U\hat{\imath} - \frac{Ua^2}{x^2}\hat{\imath} = U\left(\frac{x^2-a^2}{x^2}\right)\hat{\imath}$ so $\|\underset{\sim}{v}(x,0)\| = U(x^2-a^2)/x^2$.

On $x^2+y^2=a^2$, for $|x| \le a$, we can use the polar coordinates r, θ (i.e., $x = a\cos\theta$, $y = a\sin\theta$). Then

$$\underset{\sim}{v} = U\hat{\imath} + \frac{Ua^2}{a^4}\left[a^2(\sin^2\theta - \cos^2\theta)\hat{\imath} - 2a^2\sin\theta\cos\theta\,\hat{\jmath}\right].$$

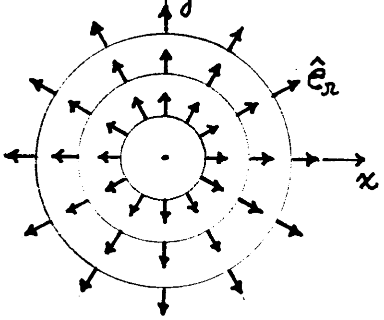
If, to complete the conversion to polar coordinates, we express $\hat{\imath} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$ and $\hat{\jmath} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$ (from Sec. 14.6), we find that the $\hat{e}_r$ component of $\underset{\sim}{v}$ is zero on that semicircle and we obtain the nice form $\underset{\sim}{v} = f(\theta)\,\hat{e}_\theta = -2U\sin\theta\,\hat{e}_\theta$. However, the problem asks us to use Cartesian coordinates, so write $y^2 = a^2-x^2$ and $y = +\sqrt{a^2-x^2}$, so

$$\underset{\sim}{v} = U\hat{\imath} + \frac{Ua^2}{a^4}\left[(a^2-2x^2)\hat{\imath} - 2x\sqrt{a^2-x^2}\,\hat{\jmath}\right],$$

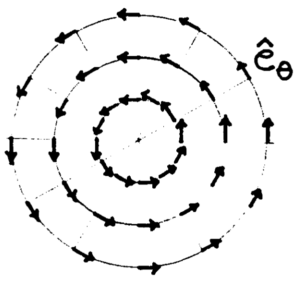
$$\|\underset{\sim}{v}\| = U\left[\left(2-\frac{2x^2}{a^2}\right)^2 + \left(-\frac{2x}{a^2}\sqrt{a^2-x^2}\right)^2\right]^{1/2} = 2\frac{U}{a}\sqrt{a^2-x^2}.$$

NOTE: The latter agrees with our polar result, where $\|\underset{\sim}{v}\| = 2U\sin\theta = 2U(a\sin\theta)/a = 2Uy/a = 2U\sqrt{a^2-x^2}/a$.

6. (a)

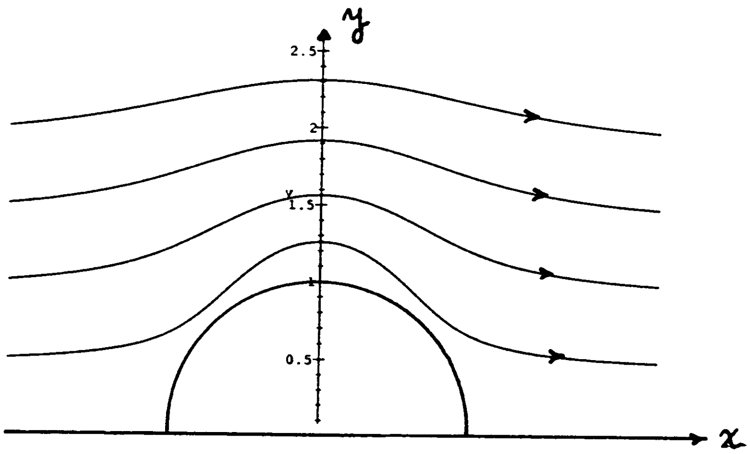


(b)



7. Let us use the Maple phaseportrait command discussed at the end of Section 7.2:

```
with(DEtools):
phaseportrait([1+(y^2-x^2)/(x^2+y^2)^2, -2*x*y/(x^2+y^2)^2],
[t,x,y], t=0..10, {[0,-3.5,.5],[0,-3.5,1],[0,-3.5,1.5],[0,-3.5,2]},
x=-3.5..3.5, y=0..3, stepsize=0.05, scene=[x,y]);
```



Section 16.3

1. (b) $\text{div}(x\hat{\imath}+y\hat{\jmath}+z\hat{k}) = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3$ everywhere

(c) $\text{div}(x\hat{\imath}-y\hat{\jmath}+z^2t\hat{k}) = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(z^2t) = 1-1+2zt = 2zt = 8t$ @$(3,-1,4)$

(g) $\text{div}(z\hat{\imath}+x^2t^2\hat{\jmath}) = \frac{\partial}{\partial x}(z)+\frac{\partial}{\partial y}(x^2t^2) = 0+0=0$ everywhere

(h) $\text{div}(\sin(x^2+y^2+z^2)\hat{\jmath}) = \frac{\partial}{\partial y}\sin(x^2+y^2+z^2) = 2y\cos(x^2+y^2+z^2) = -2\cos 26$

2. $$\text{div}\left[U\hat{\imath}+\frac{Ua^2}{(x^2+y^2)^2}((y^2-x^2)\hat{\imath}-2xy\,\hat{\jmath})\right] = \frac{Ua^2(-2)(2x)}{(x^2+y^2)^3}(y^2-x^2) - \frac{2Ua^2x}{(x^2+y^2)^2} + \frac{Ua^2(-2x)}{(x^2+y^2)^2}$$
$$+\frac{Ua^2(-2)(2y)}{(x^2+y^2)^3}(-2xy)$$
$$= \frac{2Ua^2}{(x^2+y^2)^3}\left[-2x(y^2-x^2)-2x(x^2+y^2)+4xy^2\right]$$
$= 0$ everywhere in the flow field

3. These answers will by no means be unique.

(a) $\underset{\sim}{v} = 6\hat{\imath}+x^2z\,\hat{\jmath}-(x^2+y^2)\hat{k}$

(b) $\underset{\sim}{v} = 6x\hat{\imath}$

(c) $\underset{\sim}{v} = -4y\hat{\jmath}$

(d) Set $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z} = x^2+y^2+z^2-1$. We can choose $v_x = v_y = 0$, and $v_z = \int(x^2+y^2+z^2-1)\,\partial z = (x^2+y^2-1)z+\frac{z^3}{3}$ plus an arbitrary function of y and z. So, we can take $\underset{\sim}{v} = [(x^2+y^2-1)z+\frac{z^3}{3}]\hat{k}$.

(e) Set $\nabla\cdot\underset{\sim}{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z} = x^2-1$. Let us choose $v_y = v_z = 0$ and

$v_x = \int(x^2-1)\partial x = \frac{x^3}{3}-x$ plus an arbitrary function of y and z. Let us choose $\underset{\sim}{v} = (\frac{x^3}{3}-x)\hat{\imath}$

4.(a) $\underset{\sim}{v} = 2\hat{\imath}-\hat{\jmath}+4\hat{k}$, so (3) gives

$$\begin{aligned}\operatorname{div}\underset{\sim}{v} &= \lim_{\epsilon\to 0}\frac{1}{(2\epsilon)^3}\Big\{\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{\imath}\cdot\underset{\sim}{v}\big|_{x=\epsilon}-\hat{\imath}\cdot\underset{\sim}{v}\big|_{x=-\epsilon})\,dy\,dz \\ &\quad+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{\jmath}\cdot\underset{\sim}{v}\big|_{y=\epsilon}-\hat{\jmath}\cdot\underset{\sim}{v}\big|_{y=-\epsilon})\,dx\,dz+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{k}\cdot\underset{\sim}{v}\big|_{z=\epsilon}-\hat{k}\cdot\underset{\sim}{v}\big|_{z=-\epsilon})\,dx\,dy\Big\} \\ &= \lim_{\epsilon\to 0}\frac{1}{(2\epsilon)^3}\Big\{\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\overset{0}{\cancel{2-2}})\,dy\,dz+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\overset{0}{\cancel{-1+1}})\,dx\,dz+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\overset{0}{\cancel{4-4}})\,dx\,dy\Big\} = 0,\end{aligned}$$

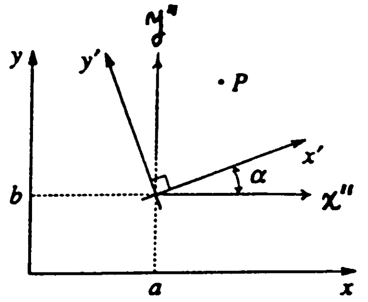
and of course (7) gives $\operatorname{div}\underset{\sim}{v} = \frac{\partial}{\partial x}(2)+\frac{\partial}{\partial y}(-1)+\frac{\partial}{\partial z}(4) = 0$ — everywhere, in fact, not just at $(0,0,0)$.

(e) $\underset{\sim}{v} = x\hat{\imath}+2y\hat{\jmath}-4z^3\hat{k}$,

$$\begin{aligned}\operatorname{div}\underset{\sim}{v} &= \lim_{\epsilon\to 0}\frac{1}{(2\epsilon)^3}\Big\{\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{\imath}\cdot\underset{\sim}{v}\big|_{x=\epsilon}-\hat{\imath}\cdot\underset{\sim}{v}\big|_{x=-\epsilon})\,dy\,dz \\ &\quad+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{\jmath}\cdot\underset{\sim}{v}\big|_{y=\epsilon}-\hat{\jmath}\cdot\underset{\sim}{v}\big|_{y=-\epsilon})\,dx\,dz+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}(\hat{k}\cdot\underset{\sim}{v}\big|_{z=\epsilon}-\hat{k}\cdot\underset{\sim}{v}\big|_{z=-\epsilon})\,dx\,dy\Big\} \\ &= \lim_{\epsilon\to 0}\frac{1}{(2\epsilon)^3}\Big\{\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}[\epsilon-(-\epsilon)]\,dy\,dz+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}[2\epsilon-(-2\epsilon)]\,dx\,dz \\ &\qquad+\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}[-4\epsilon^3+(-4\epsilon^3)]\,dx\,dy\Big\} \\ &= \lim_{\epsilon\to 0}\frac{1}{8\epsilon^3}[2\epsilon(4\epsilon^2)+4\epsilon(4\epsilon^2)-8\epsilon^3(4\epsilon^2)] = 3.\end{aligned}$$

Equation (7) gives $\operatorname{div}\underset{\sim}{v} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(2y)+\frac{\partial}{\partial z}(-4z^3)\big|_{x=y=z=0} = 1+2+0 = 3.$ ✓

5.



Equation (16) in Section 10.7 gives

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x''\\y''\end{pmatrix}$$

But $x'' = x-a$, $y'' = y-b$, so

$$x' = (\cos\alpha)(x-a)+(\sin\alpha)(y-b)$$
$$y' = (-\sin\alpha)(x-a)+(\cos\alpha)(y-b).$$

Also, with "$\underset{\sim}{x}$" changed to $\underset{\sim}{v}$, (17) in Section 10.7 gives

$$v_x = (\cos\alpha)v_{x'}-(\sin\alpha)v_{y'}$$
$$v_y = (\sin\alpha)v_{x'}+(\cos\alpha)v_{y'}.$$

Finally, $$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} = \overset{\text{“c”}}{\widetilde{\cos\alpha}}\frac{\partial}{\partial x'} - \overset{\text{“s”}}{\widetilde{\sin\alpha}}\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} = \sin\alpha\frac{\partial}{\partial x'} + \cos\alpha\frac{\partial}{\partial y'},$$

so $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = c\frac{\partial}{\partial x'}(cv_{x'} - \cancel{sv_{y'}}) - s\frac{\partial}{\partial y'}(\cancel{cv_{x'}} - sv_{y'}) + s\frac{\partial}{\partial x'}(sv_{x'} + \cancel{cv_{y'}}) + c\frac{\partial}{\partial y'}(\cancel{sv_{x'}} + cv_{y'})$$

$$= (c^2+s^2)\frac{\partial v_{x'}}{\partial x'} + (s^2+c^2)\frac{\partial v_{y'}}{\partial y'} = \frac{\partial v_{x'}}{\partial x'} + \frac{\partial v_{y'}}{\partial y'} \checkmark$$

Section 16.4

1. (b) $\underset{\sim}{\nabla}(x^2) = 2x\hat{\imath}$. at $(9,4,-1)$, it $= 18\hat{\imath}$.

(c) $\underset{\sim}{\nabla}(z\sin(x^2+y^2)) = 2xz\cos(x^2+y^2)\hat{\imath} + 2yz\cos(x^2+y^2)\hat{\jmath} + \sin(x^2+y^2)\hat{k}$. at $(9,4,-1)$ it $= -18\cos 52\,\hat{\imath} - 8\cos 52\,\hat{\jmath} + \sin 52\,\hat{k}$

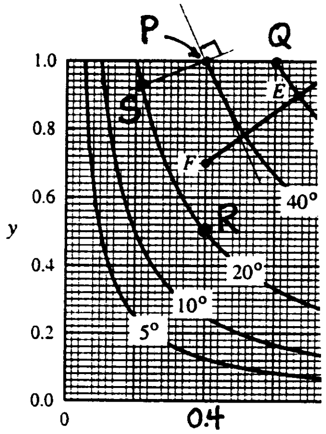
(d) $\underset{\sim}{\nabla}(x^3) = 3x^2\hat{\imath}$. at $(9,4,-1)$ it $= 243\hat{\imath}$

(f) $\underset{\sim}{\nabla}(x^2+y^2+z^2) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$. at $(9,4,-1)$ it $= 18\hat{\imath} + 8\hat{\jmath} - 2\hat{k}$

2. (b) $du/ds = \underset{\sim}{\nabla}u \cdot \underset{\sim}{\hat{v}} = (\hat{\imath}+\hat{\jmath}+3\hat{k})\cdot(2\hat{\imath}-\hat{\jmath}+5\hat{k})/\sqrt{30} = (2-1+15)/\sqrt{30} = 16/\sqrt{30}$

(c) $du/ds = \underset{\sim}{\nabla}u \cdot \underset{\sim}{\hat{v}} = (yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k})\cdot(3\hat{\imath}-\hat{k})/\sqrt{10} = (3yz - xy)/\sqrt{10}\Big|_{(1,-1,2)}$ $= (-6+1)/\sqrt{10} = -5/\sqrt{10}$

3. (a) $P = (0.4, 1)$. To compute $\underset{\sim}{\nabla}T$ at P we can use either the Cartesian formula $\underset{\sim}{\nabla}T = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath}$ or the intrinsic formula $\underset{\sim}{\nabla}T = \frac{\partial T}{\partial n}\underset{\sim}{\hat{n}}$. Let us do both. Since P is a boundary point we will need to use backward difference quotients for $\partial T/\partial y$ and $\partial T/\partial n$.

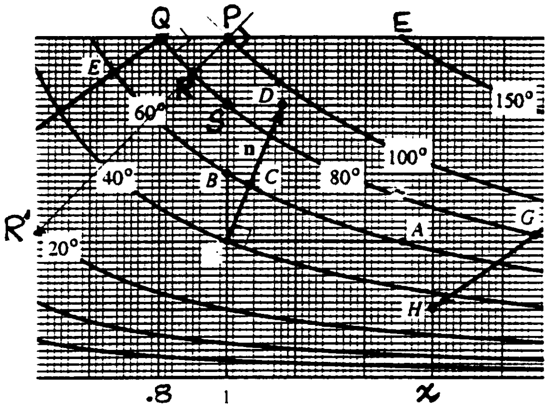


Cartesian: $$\underset{\sim}{\nabla}T\big|_P \approx \frac{T_Q - T_P}{PQ}\hat{\imath} + \frac{T_P - T_R}{RP}\hat{\jmath}$$

$$\approx \frac{60-40}{0.2}\hat{\imath} + \frac{40-20}{0.5}\hat{\jmath} = 100\hat{\imath} + 40\hat{\jmath}$$

Intrinsic: $$\underset{\sim}{\nabla}T\big|_P \approx \frac{T_P - T_S}{SP}\frac{\underset{\sim}{SP}}{\|\underset{\sim}{SP}\|} \approx \frac{40-20}{0.2}\frac{9\hat{\imath}+3.5\hat{\jmath}}{\sqrt{9^2+3.5^2}} = 93.2\hat{\imath} + 36.2\hat{\jmath}$$

Exact: $\underset{\sim}{\nabla}T\big|_P = 100y\hat{\imath} + 100x\hat{\jmath}\big|_P = 100\hat{\imath} + 40\hat{\jmath}$. (Only by luck did the Cartesian approximation prove to be exact.)



(b) $P=(1,1)$. Let us evaluate $\nabla T|_P$ both ways.

Cartesian: $\underset{\sim}{\nabla}T|_P \approx \frac{T_P-T_Q}{QP}\hat{\imath} + \frac{T_P-T_S}{SP}\hat{\jmath}$

$$\approx \frac{100-80}{0.2}\hat{\imath} + \frac{100-80}{0.2}\hat{\jmath}$$
$$= 100\hat{\imath} + 100\hat{\jmath}$$

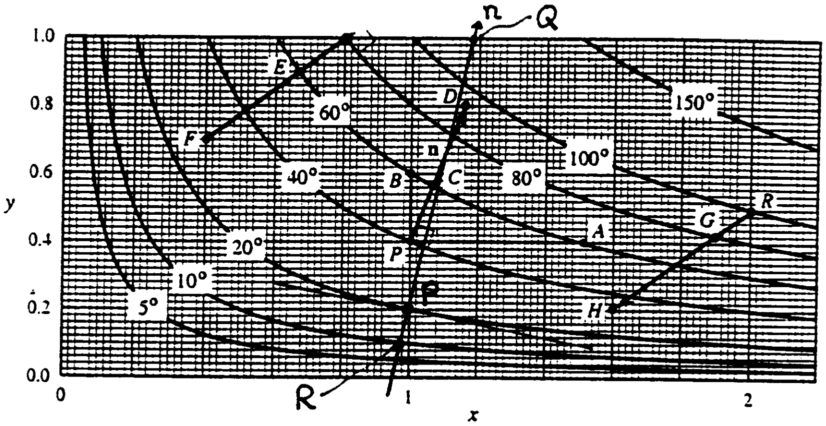
Here, we used a backward difference quotient for $\partial T/\partial x$ rather than a forward one since the point E is much further from P than Q (so we can expect the forward difference quotient to be the less accurate, except by chance)

Intrinsic: $\underset{\sim}{\nabla}T|_P \approx \frac{T_P-T_R}{RP}\frac{\underset{\approx}{RP}}{\|\underset{\sim}{RP}\|}$, but we cannot measure $\underset{\sim}{RP}$ very accurately.

Thus, extend PR back to R' and write

$$\underset{\sim}{\nabla}T|_P \approx \frac{T_P-T_R}{RP}\frac{\underset{\approx}{R'P}}{\|R'P\|} \approx \frac{100-80}{0.15}\frac{28\hat{\imath}+28\hat{\jmath}}{\sqrt{28^2+28^2}} = 94.3\hat{\imath} + 94.3\hat{\jmath}$$

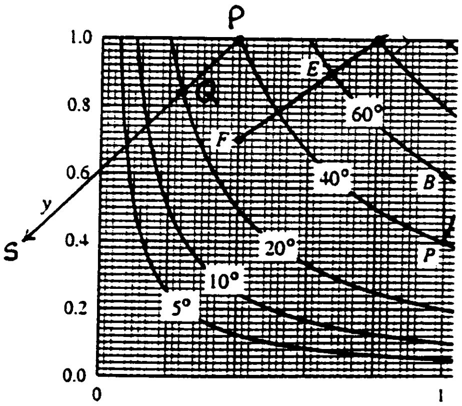
Exact: $\underset{\sim}{\nabla}T|_P = 100\hat{\imath} + 100\hat{\jmath}$.



(c) $P=(1,0.2)$. To evaluate $\underset{\sim}{\nabla}T|_P$ let us use the intrinsic formula because we can expect the estimation of $\partial T/\partial x|_P$ to be inaccurate because of the wide x-spacing. Thus, write

$$\underset{\sim}{\nabla}T|_P \approx \frac{T_P-T_R}{RP}\frac{\underset{\approx}{PQ}}{\|\underset{\approx}{PQ}\|}$$
$$\approx \frac{20-10}{0.10}\frac{9\hat{\imath}+40\hat{\jmath}}{\sqrt{9^2+40^2}} \approx 22.0\hat{\imath} + 97.6\hat{\jmath}$$

Exact: $\underset{\sim}{\nabla}T|_P = 20\hat{\imath} + 100\hat{\jmath}$.



(g) $P=(0.4,1)$. We could evaluate dT/ds at P by estimating $\underset{\sim}{\nabla}T$ there and then using $dT/ds = \underset{\sim}{\nabla}T \cdot \hat{\underset{\sim}{s}}$, but it is much simpler to merely use a difference quotient in the s direction — i.e., in the direction of $-\hat{\imath}-\hat{\jmath}$, as shown at the right. That is,

$$\frac{dT}{ds}\Big|_P \approx \frac{T_Q-T_P}{QP} \approx \frac{20-40}{0.23} \approx 87.0.$$

Exact: $\frac{dT}{ds}\Big|_P = \underset{\sim}{\nabla}T \cdot \left(\frac{-\hat{\imath}-\hat{\jmath}}{\sqrt{2}}\right) = (100\hat{\imath}+40\hat{\jmath})\cdot(-\hat{\imath}-\hat{\jmath})/\sqrt{2} = 99.0$

(h) $P=(1,1)$.

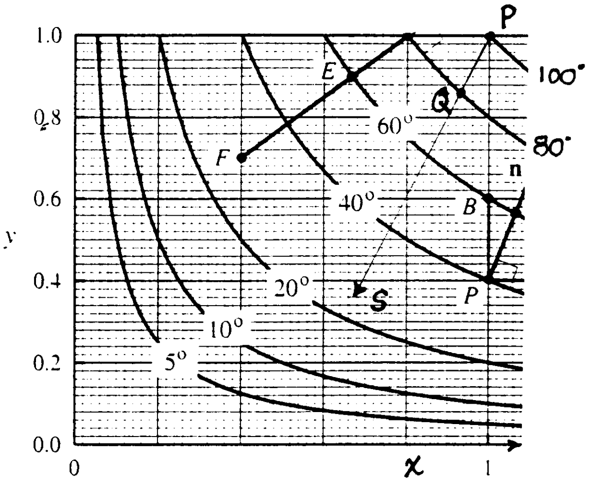
$$\left.\frac{dT}{ds}\right|_P \approx \frac{T|_Q - T|_P}{QP} \approx \frac{80-100}{0.154}$$

$$\approx -129.$$

Exact: $\left.\frac{dT}{ds}\right|_P = \underset{\sim}{\nabla} T|_P \cdot \left(\frac{-\hat{\imath}-2\hat{\jmath}}{\sqrt{5}}\right)$

$= (100\hat{\imath}+100\hat{\jmath})\cdot\left(\frac{-\hat{\imath}-2\hat{\jmath}}{\sqrt{5}}\right)$

$= -130.2.$



4.(b) $V = 3x^2y - xz$. The direction of maximum rate of voltage d<u>r</u>op is the $-\underset{\sim}{\nabla}V$ direction. $-\underset{\sim}{\nabla}V = -(6xy-z)\hat{\imath} - 3x^2\hat{\jmath} + x\hat{k} = -\hat{\imath} - 48\hat{\jmath} + 4\hat{k}$.

5.(b) $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}T =$

If $T = 2t$ and $\underset{\sim}{v} = \underset{\sim}{0}$ then $\frac{DT}{Dt} = 2 + 0 = 2$

If $T = 3x$ and $\underset{\sim}{v} = U\underset{\sim}{\hat{\imath}}$ then $\frac{DT}{Dt} = 0 + U\hat{\imath}\cdot 3\hat{\imath} = 3U$

If $T = 2t + 3x$ and $\underset{\sim}{v} = U\underset{\sim}{\hat{\imath}}$, then $\frac{DT}{Dt} = 2 + U\hat{\imath}\cdot 3\hat{\imath} = 2 + 3U$

In the first case DT/Dt is due entirely to the local rate of change of T with respect to time; in the second case it is due entirely to the fact that the fluid is moving in a spatially-varying field; in the third case it is a superposition of both effects.

(c) $(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})T = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)T = v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}$

$(\underset{\sim}{\nabla}\cdot\underset{\sim}{v})T = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)(T)$

$\Big\rangle$ n<u>o</u>t the same

6. Want dT/dt at $(2,-1,3)$ and $t=4$.

(a) $T = 100t^2 + 2x^2 + 2y^2$ and $\underset{\sim}{v} = 20xt\,\hat{\imath} - 10y\hat{\jmath}$, so

$\frac{dT}{dt}\left(\text{or } \frac{DT}{Dt}\right) = \frac{\partial T}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}T = 200t + (20xt\hat{\imath} - 10y\hat{\jmath})\cdot(4x\hat{\imath} + 4y\hat{\jmath})$

$= 200t + 80x^2t - 40y^2$

$= 2{,}040.$

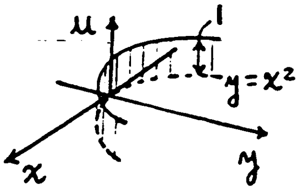
(b) $T = 25xt^2 - 2xyz$ and $\underset{\sim}{v} = 20\hat{\imath} + xy\hat{\jmath} - t\hat{k}$

$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}T = 50xt + (20\hat{\imath} + xy\hat{\jmath} - t\hat{k})\cdot[(25t^2 - 2yz)\hat{\imath} - 2xz\hat{\jmath} - 2xy\hat{k}]$

$= 400 + (20\hat{\imath} - 2\hat{\jmath} - 4\hat{k})\cdot(406\hat{\imath} - 12\hat{\jmath} + 4\hat{k})$

$= 8{,}528$

7.



$u(x,y)$ is not C^1 in any neighborhood of the origin so we cannot use chain differentiation. Rather, we fall back on the definition of du/ds:

$$\frac{du}{ds} = \lim_{h\to 0}\frac{u(\underset{\sim}{R}+h\hat{\underset{\sim}{s}})-u(\underset{\sim}{R})}{h}$$

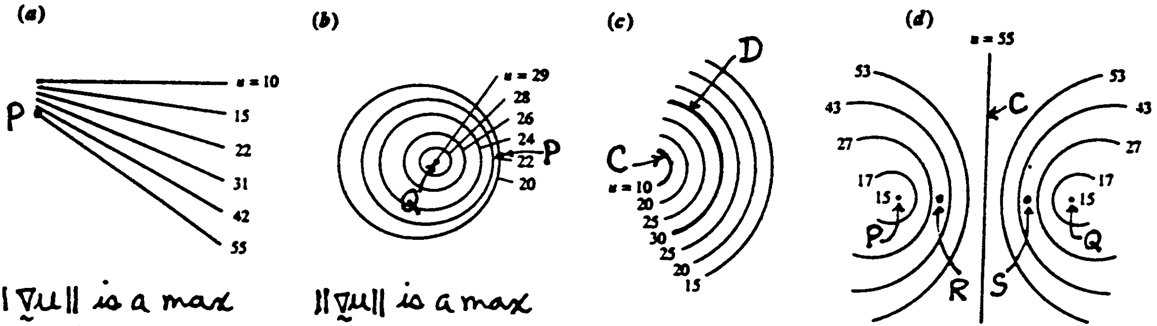
$$= \lim_{h\to 0}\frac{u(h\hat{\underset{\sim}{s}})-u(\underset{\sim}{0})}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0 \text{ for any } \hat{\underset{\sim}{s}}.$$

Yet, u is discontinuous at the origin since

$$\lim_{y\to 0} u(0,y) = \lim_{y\to 0} 0 = 0$$

whereas $\lim_{x\to 0} u(x,x^2) = \lim_{x\to 0} 1 = 1$, and these are unequal.

8.

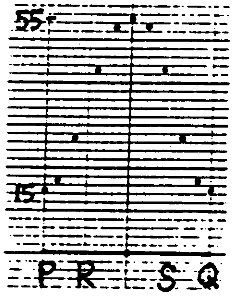


(a) $\|\underset{\sim}{\nabla} u\|$ is a max near P; $\underset{\sim}{\nabla} u \approx \underset{\sim}{0}$ nowhere.

(b) $\|\underset{\sim}{\nabla} u\|$ is a max near P; $\underset{\sim}{\nabla} u \approx \underset{\sim}{0}$ at Q

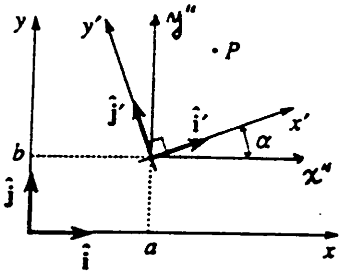
(c) $\|\underset{\sim}{\nabla} u\|$ is a max on curve C; $\underset{\sim}{\nabla} u \approx \underset{\sim}{0}$ on curve D.

(d) Let's plot u along the line PQ:



so $\|\underset{\sim}{\nabla} u\|$ is a max at R and S; $\underset{\sim}{\nabla} u = \underset{\sim}{0}$ along the line C, and at P and Q

9.



Equation (16) in Section 10.7 gives

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x''\\ y''\end{pmatrix}$$

But $x'' = x-a$ and $y'' = y-b$, so

$$x' = (\cos\alpha)(x-a)+(\sin\alpha)(y-b)$$

$$y' = (-\sin\alpha)(x-a)+(\cos\alpha)(y-b).$$

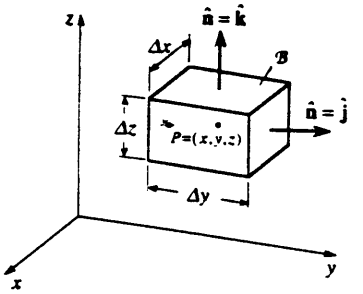
Thus,

$$\frac{\partial u}{\partial x}\hat{\imath}+\frac{\partial u}{\partial y}\hat{\jmath} = \left(\frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}\right)(\overset{\cos\alpha}{c}\hat{\imath}'-\overset{\sin\alpha}{s}\hat{\jmath}')+\left(\frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y}+\frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y}\right)(s\hat{\imath}'+c\hat{\jmath}')$$

$$= \left(c\frac{\partial u}{\partial x'}-s\frac{\partial u}{\partial y'}\right)(c\hat{\imath}'-s\hat{\jmath}')+\left(s\frac{\partial u}{\partial x'}+c\frac{\partial u}{\partial y'}\right)(s\hat{\imath}'+c\hat{\jmath}')$$

$$= (c^2+s^2)\frac{\partial u}{\partial x'}\hat{\imath}'+(c^2+s^2)\frac{\partial u}{\partial y'}\hat{\jmath}' = \frac{\partial u}{\partial x'}\hat{\imath}'+\frac{\partial u}{\partial y'}\hat{\jmath}'. \checkmark$$

10. $\operatorname{grad} u(P) = \lim_{B\to 0}\left\{\frac{\int_S \hat n u\,dA}{V}\right\}$



$$= \lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{1}{\Delta x\Delta y\Delta z}\left\{\iint_{\text{front}}(+\hat\imath)u(x+\tfrac{\Delta x}{2},y',z')\,dy'dz' + \iint_{\text{back}}(-\hat\imath)u(x-\tfrac{\Delta x}{2},y',z')\,dy'dz' + \iint_{\text{left}} + \iint_{\text{right}} + \iint_{\text{top}} + \iint_{\text{bottom}}\right\}$$

$$= \lim_{\Delta x\to 0}\lim_{\Delta y,\Delta z\to 0}\frac{\hat\imath u(x+\frac{\Delta x}{2},y_1,z_1)\cancel{\Delta y}\cancel{\Delta z} - \hat\imath u(x-\frac{\Delta x}{2},y_2,z_2)\cancel{\Delta y}\cancel{\Delta z}}{\Delta x\cancel{\Delta y}\cancel{\Delta z}} + \underbrace{\text{etc}}_{\text{left + right}} + \underbrace{\text{etc}}_{\text{top + bottom}}$$

where y_1, z_1 are the y, z coordinates on the front face where $u(x+\frac{\Delta x}{2}, y', z')$ takes on its average value and y_2, z_2 are the y, z coordinates on the back face where $u(x-\frac{\Delta x}{2}, y', z')$ takes on its average value

$$= \lim_{\Delta x\to 0}\frac{u(x+\frac{\Delta x}{2},y,z)-u(x-\frac{\Delta x}{2},y,z)}{\Delta x}\hat\imath + \underbrace{\text{etc} + \text{etc}}_{\text{similar}} = \frac{\partial u}{\partial x}\hat\imath + \frac{\partial u}{\partial y}\hat\jmath + \frac{\partial u}{\partial z}\hat k.$$

Section 16.5

1. (b) $\operatorname{curl}\underset{\sim}{v} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix} = 0\hat\imath - 0\hat\jmath + 0\hat k = \underset{\sim}{0}$

(c) $\operatorname{curl}\underset{\sim}{v} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & -y^2 & z^3\end{vmatrix} = 0\hat\imath - 0\hat\jmath + 0\hat k = \underset{\sim}{0}$

(e) $\operatorname{curl}\underset{\sim}{v} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy & 0 & -2(x^2+z^2)\end{vmatrix} = 0\hat\imath + 4x\hat\jmath - x\hat k$

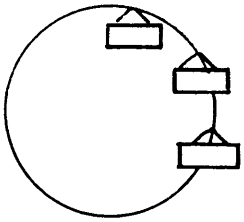
NOTE: You might wish to use Exercises 2 and 3 as lecture material. Note in particular that although the two streamline patterns are identical, one flow is rotational everywhere and the other is irrotational everywhere (except at the origin), thus adding emphasis to the point made just before Example 3.

2. (a) $\operatorname{curl}\underset{\sim}{v} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -\omega y & \omega x & 0\end{vmatrix} = 0\hat\imath - 0\hat\jmath + 2\omega\hat k = 2\omega\hat k = 2\underset{\sim}{\omega}$. This result is in line with the result derived in the text—that curl $\underset{\sim}{v}$ is twice the angular velocity of the fluid particle at that location. In the case of solid body rotation the entire field has the same angular velocity, namely, $\underset{\sim}{\omega} = \omega\hat k$.

(b) $\underset{\sim}{R} = r\hat e_r$, $\dot{\underset{\sim}{R}} = \cancelto{0}{\dot r}\hat e_r + r\dot{\hat e}_r = r\frac{d\hat e_r}{d\theta}\dot\theta = r\omega\,\hat e_\theta = r\omega(-\sin\theta\,\hat\imath + \cos\theta\,\hat\jmath)$
$= -\omega y\hat\imath + \omega x\hat\jmath$. ✓

3. $$\underset{\sim}{v} = \frac{\Gamma}{2\pi r}\hat{e}_\theta = \frac{\Gamma}{2\pi r}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) = \frac{\Gamma}{2\pi r^2}(-r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath})$$
$$= \frac{\Gamma}{2\pi}\left(-\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right)$$
$$\text{curl}\,\underset{\sim}{v} = \frac{\Gamma}{2\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]\hat{k}$$
$$= \frac{\Gamma}{2\pi}\left[\frac{1}{x^2+y^2} + \frac{x(-2x)}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} + \frac{y(-2y)}{(x^2+y^2)^2}\right]\hat{k} \quad (\text{if } x^2+y^2 \neq 0)$$
$$= \frac{\Gamma}{2\pi}\left[\frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}\right]\hat{k} = 0\hat{k} = \underset{\sim}{0} \quad (\text{if } x^2+y^2 \neq 0)$$

NOTE: In case the term "Ferris wheel" is not familiar to you, it refers to an amusement park ride. A large wheel, perhaps 70 feet in diameter rotates in a vertical plane at constant angular velocity. From that wheel are hung "buckets" for passengers.



4. Proceeding as in Exercise 5 of Section 16.3,
$$v_x = c v_{x'} - s v_{y'}, \qquad (c \text{ is } \cos\alpha,\ s \text{ is } \sin\alpha)$$
$$v_y = s v_{x'} + c v_{y'},$$
and
$$\partial/\partial x = c\,\partial/\partial x' - s\,\partial/\partial y',$$
$$\partial/\partial y = s\,\partial/\partial x' + c\,\partial/\partial y',$$
so
$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \left(c\frac{\partial}{\partial x'} - s\frac{\partial}{\partial y'}\right)(s v_{x'} + c v_{y'}) - (s\,\partial/\partial x' + c\,\partial/\partial y')(c v_{x'} - s v_{y'})$$
$$= \cancel{cs\frac{\partial v_{x'}}{\partial x'}} + c^2\frac{\partial v_{y'}}{\partial x'} - s^2\frac{\partial v_{x'}}{\partial y'} - \cancel{sc\frac{\partial v_{y'}}{\partial y'}} - \cancel{sc\frac{\partial v_{x'}}{\partial x'}} + s^2\frac{\partial v_{y'}}{\partial x'} - c^2\frac{\partial v_{x'}}{\partial y'} + \cancel{sc\frac{\partial v_{y'}}{\partial y'}}$$
$$= \frac{\partial v_{y'}}{\partial x'} - \frac{\partial v_{x'}}{\partial y'} \quad \checkmark$$

5. Not necessarily. For example, if $\underset{\sim}{v} = \hat{\imath} + y\hat{k}$, then
$$\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} v_x & v_y & v_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} 1 & 0 & y \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & 0 & y \end{vmatrix} = 1 \neq 0.$$

6. $$\text{curl}\,\underset{\sim}{v}(P) = \lim_{B\to 0}\left\{\frac{\int_S \hat{\underset{\sim}{n}}\times\underset{\sim}{v}\,dA}{V}\right\}$$

$$= \lim_{\Delta x,\Delta y,\Delta z\to 0} \frac{1}{\Delta x\Delta y\Delta z}\Big\{\iint_{\text{front}} \underset{\sim}{\hat\imath}\times\Big[v_x(x+\tfrac{\Delta x}{2},y',z')\hat\imath + v_y(x+\tfrac{\Delta x}{2},y',z')\hat\jmath + v_z(x+\tfrac{\Delta x}{2},y',z')\hat k\Big]\,dy\,dz$$

$$+\iint_{\text{back}} (-\hat\imath)\times\Big[v_x(x-\tfrac{\Delta x}{2},y',z')\hat\imath + v_y(x-\tfrac{\Delta x}{2},y',z')\hat\jmath + v_z(x-\tfrac{\Delta x}{2},y',z')\hat k\Big]\,dy\,dz$$

$$+\text{ etc. for left, right, top, bottom}\Big\}$$

$$= \lim_{\Delta x\to 0}\lim_{\Delta y,\Delta z\to 0}\frac{1}{\Delta x\Delta y\Delta z}\Big\{\Big[v_y(x+\tfrac{\Delta x}{2},y_1,z_1)\hat k - v_z(x+\tfrac{\Delta x}{2},y_2,z_2)\hat\jmath\Big]\Delta y\Delta z$$

$$+\Big[-v_y(x-\tfrac{\Delta x}{2},y_3,z_3)\hat k + v_z(x-\tfrac{\Delta x}{2},y_4,z_4)\hat\jmath\Big]\Delta y\Delta z$$

$$+\text{ etc.}\Big\}$$

where y_1,z_1 are the y,z coordinates where the integrand $v_y(x+\frac{\Delta x}{2},y',z')$ takes on its average value on the front face, and similarly for y_2,z_2 and y_3,z_3 and y_4,z_4

$$= \lim_{\Delta x\to 0}\Big\{\frac{v_y(x+\frac{\Delta x}{2},y,z)-v_y(x-\frac{\Delta x}{2},y,z)}{\Delta x}\,\underset{\sim}{\hat k} - \frac{v_z(x+\frac{\Delta x}{2},y,z)-v_z(x-\frac{\Delta x}{2},y,z)}{\Delta x}\,\hat\jmath\Big\}$$

$$+\text{ etc}$$

$$= \underbrace{\frac{\partial v_y}{\partial x}\hat k - \frac{\partial v_z}{\partial x}\hat\jmath}_{\text{from front \& back faces}} + \underbrace{\frac{\partial v_z}{\partial y}\hat\imath - \frac{\partial v_x}{\partial y}\hat k}_{\text{from left \& right faces}} + \underbrace{\frac{\partial v_x}{\partial z}\hat\jmath - \frac{\partial v_y}{\partial z}\hat k}_{\text{from top \& bottom faces}}$$

Section 16.6

1. (b) $u=x^2+y^2$. $\nabla^2 u = u_{xx}+u_{yy} = 2+2=4$, $\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}u = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2x & 2y & 0\end{vmatrix} = \underset{\sim}{0}$

(e) $u=xe^y$. $\nabla^2 u = 0+xe^y = xe^y$, $\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}u = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ e^y & xe^y & 0\end{vmatrix} = (e^y-e^y)\hat k = \underset{\sim}{0}$

NOTE: Recall from (13) that $\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}u$ is *necessarily* $\underset{\sim}{0}$, so the foregoing results are not coincidents.

(g) $u=xyz$. $\nabla^2 u = 0+0+0=0$, $\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}u = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ yz & xz & xy\end{vmatrix} = \underset{\sim}{0}$.

2. (b) $\underset{\sim}{v} = xe^y\hat\imath - 2\hat\jmath + z^2\hat k$. $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xe^y & -2 & z^2\end{vmatrix} = -xe^y\hat k$

so $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = \frac{\partial}{\partial z}(-xe^y) = 0$ (as is *always* true for $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}$, per (12)) and $\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{v}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & -xe^y \end{vmatrix} = -xe^y\hat{\imath} + e^y\hat{\jmath}$

(c) $\underset{\sim}{v} = z^3\hat{\imath}$. $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^3 & 0 & 0 \end{vmatrix} = 0\hat{\imath} + 3z^2\hat{\jmath} + 0\hat{k} = 3z^2\hat{\jmath}$

so $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = \partial(3z^2)/\partial y = 0$ (no surprise!)
and

$$\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{v}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 3z^2 & 0 \end{vmatrix} = -6z\hat{\imath}$$

(f) $\underset{\sim}{v} = f(y)\hat{\imath} + g(x)\hat{\jmath}$. $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(y) & g(x) & 0 \end{vmatrix} = [g'(x) - f'(y)]\hat{k}$

so $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = \partial[g'(x) - f'(y)]/\partial z = 0$ (no surprise!)

3. Yes, they are needed since $(\underset{\sim}{\nabla}\times\underset{\sim}{\nabla})\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} \times\underset{\sim}{v} = \underset{\sim}{0}\times\underset{\sim}{v} = \underset{\sim}{0}$

for all $\underset{\sim}{v}$'s, whereas $\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{v})$ is not necessarily $\underset{\sim}{0}$.

4. (e) $\frac{\partial}{\partial x}\ln(x^2+y^2) = \frac{2x}{x^2+y^2}$, if $x^2+y^2 \neq 0$,

$$\frac{\partial^2}{\partial x^2}\ln(x^2+y^2) = \frac{2}{x^2+y^2} - \frac{2x(2x)}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2}{\partial y^2}\ln(x^2+y^2) = \frac{2}{x^2+y^2} - \frac{2y(2y)}{(x^2+y^2)} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$$

and $\frac{\partial^2}{\partial z^2}\ln(x^2+y^2) = 0$, so $\nabla^2[\ln(x^2+y^2)] = \frac{-2x^2+2y^2}{(x^2+y^2)^2} + \frac{-2y^2+2x^2}{(x^2+y^2)^2} + 0 = 0$

5. $(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{R} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} = \underset{\sim}{v}$.

6. (a) $\underset{\sim}{\nabla}\times(u\underset{\sim}{v}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ uv_x & uv_y & uv_z \end{vmatrix} = \hat{\imath}\left[\frac{\partial}{\partial y}uv_z - \frac{\partial}{\partial z}uv_y\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}uv_z - \frac{\partial}{\partial z}uv_x\right]$
$+ \hat{k}\left[\frac{\partial}{\partial x}uv_y - \frac{\partial}{\partial y}uv_x\right]$

$$= u\left[\hat{\imath}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$+ \hat{\imath}\left(v_z\frac{\partial u}{\partial y} - v_y\frac{\partial u}{\partial z}\right) - \hat{\jmath}\left(v_z\frac{\partial u}{\partial x} - v_x\frac{\partial u}{\partial z}\right) + \hat{k}\left(v_y\frac{\partial u}{\partial x} - v_x\frac{\partial u}{\partial y}\right)$$

$$= u\underset{\sim}{\nabla}\times\underset{\sim}{v} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ v_x & v_y & v_z \end{vmatrix} = u\underset{\sim}{\nabla}\times\underset{\sim}{v} + \underset{\sim}{\nabla}u\times\underset{\sim}{v}.$$

(d) $\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = \frac{\partial}{\partial x}\left(\cancel{\frac{\partial v_z}{\partial y}} - \cancel{\frac{\partial v_y}{\partial z}}\right) + \frac{\partial}{\partial y}\left(\cancel{\frac{\partial v_x}{\partial z}} - \cancel{\frac{\partial v_z}{\partial x}}\right) + \frac{\partial}{\partial z}\left(\cancel{\frac{\partial v_y}{\partial x}} - \cancel{\frac{\partial v_x}{\partial y}}\right) = 0.$

7. (a) $\frac{\partial \sigma}{\partial t} + \underset{\sim}{\nabla}\cdot(\sigma \underset{\sim}{v}) = 0$, (b) $\sigma \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} = \mu \nabla^2 \underset{\sim}{u} + (\lambda+\mu)\underset{\sim}{\nabla}\Theta + \underset{\sim}{F}$, (c) $\underset{\sim}{\nabla}\times\underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$

8. (a) $\nabla^4 u = \nabla^2(\nabla^2 u) = \frac{\partial^2}{\partial x^2}(u_{xx}+u_{yy}) + \frac{\partial^2}{\partial y^2}(u_{xx}+u_{yy}) = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$ assuming that $u_{xxyy} = u_{yyxx}$.

(b) $\nabla^4 u = \nabla^2(\nabla^2 u) = \frac{\partial^2}{\partial x^2}(u_{xx}+u_{yy}+u_{zz}) + \frac{\partial^2}{\partial y^2}(u_{xx}+u_{yy}+u_{zz}) + \frac{\partial^2}{\partial z^2}(u_{xx}+u_{yy}+u_{zz})$
$= u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz}$

assuming (as usual) that $u_{xxyy} = u_{yyxx}$, $u_{xxzz} = u_{zzxx}$, $u_{yyzz} = u_{zzyy}$.

10. Let us eliminate $\underset{\sim}{w}$, say, between (10.1) and (10.2). First, take $\underset{\sim}{\nabla}\times$ of (10.1) and then use (10.2):

$$\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{v}) = \underset{\sim}{\nabla}\times\left(-\frac{1}{c}\frac{\partial \underset{\sim}{w}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}(\underset{\sim}{\nabla}\times\underset{\sim}{w}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \underset{\sim}{v}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \underset{\sim}{v}}{\partial t^2}.$$

Then use (14) and (10.3):

$$\underset{\sim}{\nabla}(\underset{0}{\cancel{\underset{\sim}{\nabla}\cdot\underset{\sim}{v}}}) - \nabla^2 \underset{\sim}{v} = -\frac{1}{c^2}\frac{\partial^2 \underset{\sim}{v}}{\partial t^2}, \quad \text{so} \quad c^2\nabla^2\underset{\sim}{v} = \frac{\partial^2 \underset{\sim}{v}}{\partial t^2}.$$

Similarly, we can eliminate $\underset{\sim}{v}$ between (10.1) and (10.2) by taking $\underset{\sim}{\nabla}\times$ of (10.2) and then using (10.1), (14), and (10.4), obtaining $c^2\nabla^2\underset{\sim}{w} = \frac{\partial^2 \underset{\sim}{w}}{\partial t^2}$.

11. (b) $\nabla^2 \underset{\sim}{v} = \nabla^2(x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) = \frac{\partial^2}{\partial x^2}(x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) + \frac{\partial^2}{\partial y^2}(\text{etc}) + \frac{\partial^2}{\partial z^2}(\text{etc})$
$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

(c) $\nabla^2 \underset{\sim}{v} = \underset{\sim}{0}$

(d) $\nabla^2 \underset{\sim}{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\sin xz\,\hat{\jmath}) = -(x^2+z^2)\sin xz\,\hat{\jmath}$.

Section 16.7

1. (b) $u = r\sin 2\theta$: $\underset{\sim}{\nabla}u = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)(r\sin 2\theta) = (\sin 2\theta)\hat{e}_r + (2\cos 2\theta)\hat{e}_\theta$

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = \frac{1}{r}\sin 2\theta - \frac{4r}{r^2}\sin 2\theta = -\frac{3}{r}\sin 2\theta$

$\underset{\sim}{v} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$: $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\cos\theta) + \frac{1}{r}\frac{\partial}{\partial\theta}(-\sin\theta) + \frac{\partial}{\partial z}(0) = \frac{\cos\theta}{r} - \frac{\cos\theta}{r} = 0$

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left(\frac{1}{r}\frac{\partial}{\partial\theta}0 - \frac{\partial}{\partial z}(-\sin\theta)\right)\hat{e}_r + \left(\frac{\partial}{\partial z}\cos\theta - \frac{\partial}{\partial r}0\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(-r\sin\theta) - \frac{\partial}{\partial\theta}\cos\theta\right)\hat{e}_z$
$= 0\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_z = \underset{\sim}{0}$.

The fact that both $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$ and $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{0}$ is not surprising if we notice that $\underset{\sim}{v}$ is simply $= \hat{\imath}$; i.e., it is a <u>constant</u> vector field.

(c) $u = r^2 \sin\theta$: $\underset{\sim}{\nabla} u = 2r\sin\theta\, \hat{e}_r + r\cos\theta\, \hat{e}_\theta$

$$\nabla^2 u = 2\sin\theta + \frac{1}{r}(2r\sin\theta) + \frac{1}{r^2}(-r^2\sin\theta) = 3\sin\theta$$

$\underset{\sim}{v} = z^2 \hat{e}_r$: $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{1}{r}\frac{\partial}{\partial r}(rz^2) + 0 + 0 = z^2/r$

$$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{0} + \left(\frac{\partial}{\partial z}z^2 - 0\right)\hat{e}_\theta + \underset{\sim}{0} = 2z\,\hat{e}_\theta$$

(e) $u = 1/r$: $\underset{\sim}{\nabla} u = -\frac{1}{r^2}\hat{e}_r$

$$\nabla^2 u = \frac{2}{r^3} + \frac{1}{r}\left(-\frac{1}{r^2}\right) + 0 + 0 = \frac{1}{r^3}$$

$\underset{\sim}{v} = r\sin\theta\,\hat{e}_z$: $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 + 0 + \frac{\partial}{\partial z}(r\sin\theta) = 0$

$$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left(\frac{1}{r}\frac{\partial}{\partial\theta}(r\sin\theta) - 0\right)\hat{e}_r + \left(0 - \frac{\partial}{\partial r}(r\sin\theta)\right)\hat{e}_\theta + 0\,\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

These results are not surprising since $\underset{\sim}{v} = y\hat{k}$ easily gives $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$ and $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{\hat{\imath}}$

2.(b) $\underset{\sim}{v} = \hat{e}_\theta$. (16) gives $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 + \frac{1}{r}\frac{\partial}{\partial\theta}(1) + 0 = 0$.

Alternatively, $\underset{\sim}{v} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \frac{-r\sin\theta}{r}\hat{\imath} + \frac{r\cos\theta}{r}\hat{\jmath} = -\frac{y}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{x}{\sqrt{x^2+y^2}}\hat{\jmath}$

so $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{\partial}{\partial x}\left(-y(x^2+y^2)^{-1/2}\right) + \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1/2}\right) = \frac{-y(-\frac{1}{2})2x + x(-\frac{1}{2})2y}{(x^2+y^2)^{3/2}} = 0$ ✓

(c) $\underset{\sim}{v} = r\hat{e}_r + z\hat{e}_z$. (16) gives $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{1}{r}\frac{\partial}{\partial r}(r^2) + 0 + \frac{\partial}{\partial z}(z) = 2 + 0 + 1 = 3$.

Alternatively, $\underset{\sim}{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ so $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 1+1+1 = 3$. ✓

3. (a) See (b), below.

(b) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\times\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$

$= \frac{\partial v_\theta}{\partial r}\hat{e}_r\times\hat{e}_\theta + \frac{\partial v_z}{\partial r}\hat{e}_r\times\hat{e}_z$ (omitting Terms that give $\underset{\sim}{0}$)

$$+ \frac{1}{r}\frac{\partial v_r}{\partial\theta}\hat{e}_\theta\times\hat{e}_r + \frac{v_r}{r}\hat{e}_\theta\times\cancelto{\hat{e}_\theta}{\frac{\partial\hat{e}_r}{\partial\theta}} + \frac{v_\theta}{r}\hat{e}_\theta\times\cancelto{-\hat{e}_r}{\frac{\partial\hat{e}_\theta}{\partial\theta}} + \frac{v_z}{r}\hat{e}_\theta\times\cancelto{0}{\frac{\partial\hat{e}_z}{\partial\theta}}$$

$$+ \frac{1}{r}\frac{\partial v_z}{\partial\theta}\hat{e}_\theta\times\hat{e}_z + \frac{\partial v_r}{\partial z}\hat{e}_z\times\hat{e}_r + \frac{\partial v_\theta}{\partial z}\hat{e}_z\times\hat{e}_\theta$$

$$= \frac{\partial v_\theta}{\partial r}\hat{e}_z - \frac{\partial v_z}{\partial r}\hat{e}_\theta - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\hat{e}_z + \frac{v_\theta}{r}\hat{e}_z + \frac{\partial v_r}{\partial z}\hat{e}_\theta - \frac{\partial v_\theta}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\hat{e}_r$$

$$= \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\hat{e}_z,$$

which agrees with (17). For the plane polar case $v_z = 0$ and v_r and v_θ do not vary with z. Thus, setting all v_z's and $\partial(\ )/\partial z$'s $= 0$, the preceding result gives (26).

4. (a) $\nabla^2 u = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} u = \underset{\sim}{\nabla} \cdot \left(u_r \hat{e}_r + \frac{1}{r} u_\theta \hat{e}_\theta + u_z \hat{e}_z \right)$ per (7)

$$= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r} u_\theta\right) + \frac{\partial}{\partial z} u_z \quad \text{per (16)}$$

$$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}, \quad \text{as in (18).} \checkmark$$

(b) $\nabla^2 u = \left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial \theta}\right)\left(c u_r - \frac{s}{r} u_\theta\right) + \left(s\frac{\partial}{\partial r} + \frac{c}{r}\frac{\partial}{\partial \theta}\right)\left(s u_r + \frac{c}{r} u_\theta\right) + u_{zz}$

$$= c^2 u_{rr} + \cancel{\frac{cs}{r^2} u_\theta} - \cancel{\frac{cs}{r} u_{r\theta}} + \frac{s^2}{r} u_r - \cancel{\frac{cs}{r} u_{r\theta}} + \cancel{\frac{cs}{r^2} u_\theta} + \frac{s^2}{r^2} u_{\theta\theta}$$
$$+ s^2 u_{rr} - \cancel{\frac{cs}{r^2} u_\theta} + \cancel{\frac{cs}{r} u_{r\theta}} + \frac{c^2}{r} u_r + \cancel{\frac{cs}{r} u_{r\theta}} - \cancel{\frac{cs}{r^2} u_\theta} + \frac{c^2}{r^2} u_{\theta\theta} + u_{zz}$$

$$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}, \text{ in agreement with (18).}$$

5. (a) $\operatorname{div} \underset{\sim}{v} = \lim_{\Delta r, \Delta\theta, \Delta z \to 0} \frac{1}{[\pi(r+\Delta r)^2 - \pi r^2]\frac{\Delta\theta}{2\pi}\Delta z} \Bigg\{ \int_z^{z+\Delta z}\int_\theta^{\theta+\Delta\theta} \hat{e}_r(\theta') \cdot$

$$[v_r(r+\Delta r, \theta', z')\hat{e}_r(\theta') + v_\theta \cancel{\hat{e}_\theta} + v_z \cancel{\hat{e}_z}](r+\Delta r)\,d\theta' dz' \qquad \text{(right)}$$

$$+ \int_z^{z+\Delta z}\int_\theta^{\theta+\Delta\theta} -\hat{e}_r(\theta') \cdot [v_r(r,\theta',z')\hat{e}_r(\theta') + \cancel{\text{etc}}]\, r\,d\theta' dz' \qquad \text{(left)}$$

$$+ \int_z^{z+\Delta z}\int_r^{r+\Delta r} -\hat{e}_\theta(\theta) \cdot [v_\theta(r',\theta,z')\hat{e}_\theta(\theta) + \cancel{\text{etc}}]\, dr' dz' \qquad \text{(front)}$$

$$+ \int_z^{z+\Delta z}\int_r^{r+\Delta r} \hat{e}_\theta(\theta+\Delta\theta) \cdot [v_\theta(r',\theta+\Delta\theta,z')\hat{e}_\theta(\theta+\Delta\theta) + \cancel{\text{etc}}]\, dr' dz' \qquad \text{(back)}$$

$$+ \int_\theta^{\theta+\Delta\theta}\int_r^{r+\Delta r} \hat{e}_z \cdot [\cancel{\text{etc}} + v_z(r',\theta',z+\Delta z)\hat{e}_z]\, r' dr' d\theta' \qquad \text{(top)}$$

$$+ \int_\theta^{\theta+\Delta\theta}\int_r^{r+\Delta r} -\hat{e}_z \cdot [\cancel{\text{etc}} + v_z(r',\theta',z)\hat{e}_z]\, r' dr' d\theta' \Bigg\} \qquad \text{(bottom)}$$

$$= \lim_{\Delta r, \Delta\theta, \Delta z \to 0} \frac{1}{[r\Delta r + \frac{(\Delta r)^2}{2}]\Delta\theta\Delta z} \Big\{ v_r(r+\Delta r, \theta_1, z_1)(r+\Delta r)\Delta\theta\Delta z$$
$$- v_r(r,\theta_2,z_2)\, r\Delta\theta\Delta z - v_\theta(r_3,\theta,z_3)\,\Delta r\Delta z$$
$$+ v_\theta(r_4,\theta+\Delta\theta,z_4)\,\Delta r\Delta z + v_z(r_5,\theta_5,z+\Delta z)\, r_5\,\Delta r\Delta\theta$$
$$- v_z(r_6,\theta_6,z)\, r_6\,\Delta r\Delta\theta \Big\}$$

$$= \lim_{\Delta r \to 0} \frac{1}{r + \frac{\Delta r}{2}} \frac{v_r(r+\Delta r,\theta,z)(r+\Delta r) - v_r(r,\theta,z)r}{\Delta r}$$
$$+ \lim_{\Delta\theta \to 0} \frac{1}{r} \frac{v_\theta(r,\theta+\Delta\theta,z) - v_\theta(r,\theta,z)}{\Delta\theta} + \lim_{\Delta z \to 0} \frac{r}{r} \frac{v_z(r,\theta,z+\Delta z) - v_z(r,\theta,z)}{\Delta z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}. \checkmark$$

(b) grad u

$$= \lim_{\Delta r,\Delta\theta,\Delta z\to 0} \frac{1}{[\pi(r+\Delta r)^2-\pi r^2]\frac{\Delta\theta}{2\pi}\Delta z}\Big\{\int_z^{z+\Delta z}\int_\theta^{\theta+\Delta\theta} \hat{e}_r(\theta')\,u(r+\Delta r,\theta',z')(r+\Delta r)\,d\theta' dz' \quad \text{(right surface)}$$

$$+\int_z^{z+\Delta z}\int_\theta^{\theta+\Delta\theta} -\hat{e}_r(\theta')\,u(r,\theta',z')\,r\,d\theta' dz' \qquad \text{(left)}$$

$$+\int_z^{z+\Delta z}\int_\theta^{\theta+\Delta\theta} -\hat{e}_\theta(\theta)\,u(r',\theta,z')\,dr' dz' \qquad \text{(front)}$$

$$+\int_z^{z+\Delta z}\int_r^{r+\Delta r} \hat{e}_\theta(\theta+\Delta\theta)\,u(r',\theta+\Delta\theta,z')\,dr' dz' \qquad \text{(back)}$$

$$+\int_\theta^{\theta+\Delta\theta}\int_r^{r+\Delta r} \hat{e}_z\,u(r',\theta',z+\Delta z)\,r' dr' d\theta' \qquad \text{(top)}$$

$$+\int_\theta^{\theta+\Delta\theta}\int_r^{r+\Delta r} -\hat{e}_z\,u(r',\theta',z)\,r' dr' d\theta' \Big\} \qquad \text{(bottom)}$$

$$= \lim_{\Delta r,\Delta\theta,\Delta z\to 0} \frac{1}{[r\Delta r+\frac{(\Delta r)^2}{2}]\Delta\theta\Delta z}\Big\{ \hat{e}_r(\theta_1)\,u(r+\Delta r,\theta_1,z_1)(r+\Delta r)\Delta\theta\Delta z$$

$$- \hat{e}_r(\theta_2)\,u(r,\theta_2,z_2)\,r\Delta\theta\Delta z$$

$$- \hat{e}_\theta(\theta)\,u(r_3,\theta,z_3)\,\Delta r\Delta z$$

$$+ \hat{e}_\theta(\theta+\Delta\theta)\,u(r_4,\theta+\Delta\theta,z_4)\,\Delta r\Delta z$$

$$+ \hat{e}_z\,u(r_5,\theta_5,z+\Delta z)\,r_5\Delta r\Delta\theta$$

$$- \hat{e}_z\,u(r_6,\theta_6,z)\,r_6\Delta r\Delta\theta \Big\}$$

$$= \lim_{\Delta r\to 0} \frac{u(r+\Delta r,\theta,z)(r+\Delta r)-u(r,\theta,z)r}{r\Delta r+\frac{(\Delta r)^2}{2}}\,\hat{e}_r(\theta)$$

$$+\lim_{\Delta\theta\to 0} \frac{\hat{e}_\theta(\theta+\Delta\theta)u(r,\theta+\Delta\theta,z)-\hat{e}_\theta(\theta)u(r,\theta,z)}{r\Delta\theta}$$

$$+\lim_{\Delta z\to 0} \frac{u(r,\theta,z+\Delta z)-u(r,\theta,z)}{\Delta z}\,\hat{e}_z$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(ru)\,\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(u\hat{e}_\theta) + \frac{\partial u}{\partial z}\,\hat{e}_z$$

$$= \cancel{\frac{u}{r}\hat{e}_r} + u_r\hat{e}_r + \frac{1}{r}u_\theta\,\hat{e}_\theta + \cancel{\frac{u}{r}(-\hat{e}_r)} + u_z\hat{e}_z = \frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\,\hat{e}_\theta + \frac{\partial u}{\partial z}\,\hat{e}_z.\ \checkmark$$

7. $$\nabla^2 \underset{\sim}{v} = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(v_r\hat{e}_r+v_\theta\,\hat{e}_\theta)$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}\right)\hat{e}_r + \left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}\right)\hat{e}_\theta + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial v_r}{\partial\theta}\hat{e}_r + v_r\,\overset{\nearrow\hat{e}_\theta}{\cancel{\frac{\partial\hat{e}_r}{\partial\theta}}}\right)$$

$$+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial\theta}\,\hat{e}_\theta + v_\theta\,\overset{\nearrow -\hat{e}_r}{\cancel{\frac{\partial\hat{e}_\theta}{\partial\theta}}}\right)$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}\right)\hat{e}_r + \left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}\right)\hat{e}_\theta + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial\theta^2}\hat{e}_r + \frac{\partial v_r}{\partial\theta}\hat{e}_\theta + \frac{\partial v_r}{\partial\theta}\hat{e}_\theta - v_r\hat{e}_r\right)$$

$$+\frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial\theta^2}\,\hat{e}_\theta - \frac{\partial v_\theta}{\partial\theta}\,\hat{e}_r - \frac{\partial v_\theta}{\partial\theta}\,\hat{e}_r - v_\theta\,\hat{e}_\theta\right)$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 v_r}{\partial\theta^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\,\frac{\partial v_\theta}{\partial\theta}\right)\hat{e}_r$$

$$+\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}-\frac{1}{r^2}v_\theta\right)\hat{e}_\theta$$

9. Cartesian: $\nabla^4 u = (\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})(u_{xx}+u_{yy}+u_{zz})$

$= u_{xxxx}+u_{yyyy}+u_{zzzz}+2u_{xxyy}+2u_{xxzz}+2u_{yyzz}$

Cylindrical: $\nabla^4 u = (\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2})(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz})$

$= u_{rrrr}+(\frac{1}{r}u_r)_{rr}+(\frac{1}{r^2}u_{\theta\theta})_{rr}+u_{rrzz}$

$+\frac{1}{r}u_{rrr}+\frac{1}{r}(\frac{1}{r}u_r)_r+\frac{1}{r}(\frac{1}{r^2}u_{\theta\theta})_r+\frac{1}{r}u_{rzz}$

$+\frac{1}{r^2}u_{rr\theta\theta}+\frac{1}{r^3}u_{r\theta\theta}+\frac{1}{r^4}u_{\theta\theta\theta\theta}+\frac{1}{r^2}u_{zz\theta\theta}$

$+u_{rrzz}+\frac{1}{r}u_{rzz}+\frac{1}{r^2}u_{zz\theta\theta}+u_{zzzz}$,

which could be simplified a bit.

10. $q_1=r,\ q_2=\theta,\ q_3=z$. Since $\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{matrix}$,

(38) gives $h_1=\sqrt{\cos^2\theta+\sin^2\theta+0}=1$

$h_2=\sqrt{r^2\sin^2\theta+r^2\cos^2\theta+0}=r$

$h_3=\sqrt{0+0+1}=1$

Then,

(39): $\underset{\sim}{\nabla}=\hat{e}_r\frac{1}{1}\frac{\partial}{\partial r}+\hat{e}_2\frac{1}{r}\frac{\partial}{\partial \theta}+\hat{e}_3\frac{1}{1}\frac{\partial}{\partial z}=\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\hat{e}_z\frac{\partial}{\partial z}$ ✓

(40): From the preceding, $\underset{\sim}{\nabla}u=\hat{e}_r\frac{\partial u}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial u}{\partial \theta}+\hat{e}_z\frac{\partial u}{\partial z}$ ✓

(41): $\nabla^2 u=\frac{1}{r}\left[\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial u}{\partial \theta})+\frac{\partial}{\partial z}(r\frac{\partial u}{\partial z})\right]=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$ ✓

(42): $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rv_r)+\frac{\partial}{\partial \theta}v_\theta+\frac{\partial}{\partial z}(rv_z)\right]$ ✓

(43): $\underset{\sim}{\nabla}\times\underset{\sim}{v}=\frac{1}{r}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ v_r & rv_\theta & v_z\end{vmatrix}=\frac{1}{r}\left(\frac{\partial v_z}{\partial \theta}-r\frac{\partial v_\theta}{\partial z}\right)\hat{e}_r-\frac{r}{r}\hat{e}_\theta\left(\frac{\partial v_z}{\partial r}-\frac{\partial v_r}{\partial z}\right)$

$+\frac{1}{r}\left(\frac{\partial}{\partial r}(rv_\theta)-\frac{\partial v_r}{\partial \theta}\right)\hat{e}_z$ ✓

11. (b) $u=\rho^2$: $\underset{\sim}{\nabla}u=2\rho\hat{e}_\rho$

$\nabla^2 u=\frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}(\rho^2 2\rho)+0+0\right]=6$ Check: $u=\rho^2=x^2+y^2+z^2$

and $\nabla^2(\quad \text{"} \quad)=2+2+2=6$ ✓

$\underset{\sim}{v}=3\hat{e}_\phi$: $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=0+\frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(3\sin\phi)+0=\frac{3}{\rho}\cot\phi$

$\underset{\sim}{\nabla}\times\underset{\sim}{v}=0\hat{e}_\rho+\hat{e}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(3\rho)-0\right)\hat{e}_\theta=\frac{3}{\rho}\hat{e}_\theta$

(c) $u=\sin\phi$: $\underset{\sim}{\nabla}u=\underset{\sim}{0}+\frac{1}{\rho}\frac{\partial}{\partial \phi}\sin\phi\,\hat{e}_\phi+\underset{\sim}{0}=\frac{\cos\phi}{\rho}\hat{e}_\phi$

$\nabla^2 u=0+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\cos\phi)+0=\frac{\cos^2\phi-\sin^2\phi}{\rho^2\sin\phi}$

$$\underset{\sim}{v} = \rho\hat{e}_\phi: \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\rho\sin\phi) + 0 = \cot\phi$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{0} + \underset{\sim}{0} + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2)\,\hat{e}_\theta = 2\hat{e}_\theta$$

12. (a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{e}_\theta\frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}\right)\cdot\left(v_\rho\hat{e}_\rho + v_\phi\hat{e}_\phi + v_\theta\hat{e}_\theta\right)$

$$= \hat{e}_\rho\cdot\left(\frac{\partial v_\rho}{\partial\rho}\hat{e}_\rho + v_\rho\cancel{\frac{\partial\hat{e}_\rho}{\partial\rho}}\right) + \hat{e}_\rho\cdot\left(\frac{\partial v_\phi}{\partial\rho}\hat{e}_\phi + v_\phi\cancel{\frac{\partial\hat{e}_\phi}{\partial\rho}}\right) + \hat{e}_\rho\cdot\left(\frac{\partial v_\theta}{\partial\rho}\hat{e}_\theta + v_\theta\cancel{\frac{\partial\hat{e}_\theta}{\partial\rho}}\right)$$

$$+ \frac{1}{\rho}\hat{e}_\phi\cdot\left(\frac{\partial v_\rho}{\partial\phi}\hat{e}_\rho + v_\rho\cancelto{\hat{e}_\phi}{\frac{\partial\hat{e}_\rho}{\partial\phi}}\right) + \frac{1}{\rho}\hat{e}_\phi\cdot\left(\frac{\partial v_\phi}{\partial\phi}\hat{e}_\phi + v_\phi\cancelto{-\hat{e}_\rho}{\frac{\partial\hat{e}_\phi}{\partial\phi}}\right) + \frac{1}{\rho}\hat{e}_\phi\cdot\left(\frac{\partial v_\theta}{\partial\phi}\hat{e}_\theta + v_\theta\cancel{\frac{\partial\hat{e}_\theta}{\partial\phi}}\right)$$

$$+ \frac{1}{\rho\sin\phi}\left[\hat{e}_\theta\cdot\left(\frac{\partial v_\rho}{\partial\theta}\hat{e}_\rho + v_\rho\cancelto{\sin\phi\,\hat{e}_\theta}{\frac{\partial\hat{e}_\rho}{\partial\theta}}\right) + \hat{e}_\theta\cdot\left(\frac{\partial v_\phi}{\partial\theta}\hat{e}_\phi + v_\phi\cancelto{\cos\phi\,\hat{e}_\theta}{\frac{\partial\hat{e}_\phi}{\partial\theta}}\right) + \hat{e}_\theta\cdot\left(\frac{\partial v_\theta}{\partial\theta}\hat{e}_\theta + v_\theta\cancelto{-\sin\phi\,\hat{e}_\rho-\cos\phi\,\hat{e}_\phi}{\frac{\partial\hat{e}_\theta}{\partial\theta}}\right)\right]$$

$$= \frac{\partial v_\rho}{\partial\rho} + \frac{v_\rho}{\rho} + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{\rho\sin\phi}\left(v_\rho\sin\phi + v_\phi\cos\phi + \frac{\partial v_\theta}{\partial\theta}\right) \checkmark$$

(b) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{e}_\theta\frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}\right)\times\left(v_\rho\hat{e}_\rho + v_\phi\hat{e}_\phi + v_\theta\hat{e}_\theta\right)$

$$= \hat{e}_\rho\times\left(\frac{\partial v_\rho}{\partial\rho}\hat{e}_\rho + v_\rho\cancel{\frac{\partial\hat{e}_\rho}{\partial\rho}}\right) + \hat{e}_\rho\times\left(\frac{\partial v_\phi}{\partial\rho}\hat{e}_\phi + v_\phi\cancel{\frac{\partial\hat{e}_\phi}{\partial\rho}}\right) + \hat{e}_\rho\times\left(\frac{\partial v_\theta}{\partial\rho}\hat{e}_\theta + v_\theta\cancel{\frac{\partial\hat{e}_\theta}{\partial\rho}}\right)$$

$$+ \frac{1}{\rho}\hat{e}_\phi\times\left(\frac{\partial v_\rho}{\partial\phi}\hat{e}_\rho + v_\rho\cancelto{\hat{e}_\phi}{\frac{\partial\hat{e}_\rho}{\partial\phi}}\right) + \frac{1}{\rho}\hat{e}_\phi\times\left(\frac{\partial v_\phi}{\partial\phi}\hat{e}_\phi + v_\phi\cancelto{-\hat{e}_\rho}{\frac{\partial\hat{e}_\phi}{\partial\phi}}\right) + \frac{1}{\rho}\hat{e}_\phi\times\left(\frac{\partial v_\theta}{\partial\phi}\hat{e}_\theta + v_\theta\cancel{\frac{\partial\hat{e}_\theta}{\partial\phi}}\right)$$

$$+ \frac{1}{\rho\sin\phi}\left[\hat{e}_\theta\times\left(\frac{\partial v_\rho}{\partial\theta}\hat{e}_\rho + v_\rho\cancelto{\sin\phi\,\hat{e}_\theta}{\frac{\partial\hat{e}_\rho}{\partial\theta}}\right) + \hat{e}_\theta\times\left(\frac{\partial v_\phi}{\partial\theta}\hat{e}_\phi + v_\phi\cancelto{\cos\phi\,\hat{e}_\theta}{\frac{\partial\hat{e}_\phi}{\partial\theta}}\right) + \hat{e}_\theta\times\left(\frac{\partial v_\theta}{\partial\theta}\hat{e}_\theta + v_\theta\cancelto{-\sin\phi\,\hat{e}_\rho-\cos\phi\,\hat{e}_\phi}{\frac{\partial\hat{e}_\theta}{\partial\theta}}\right)\right]$$

$$= \frac{\partial v_\phi}{\partial\rho}\hat{e}_\theta - \frac{\partial v_\theta}{\partial\rho}\hat{e}_\phi - \frac{1}{\rho}\frac{\partial v_\rho}{\partial\phi}\hat{e}_\theta + \frac{1}{\rho}v_\phi\hat{e}_\theta + \frac{1}{\rho}\frac{\partial v_\theta}{\partial\phi}\hat{e}_\rho$$

$$+ \frac{1}{\rho\sin\phi}\frac{\partial v_\rho}{\partial\theta}\hat{e}_\phi - \frac{1}{\rho\sin\phi}\frac{\partial v_\phi}{\partial\theta}\hat{e}_\rho - \frac{1}{\rho}\frac{\partial v_\theta}{\partial\theta}\hat{e}_\phi + \frac{\cot\phi}{\rho}v_\theta\hat{e}_\rho \checkmark$$

13. $\nabla^2\rho^\alpha = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}(\rho^2\alpha\rho^{\alpha-1}) + 0 + 0\right] = 0$ requires that $\rho^2\alpha\rho^{\alpha-1} = \text{constant}$; i.e., we need $\alpha = 0$ or $\alpha = -1$. Thus, we have the two solutions $u = 1$ and $u = \frac{1}{\rho} = \frac{1}{\sqrt{x^2+y^2+z^2}}$ (for $\rho \neq 0$).

15. (a) x: $\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = -\frac{1}{\sigma}\frac{\partial p}{\partial x}$

y: $\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} = -\frac{1}{\sigma}\frac{\partial p}{\partial y}$

z: $\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\sigma}\frac{\partial p}{\partial z}$

(b) Then (15.1) is $\frac{\partial}{\partial t}(v_r \hat e_r + v_\theta \hat e_\theta + v_z \hat e_z) + \left(v_r \frac{\partial}{\partial r} + v_\theta \frac{1}{r}\overset{*}{\frac{\partial}{\partial \theta}} + v_z \frac{\partial}{\partial z}\right)\overbrace{(v_r \hat e_r + v_\theta \hat e_\theta + v_z \hat e_z)}^{\underset{\sim}{v}}$

$$= -\frac{1}{\sigma}\left(\frac{\partial p}{\partial r}\hat e_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat e_\theta + \frac{\partial p}{\partial z}\hat e_z\right)$$

Remembering that the $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial z$ derivatives in $*$ act not only on the coefficients v_r, v_θ, v_z in $\underset{\sim}{v}$ but also on the base vectors in $\underset{\sim}{v}$ (in particular, $\partial \hat e_r/\partial\theta = \hat e_\theta$ and $\partial \hat e_\theta/\partial\theta = -\hat e_r$), the latter gives, for the $\hat e_r, \hat e_\theta, \hat e_z$ components

$$r:\quad \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{1}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \boxed{- \frac{v_\theta^2}{r}} = -\frac{1}{\sigma}\frac{\partial p}{\partial r},$$

$$\theta:\quad \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \boxed{\frac{v_\theta v_r}{r}} = -\frac{1}{\sigma}\frac{1}{r}\frac{\partial p}{\partial \theta},$$

$$z:\quad \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_\theta \frac{1}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\sigma}\frac{\partial p}{\partial z},$$

where we've dotted the "extra" terms mentioned above.

(c) Then (15.1) is $\frac{\partial}{\partial t}(v_\rho \hat e_\rho + v_\phi \hat e_\phi + v_\theta \hat e_\theta) + \left(v_\rho \frac{\partial}{\partial \rho} + v_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + v_\theta \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}\right)(v_\rho \hat e_\rho + v_\phi \hat e_\phi + v_\theta \hat e_\theta)$

$$= -\frac{1}{\sigma}\left(\frac{\partial p}{\partial \rho}\hat e_\rho + \frac{1}{\rho}\frac{\partial p}{\partial \phi}\hat e_\phi + \frac{1}{\rho\sin\phi}\frac{\partial p}{\partial \theta}\hat e_\theta\right)$$

or, excluding ρ derivatives of base vectors (which are $\underset{\sim}{0}$),

$$\frac{\partial}{\partial t}(v_\rho \hat e_\rho + v_\phi \hat e_\phi + v_\theta \hat e_\theta) + v_\rho \frac{\partial v_\rho}{\partial \rho}\hat e_\rho + v_\rho \frac{\partial v_\phi}{\partial \rho}\hat e_\phi + v_\rho \frac{\partial v_\theta}{\partial \rho}\hat e_\theta$$

$$+ v_\phi \frac{1}{\rho}\left(\frac{\partial v_\rho}{\partial \phi}\hat e_\rho + v_\rho \overset{\hat e_\phi}{\cancel{\frac{\partial \hat e_\rho}{\partial \phi}}} + \frac{\partial v_\phi}{\partial \phi}\hat e_\phi + v_\phi \overset{-\hat e_\rho}{\cancel{\frac{\partial \hat e_\phi}{\partial \phi}}} + \frac{\partial v_\theta}{\partial \phi}\hat e_\theta + v_\theta \cancel{\frac{\partial \hat e_\theta}{\partial \phi}}\right)$$

$$+ v_\theta \frac{1}{\rho\sin\phi}\left(\frac{\partial v_\rho}{\partial \theta}\hat e_\rho + v_\rho \underset{\sin\phi\,\hat e_\theta}{\cancel{\frac{\partial \hat e_\rho}{\partial \theta}}} + \frac{\partial v_\phi}{\partial \theta}\hat e_\phi + v_\phi \underset{\cos\phi\,\hat e_\theta}{\cancel{\frac{\partial \hat e_\phi}{\partial \theta}}} + \frac{\partial v_\theta}{\partial \theta}\hat e_\theta + v_\theta \underset{-\sin\phi\,\hat e_\rho - \cos\phi\,\hat e_\phi}{\cancel{\frac{\partial \hat e_\theta}{\partial \theta}}}\right)$$

$$= -\frac{1}{\sigma}\left(\frac{\partial p}{\partial \rho}\hat e_\rho + \frac{1}{\rho}\frac{\partial p}{\partial \phi}\hat e_\phi + \frac{1}{\rho\sin\phi}\frac{\partial p}{\partial \theta}\hat e_\theta\right)$$

so, $\rho:\quad \frac{\partial v_\rho}{\partial t} + v_\rho \frac{\partial v_\rho}{\partial \rho} + v_\phi \frac{1}{\rho}\frac{\partial v_\rho}{\partial \phi} - \frac{v_\phi^2}{\rho} + v_\theta \frac{1}{\rho\sin\phi}\frac{\partial v_\rho}{\partial \theta} - \frac{v_\theta^2}{\rho} = -\frac{1}{\sigma}\frac{\partial p}{\partial \rho}$

$$\phi:\quad \frac{\partial v_\phi}{\partial t} + v_\rho \frac{\partial v_\phi}{\partial \rho} + v_\phi \frac{1}{\rho} v_\rho + v_\phi \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + v_\theta \frac{1}{\rho\sin\phi}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\theta^2}{\rho}\cot\phi = -\frac{1}{\sigma}\frac{1}{\rho}\frac{\partial p}{\partial \phi}$$

$$\theta:\quad \frac{\partial v_\theta}{\partial t} + v_\rho \frac{\partial v_\theta}{\partial \rho} + v_\phi \frac{1}{\rho}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{\rho} v_\theta v_\rho + \frac{v_\theta v_\phi}{\rho}\cot\phi + v_\theta \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial \theta} = -\frac{1}{\sigma}\frac{1}{\rho\sin\phi}\frac{\partial p}{\partial \theta}$$

NOTE: Consider this as an examination question. Give the full Navier-Stokes equation $\sigma\frac{D\underset{\sim}{v}}{Dt}=\underset{\sim}{F}-\underset{\sim}{\nabla}p+\mu\nabla^2\underset{\sim}{v}$, where $\frac{D\underset{\sim}{v}}{Dt}=\frac{\partial\underset{\sim}{v}}{\partial t}+\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}$, for an incompressible fluid with velocity $\underset{\sim}{v}$, constant density σ, constant viscosity μ, pressure p, and body-force field $\underset{\sim}{F}$. Also give the Laplacian in cylindrical coordinates and possibly the gradient and the derivatives of the $\hat{e}_\rho,\hat{e}_\theta$ base vectors. Then, ask the student to write out the r,θ,z components of the equation. Or, one could do that for spherical coordinates. A shorter version would be to use plane polar rather than cylindrical coordinates, or spherical coordinates where there is axisymmetry such that there is no variation of the flow field with θ. As another possible question (similar to Exercise 7 in Section 16.6), one could give the Cartesian form of the x,y,z components of the Navier-Stokes equation and ask the student to infer the compact vector form (as given above).

Section 16.8

1. (a) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_V 0\,dV = 0.$

$$\begin{aligned}\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA &= \hat{\imath}\cdot(2\hat{\imath}-\hat{\jmath}+4\hat{k})(3)(2)+(-\hat{\imath})\cdot(2\hat{\imath}-\hat{\jmath}+4\hat{k})(3)(2)\\ &\quad+\hat{\jmath}\cdot(\quad '' \quad)(1)(2)+(-\hat{\jmath})\cdot(\quad '' \quad)(1)(2)\\ &\quad+\hat{k}\cdot(\quad '' \quad)(1)(3)+(-\hat{k})\cdot(\quad '' \quad)(1)(3)\\ &= 2(6)-2(6)-1(2)+1(2)+4(3)-4(3)=0.\ \checkmark\end{aligned}$$

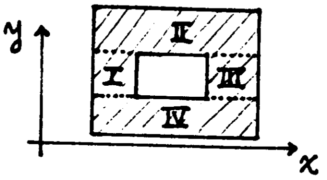
(b) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_V 3\,dV = 3(8) = 24.$

$$\begin{aligned}\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA &= \int_{-1}^{1}\int_{1}^{1}\hat{\imath}\cdot(x\hat{\imath}+2y\hat{\jmath})\Big|_{x=1}dy\,dz+\int_{-1}^{1}\int_{-1}^{1}-\hat{\imath}\cdot(x\hat{\imath}+2y\hat{\jmath})\Big|_{x=-1}dy\,dz\\ &\quad+\int_{-1}^{1}\int_{-1}^{1}\hat{\jmath}\cdot(\quad '' \quad)\Big|_{y=1}dx\,dz+\int_{-1}^{1}\int_{-1}^{1}-\hat{\jmath}\cdot(\quad '' \quad)\Big|_{y=-1}dx\,dz\\ &\quad+\int_{-1}^{1}\int_{-1}^{1}\hat{k}\cdot(\quad\quad)\Big|_{z=1}dx\,dy+\int_{-1}^{1}\int_{-1}^{1}-\hat{k}\cdot(\quad '' \quad)\Big|_{z=-1}dx\,dy\\ &= 4+4+8+8+0+0=24.\ \checkmark\end{aligned}$$

(d) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^1\int_0^1\int_0^1 x^2\,dx\,dy\,dz = 1/3.$

$$\begin{aligned}\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA &= \int_0^1\int_0^1\hat{\imath}\cdot(\hat{\jmath}+x^2z\hat{k})\Big|_{x=1}dy\,dz+\int_0^1\int_0^1-\hat{\imath}\cdot(\hat{\jmath}+x^2z\hat{k})\Big|_{x=0}dy\,dz\\ &\quad+\int_0^1\int_0^1\hat{\jmath}\cdot(\hat{\jmath}+x^2z\hat{k})\Big|_{y=1}dx\,dz+\int_0^1\int_0^1-\hat{\jmath}\cdot(\hat{\jmath}+x^2z\hat{k})\Big|_{y=0}dx\,dz\\ &\quad+\int_0^1\int_0^1\hat{k}\cdot(\quad '' \quad)\Big|_{z=1}dx\,dy+\int_0^1\int_0^1-\hat{k}\cdot(\quad '' \quad)\Big|_{z=0}dx\,dy\\ &= 0-0+1-1+\int_0^1\int_0^1 x^2\,dx\,dy-0=1/3.\ \checkmark\end{aligned}$$

(f) To see how to hande the cavity, consider the 2-dim. version $\int_S f\,dA$ where S is the region shown at the right. We could write $\int_S = \int_{I} + \int_{II} + \int_{III} + \int_{IV}$, but it is simpler to write $\int_S$ = integral over the full rectangle minus the integral over the cut-out rectangle. Similarly in the 3-dim. case.



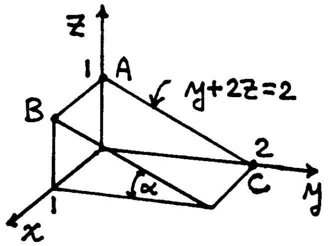
Thus,

$$\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_{-3}^{3}\int_{-2}^{2}\int_{-2}^{2} 2x\,dx\,dy\,dz - \int_0^1\int_0^1\int_0^1 2x\,dx\,dy\,dz = 0-1=-1$$

$$\begin{aligned}\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA &= \int_{-3}^{3}\int_{-2}^{2}\hat{\imath}\cdot(x^2\hat{\imath}-2z\hat{\jmath})\Big|_{x=2}dy\,dz + \int_{-3}^{3}\int_{-2}^{2}-\hat{\imath}\cdot(x^2\hat{\imath}-2z\hat{\jmath})\Big|_{x=-2}dy\,dz\\ &+\int_{-3}^{3}\int_{-2}^{2}\hat{\jmath}\cdot(\quad\text{''}\quad)\Big|_{y=2}dx\,dz + \int_{-3}^{3}\int_{-2}^{2}-\hat{\jmath}\cdot(\quad\text{''}\quad)\Big|_{y=-2}dx\,dz\\ &+\int_{-2}^{2}\int_{-2}^{2}\hat{k}\cdot(\quad\text{''}\quad)\Big|_{z=3}dx\,dy + \int\int-\hat{k}\cdot(\quad\text{''}\quad)\Big|_{z=-3}dx\,dy\\ &+\int_0^1\int_0^1\hat{\imath}\cdot(\quad\text{''}\quad)\Big|_{x=0}dy\,dz + \int_0^1\int_0^1-\hat{\imath}\cdot(\quad\text{''}\quad)\Big|_{x=1}dy\,dz\\ &+\int_0^1\int_0^1\hat{\jmath}\cdot(\quad\text{''}\quad)\Big|_{y=0}dx\,dz + \int_0^1\int_0^1-\hat{\jmath}\cdot(\quad\text{''}\quad)\Big|_{y=1}dx\,dz\\ &+\int_0^1\int_0^1\hat{k}\cdot(\quad\text{''}\quad)\Big|_{z=0}dx\,dy + \int_0^1\int_0^1-\hat{k}\cdot(\quad\text{''}\quad)\Big|_{z=1}dx\,dy\end{aligned}$$

$$= 4(24)-4(24)+0-0+0-0+0-1-1+1+0-0 = -1. \checkmark$$

(g) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^1\int_0^{2-2z}\int_0^1 x\,dx\,dy\,dz = \frac{1}{2}\int_0^1(2-2z)\,dz = \frac{1}{2}$



$$\begin{aligned}\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA &= \int_0^1\int_0^{2-2z}\cancel{\hat{\imath}\cdot xy}\hat{\jmath}\,\Big|_{x=1}dy\,dz\\ &+\int_0^1\int_0^{2-2z}\cancel{-\hat{\imath}\cdot xy}\hat{\jmath}\,\Big|_{x=0}dy\,dz\\ &+\int_0^1\int_0^1\cancel{-\hat{\jmath}\cdot xy}\hat{\jmath}\,\Big|_{y=0}dx\,dz + \int_0^2\int_0^1\cancel{-\hat{k}\cdot xy}\hat{\jmath}\,\Big|_{z=0}dx\,dy\\ &+\int_0^2\int_0^1 \underset{\nwarrow\hat{n}}{\boxed{\frac{\hat{\jmath}+2\hat{k}}{\sqrt{5}}}}\cdot xy\hat{\jmath}\,\Big|_{z=1-\frac{1}{2}y}\boxed{\frac{\sqrt{5}}{2}\,dx\,dy}\leftarrow dA\\ &= 0+0+0+0+\frac{1}{2}\int_0^2\int_0^1 xy\,dx\,dy = \frac{1}{2}.\ \checkmark\end{aligned}$$

NOTE: We can get $\underset{\sim}{n}$ as $\underset{\sim}{n} = \underset{\sim}{AB}\times\underset{\sim}{AC} = \hat{\imath}\times(-\hat{k}+2\hat{\jmath}) = \hat{\jmath}+2\hat{k}$, so $\hat{n} = (\hat{\jmath}+2\hat{k})/\sqrt{5}$, or we could note that the equation of the slanted face is $y+2z=2$ and get $\underset{\sim}{n}$ as $\underset{\sim}{n} = \underset{\sim}{\nabla}(y+2z) = \hat{\jmath}+2\hat{k}$. For dA we used (11), where the surface is given by $z=1-\frac{1}{2}y \equiv f(x,y)$, so $dA = \sqrt{1+0^2+(-1/2)^2}\,dx\,dy = \frac{\sqrt{5}}{2}\,dx\,dy$.

(h) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^3\int_0^{4z/3}\int_0^2 y^2\,dx\,dy\,dz = \frac{2}{3}\int_0^3\left(\frac{4z}{3}\right)^3 dz = 32.$

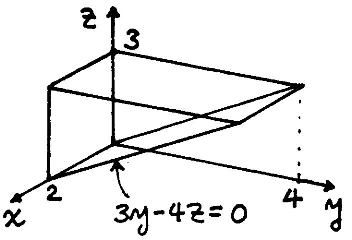
$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int\int \cancel{\hat{\imath}\cdot y^2} z\hat{k}\Big|_{x=2} dy\,dz$$

$$+\int\int -\cancel{\hat{\imath}\cdot y^2} z\,\hat{k}\Big|_{x=0} dy\,dz$$

$$+\int\int \cancel{-\hat{\jmath}\cdot y^2} z\hat{k}\Big|_{y=0} dx\,dz + \int_0^4\int_0^2 \hat{k}\cdot y^2 z\,\hat{k}\Big|_{z=3} dx\,dy$$

$$+\int_0^4\int_0^2 \left(\frac{3\hat{\jmath}-4\hat{k}}{5}\right)\cdot y^2 z\hat{k}\Big|_{z=3y/4} \frac{5}{4}dx\,dy$$

$$= 0+0+0+128-96 = 32. \checkmark$$



$\underset{\sim}{n} = \nabla(3y-4z)$
$= 3\hat{\jmath}-4\hat{k},$
$\underset{\sim}{\hat{n}} = (3\hat{\jmath}-4\hat{k})/5$
$dA = \sqrt{1+z_x^2+z_y^2}\;dxdy$
$= \sqrt{1+0+9/16}\;dxdy$
$= \frac{5}{4}dxdy$

(j) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^3\int_0^{4z/3}\int_0^2 -2z\,dx\,dy\,dz$
$= -4\int_0^3 z\left(\frac{4z}{3}\right)dz = -48.$ (The figure, $\underset{\sim}{\hat{n}}$, and dA are the same as above.)

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^3\int_0^2 -\hat{\jmath}\cdot(3x^2-2yz)\hat{\jmath}\Big|_{y=0} dx\,dz$$

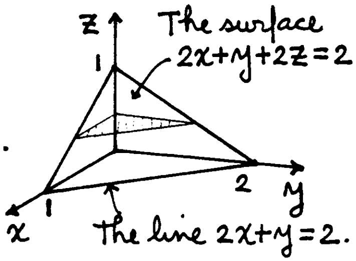
$$+\int_0^4\int_0^2\left(\frac{3\hat{\jmath}-4\hat{k}}{5}\right)\cdot(3x^2-2yz)\hat{\jmath}\Big|_{z=3y/4}\frac{5}{4}dx\,dy + \overset{\overset{x=0\text{ face}}{\downarrow}}{0} + \overset{x=2\text{ face}}{0}$$

$$= \int_0^3\int_0^2 -3x^2\,dx\,dz + \int_0^4\int_0^2 \frac{3}{5}\left(3x^2-2y\frac{3}{4}y\right)\frac{5}{4}dx\,dy = -24-24 = -48. \checkmark$$

(l) Find that $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \frac{1}{6}$

(m) $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^1\int_0^{2-2z}\int_0^{1-\frac{1}{2}y-z} z(2x)\,dx\,dy\,dz$

$$= \int_0^1\int_0^{2-2z}\left(1-\frac{1}{2}y-z\right)^2 z\,dy\,dz = \int_0^1\frac{2}{3}(1-z)^3 z\,dz = \frac{1}{30}.$$



$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_{y=0\text{ face}} \cancel{-\hat{\jmath}\cdot x^2} z\hat{\imath}\Big|_{y=0} dx\,dz$$

$$+\int_{z=0\text{ face}} \cancel{-\hat{k}\cdot x^2} z\hat{\imath}\Big|_{z=0} dx\,dy$$

$$+\int_{x=0\text{ face}} -\hat{\imath}\cdot\cancel{x^2} z\hat{\imath}\Big|_{x=0} dy\,dz$$

$$+\int_0^2\int_0^{1-\frac{1}{2}y} \frac{2\hat{\imath}+\hat{\jmath}+2\hat{k}}{3}\cdot x^2 z\hat{\imath}\Big|_{z=1-x-\frac{1}{2}y} \frac{3}{2}dx\,dy$$

$$= \int_0^2\int_0^{1-\frac{1}{2}y} x^2\left(1-x-\frac{1}{2}y\right)dx\,dy = \int_0^1\int_0^{2-2x} x^2\left(1-x-\frac{1}{2}y\right)dy\,dx$$

(I've switched the order because it's easier to integrate first on y.)

$$= \int_0^1\left[2x^2(1-x)^2 - x^2(1-x)^2\right]dx = \frac{1}{30}. \checkmark$$

On the slanted face,
$\underset{\sim}{n} = \nabla(2x+y+2z)$
$= 2\hat{\imath}+\hat{\jmath}+2\hat{k}$
so $\underset{\sim}{\hat{n}} = (2\hat{\imath}+\hat{\jmath}+2\hat{k})/3,$
and $dA = \sqrt{1+f_x^2+f_y^2}\;dxdy$
$= \sqrt{1+(-1)^2+(-\frac{1}{2})^2}\;dxdy$
$= \frac{3}{2}dxdy$

2.(a) $\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{a}\,dA = \int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{a}\,dV = \int_V 0\,dV = 0$ so $\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{a}\,dA = a\cdot\int_S \hat{n}\,dA = 0$ for every $\underset{\sim}{a}$. Thus, we can let $\underset{\sim}{a} = \int_S \hat{n}\,dA$ so $\underset{\sim}{a}\cdot\int_S \underset{\sim}{\hat{n}}\,dA = \|\int_S \underset{\sim}{\hat{n}}\,dA\|^2 = 0$. Thus, $\int_S \hat{n}\,dA = \underset{\sim}{0}$.

NOTE: If we look ahead to Exercise 9 we can proceed more easily:

$$\int_S \hat{n}\,dA = \int_V \underset{\sim}{\nabla}(1)\,dV = \int_V \underset{\sim}{0}\,dV = \underset{\sim}{0}.$$

(b) $\int_S \underset{\sim}{\hat{n}}\cdot x\underset{\sim}{\hat{\imath}}\,dA = \int_V \underset{\sim}{\nabla}\cdot x\hat{\imath}\,dV = \int_V dV = V.$

(c) $\int_S \underset{\sim}{\hat{n}}\cdot(x\hat{\imath}+y\hat{\jmath})\,dA = \int_V \underset{\sim}{\nabla}\cdot(x\hat{\imath}+y\hat{\jmath})\,dV = \int_V 2\,dV = 2V.$

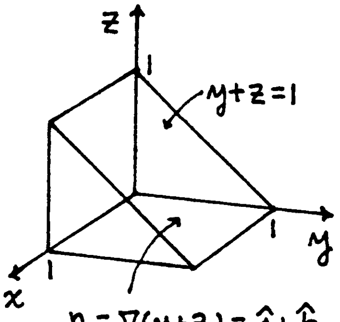
3. Green's first identity: $\int_V (\underset{\sim}{\nabla}u\cdot\underset{\sim}{\nabla}v + u\nabla^2 v)\,dV = \int_S u\frac{\partial v}{\partial n}\,dA.$

(a) LHS $= \int_0^1\int_0^3\int_0^2 (6x^2y^2+4x^4)\,dx\,dy\,dz = \int_0^1\int_0^3 (16y^2+\frac{128}{5})\,dy\,dz = \frac{1104}{5}$

$$\text{RHS} = \int_0^1\int_0^3 2x^3y^2\Big|_{x=2} dy\,dz + \int_0^1\int_0^3 2x^3(-y^2)\Big|_{x=0} dy\,dz + \int_0^1\int_0^2 2x^3(2xy)\Big|_{y=3} dx\,dz$$
$$+ \int_0^1\int_0^2 2x^3(-2xy)\Big|_{y=0} dx\,dz + \int_0^3\int_0^2 2x^3\cdot 0\,dx\,dy - \int_0^3\int_0^2 2x^3\cdot 0\,dx\,dy$$
$$= (16)(9)+0+\frac{384}{5}-0+0-0 = \frac{1104}{5}.\ \checkmark$$

(c) LHS $= \int_0^1\int_0^{1-z}\int_0^1 6\,dx\,dy\,dz = 3$

$$\text{RHS} = \int_0^1\int_0^{1-z} 2x\Big|_{x=1} dy\,dz + \int_0^1\int_0^{1-z} -2x\Big|_{x=0} dy\,dz$$
$$+ \int_0^1\int_0^1 -2y\Big|_{y=0} dx\,dz + \int_0^1\int_0^1 -2z\Big|_{z=0} dx\,dy$$
$$+ \int_0^1\int_0^1 (2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k})\cdot\frac{\hat{\jmath}+\hat{k}}{\sqrt{2}}\Big|_{z=1-y} \sqrt{2}\,dx\,dy$$
$$= 1-0-0-0+\int_0^1\int_0^1 [2y+2(1-y)]\,dx\,dy = 1+2 = 3.\ \checkmark$$



$\underset{\sim}{n} = \underset{\sim}{\nabla}(y+z) = \hat{\jmath}+\hat{k}$
so $\underset{\sim}{\hat{n}} = (\hat{\jmath}+\hat{k})/\sqrt{2}$.
$dA = \sqrt{1+f_x^2+f_y^2}\,dx\,dy$
$= \sqrt{1+0+(-1)^2}\,dx\,dy$
$= \sqrt{2}\,dx\,dy$

4. $\int_R \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dA = \int_C \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,ds$

(a) $\int_R \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dA = \int_R 2\,dA = 2A = 2ab$

$$\int_C \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,ds = \int_0^b \hat{\imath}\cdot(x\hat{\jmath}+y\hat{\jmath})\Big|_{x=a} dy + \int_0^b -\hat{\imath}\cdot(x\hat{\imath}+y\hat{\jmath})\Big|_{x=0} dy$$
$$+ \int_0^a \hat{\jmath}\cdot(x\hat{\imath}+y\hat{\jmath})\Big|_{y=b} dx + \int_0^a -\hat{\jmath}\cdot(x\hat{\imath}+y\hat{\jmath})\Big|_{y=0} dx$$
$$= \int_0^b a\,dy - 0 + \int_0^a b\,dx - 0 = 2ab.\ \checkmark$$

(c) $\int_R \nabla\cdot\mathbf{v}\,dA = \int_0^1\int_{2y}^2 y^2\,dx\,dy = \int_0^1(2y^2-2y^3)\,dy = \frac{1}{6}$.

$$\int_C \hat{\mathbf{n}}\cdot\mathbf{v}\,ds = \int_0^1 \hat{\imath}\cdot xy^2\hat{\imath}\Big|_{x=2}dy + \int_0^2 -\hat{\jmath}\cdot xy^2\hat{\imath}\Big|_{y=0}dx$$

$$+\int_0^2 \frac{-\hat{\imath}+2\hat{\jmath}}{\sqrt{5}}\cdot xy^2\hat{\imath}\Big|_{y=x/2}\frac{\sqrt{5}}{2}dx$$

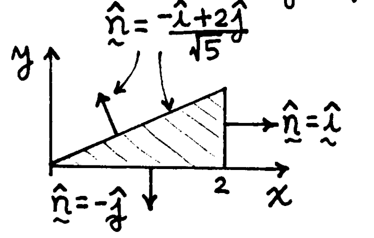
$$=\int_0^1 2y^2dy - 0 - \int_0^2 \frac{xy^2}{\sqrt{5}}\frac{\sqrt{5}}{2}\Big|_{y=x/2}dx$$

$$=\frac{2}{3}-\int_0^2\frac{x^3}{8}dx = \frac{1}{6}.\ \checkmark$$

$x-2y=0$

$\mathbf{n}=\nabla(x-2y)=\hat{\imath}-2\hat{\jmath}$

Need To change sign:



5. (a) $\mathcal{I} = \int_0^1\int_0^1\int_0^1 x^2yz\,dx\,dy\,dz = \frac{1}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{12}$.

Seek a $\mathbf{v}$ such that $\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z} = x^2yz$. Lots of freedom in doing so. For ex., we can set $v_y = v_z = 0$ and take $\frac{\partial v_x}{\partial x} = x^2yz$. Even so, we obtain $v_x = \frac{x^3}{3}yz + A(y,z)$ where $A(y,z)$ is arbitrary. Let $A=0$. Then $\mathbf{v} = \frac{1}{3}x^3yz\hat{\imath}$ so

$$\mathcal{I} = \int_V \nabla\cdot\mathbf{v}\,dV = \int_S \hat{\mathbf{n}}\cdot\mathbf{v}\,dA = \int_0^1\int_0^1 \hat{\imath}\cdot\frac{1}{3}x^3yz\hat{\imath}\Big|_{x=1}dy\,dz$$

$$+\int_0^1\int_0^1 -\hat{\imath}\cdot\frac{1}{3}x^2yz\hat{\imath}\Big|_{x=0}dy\,dz$$

plus 0 on each of the other four faces since $\pm\hat{\jmath}\cdot\hat{\imath}=0$ and $\pm\hat{k}\cdot\hat{\imath}=0$

$$=\int_0^1\int_0^1\frac{1}{3}yz\,dy\,dz + 0 = \frac{1}{12}.\ \checkmark$$

(c) $\mathcal{I} = \int_0^1\int_0^1\int_0^1 4\,dx\,dy\,dz = 4$

Seek $\mathbf{v}$ such that $\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z} = 4$. Let $\mathbf{v}=4x\hat{\imath}$, say. Then

$$\mathcal{I} = \int_V \nabla\cdot\mathbf{v}\,dV = \int_S \hat{\mathbf{n}}\cdot 4x\hat{\imath}\,dA = \int_0^1\int_0^1 \hat{\imath}\cdot 4x\hat{\imath}\Big|_{x=1}dy\,dz + \int_0^1\int_0^1 -\hat{\imath}\cdot 4x\hat{\imath}\Big|_{x=0}dy\,dz$$

plus 0 on each of the other four faces since $\pm\hat{\jmath}\cdot\hat{\imath}=0$ and $\pm\hat{k}\cdot\hat{\imath}=0$

$$= 4\int_0^1\int_0^1 dy\,dz$$

$$= 4.\ \checkmark$$

6. (a) Setting $v=1$ in (45), obtain $\int_V -\nabla^2u\,dV = \int_S -\frac{\partial u}{\partial n}dA$ so $\int_V \nabla^2u\,dV = \int_S \frac{\partial u}{\partial n}dA$. Alternatively,

$$\int_V \nabla^2u\,dV = \int_V \nabla\cdot\nabla u\,dV = \int_S \hat{\mathbf{n}}\cdot\nabla u\,dA \text{ by divergence theorem}$$

$$= \int_S \frac{\partial u}{\partial n}dA \text{ since } \frac{du}{ds} = \nabla u\cdot\hat{\mathbf{s}}.$$

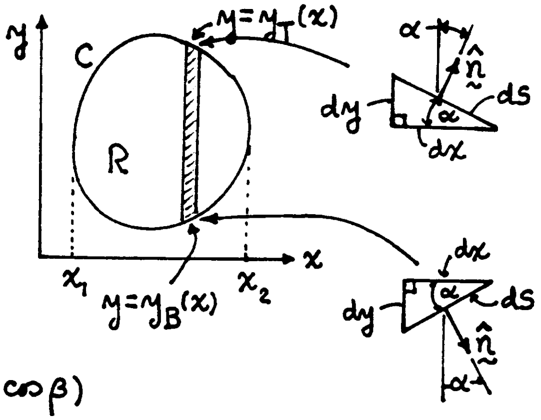
(b) Set $v=u$ in (44).

7. The result is

(a) $\int_R (\underset{\sim}{\nabla} u\cdot\underset{\sim}{\nabla} v + u\nabla^2 v)\,dA = \int_C u\frac{\partial v}{\partial n}\,ds$

(b) $\int_R (u\nabla^2 v - v\nabla^2 u)\,dA = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$

8. $\int_R \frac{\partial v_x}{\partial x}\,dA = \int_C (v_x\hat{\imath})\cdot\hat{\underset{\sim}{n}}\,ds$ per (8.1).



$$\int_R \frac{\partial v_y}{\partial y}\,dA = \int_R \frac{\partial v_y}{\partial y}\,dy\,dx$$

$$= \int_{x_1}^{x_2}\int_{y_B(x)}^{y_T(x)} \frac{\partial v_y}{\partial y}\,dy\,dx$$

$$= \int_{x_1}^{x_2} v_y\Big|_{y_T(x)} dx - \int_{x_1}^{x_2} v_y\Big|_{y_B(x)} dx$$

$$= \int_{x_1}^{x_2} v_y\Big|_{y_T(x)} (ds\cos\alpha) - \int_{x_1}^{x_2} v_y\Big|_{y_B(x)} (ds\cos\beta)$$

$$= \int_{top} v_y(\hat{\jmath}\cdot\hat{n})\,ds - \int_{bottom} v_y(-\hat{\jmath}\cdot\hat{n})\,ds$$

$$= \int_{top}(v_y\hat{\jmath})\cdot\hat{n}\,ds + \int_{bottom}(v_y\hat{\jmath})\cdot\hat{n}\,ds = \int_C (v_y\hat{\jmath})\cdot\hat{n}\,ds$$

Thus, $\int_R \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dA = \int_R\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)dA = \int_C (v_x\hat{\imath}+v_y\hat{\jmath})\cdot\hat{n}\,ds = \int_C \underset{\sim}{v}\cdot\hat{\underset{\sim}{n}}\,ds.$

9.(a) $$\int_S \hat{n}u\,dA = \hat{\imath}\int_S \hat{n}\cdot(u\hat{\imath})\,dA + \hat{\jmath}\int_S \hat{n}\cdot(u\hat{\jmath})\,dA + \hat{k}\int_S \hat{n}\cdot(u\hat{k})\,dA$$

$$= \hat{\imath}\int_V \underset{\sim}{\nabla}\cdot(u\hat{\imath})\,dV + \hat{\jmath}\int_V \underset{\sim}{\nabla}\cdot(u\hat{\jmath})\,dV + \hat{k}\int_V \underset{\sim}{\nabla}\cdot(u\hat{k})\,dV$$

$$= \hat{\imath}\int_V \frac{\partial u}{\partial x}\,dV + \hat{\jmath}\int_V \frac{\partial u}{\partial y}\,dV + \hat{k}\int_V \frac{\partial u}{\partial z}\,dV$$

$$= \int_V\left(\frac{\partial u}{\partial x}\hat{\imath}+\frac{\partial u}{\partial y}\hat{\jmath}+\frac{\partial u}{\partial z}\hat{k}\right)dV = \int_V \underset{\sim}{\nabla}u\,dV. \checkmark$$

(b) $$\int_S \hat{\underset{\sim}{n}}\times\underset{\sim}{v}\,dA = \hat{\imath}\int_S \hat{n}\cdot(v_z\hat{\jmath}-v_y\hat{k})\,dA + \hat{\jmath}\int_S \hat{n}\cdot(v_x\hat{k}-v_z\hat{\imath})\,dA + \hat{k}\int_S \hat{n}\cdot(v_y\hat{\imath}-v_x\hat{\jmath})\,dA$$

$$= \hat{\imath}\int_V \underset{\sim}{\nabla}\cdot(v_z\hat{\jmath}-v_y\hat{k})\,dV + \hat{\jmath}\int_V \underset{\sim}{\nabla}\cdot(v_x\hat{k}-v_z\hat{\imath})\,dV + \hat{k}\int_V \underset{\sim}{\nabla}\cdot(v_y\hat{\imath}-v_x\hat{\jmath})\,dV$$

$$= \int_V\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\hat{k}\right]dV = \int_V \underset{\sim}{\nabla}\times\underset{\sim}{v}\,dV. \checkmark$$

(c) With $u=1$, (9.1) gives $\int_S \hat{n}\,dA = \int_V \underset{\sim}{\nabla}(1)\,dV = \int_V \underset{\sim}{0}\,dV = \underset{\sim}{0}$.
For example, if S is the surface of the rectangular prism $0<x<a$, $0<y<b$, $0<z<c$, then

$$\int_S \hat{n}\,dA = \underset{\substack{\text{from}\\ x=a\text{ face}}}{(\hat{\imath})(bc)} + \underbrace{(-\hat{\imath})(bc)}_{x=0} + \underbrace{(\hat{\jmath})(ac)}_{y=b} + \underbrace{(-\hat{\jmath})(ac)}_{y=0} + \underbrace{(\hat{k})(ab)}_{z=c} + \underbrace{(-\hat{k})(ab)}_{z=0} = \underset{\sim}{0}\ \checkmark$$

10. $\underset{\sim}{\nabla}\cdot(\sigma\underset{\sim}{v})=0, \quad \int_V \underset{\sim}{\nabla}\cdot(\sigma\underset{\sim}{v})\,dV = \int_V 0\,dV$

$$\int_S \underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}\,dA = 0$$

$$\int_{S_1}\underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}\,dA+\int_{S_2}\underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}\,dA+\int_{S_3}\cancel{\underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}}\,dA=0$$

$\underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}=0$ on S_3

so

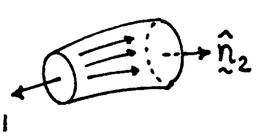
$$\int_{S_1}\underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}\,dA+\int_{S_2}\underset{\sim}{\hat{n}}\cdot\sigma\underset{\sim}{v}\,dA=0.$$

If $\sigma,\underset{\sim}{v}$ are constant on S_1 and on S_2 then the latter becomes

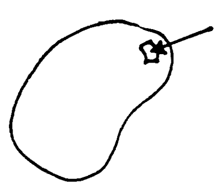
$$\underset{\sim}{\hat{n}}_1\cdot\sigma_1\underset{\sim}{v}_1\int_{S_1}dA+\underset{\sim}{\hat{n}}_2\cdot\sigma_2\underset{\sim}{v}_2\int_{S_2}dA=0,$$

$$-\sigma_1V_1A_1+\sigma_2V_2A_2=0,$$

$$\sigma_1A_1V_1=\sigma_2A_2V_2.$$



11.



$$d\underset{\sim}{F}=-\underset{\sim}{\hat{n}}p\,dA=-\underset{\sim}{\hat{n}}(-\sigma gz)\,dA=\underset{\sim}{\hat{n}}\sigma gz\,dA$$

$$\text{so } \underset{\sim}{F}=\int_S d\underset{\sim}{F}=\int_S\underset{\sim}{\hat{n}}\sigma gz\,dA=\sigma g\int_S\underset{\sim}{\hat{n}}z\,dA=\sigma g\int_V\underset{\sim}{\nabla}(z)\,dV$$

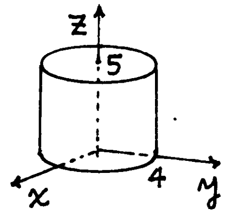
$$=\sigma g\int_V\hat{k}\,dV=\sigma g\hat{k}\int_V dV=\sigma Vg\hat{k}$$

= mass of displaced water times g (acceleration of gravity) $\hat{k}$ = (displaced water weight) times $\hat{k}$. ✓

12. (a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\underset{\sim}{\nabla}\cdot(3r^2\hat{e}_r-r\hat{e}_\theta+2\hat{e}_z)$

$$=\frac{1}{r}\frac{\partial}{\partial r}(3r^3)+\frac{1}{r}\frac{\partial}{\partial\theta}(-r)+\frac{\partial}{\partial z}(2)=9r$$

$$\text{so } \int_V\underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV=\int_0^5\int_0^{2\pi}\int_0^4 9r(r\,dr\,d\theta\,dz)=(9)(2\pi)(5)\frac{4^3}{3}=1920\pi$$



$$\int_S\underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA=\int_0^{2\pi}\int_0^4\hat{e}_z\cdot\underset{\sim}{v}\Big|_{z=5}r\,dr\,d\theta+\int_0^{2\pi}\int_0^4-\hat{e}_z\cdot\underset{\sim}{v}\Big|_{z=0}r\,dr\,d\theta$$

$$+\int_0^5\int_0^{2\pi}\hat{e}_r\cdot\underset{\sim}{v}\Big|_{r=4}r\,d\theta\,dz$$

$$=\cancel{\int_0^{2\pi}\int_0^4 2r\,dr\,d\theta}-\cancel{\int_0^{2\pi}\int_0^4 2r\,dr\,d\theta}+\int_0^5\int_0^{2\pi}3(4)^3\,d\theta\,dz=3(4)^3(2\pi)(5)=1920\pi.\ ✓$$

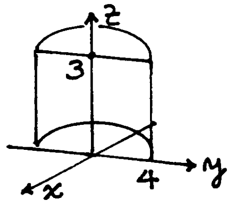
(b) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=1$, so $\int_V\underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV=\int_V 1\,dV=V=\pi(2)^2(9)=36\pi.$

$$\int_S\underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA=\int_{\text{Top}}\hat{e}_z\cdot z\hat{e}_z\Big|_{z=6}dA+\int_{\text{bottom}}-\hat{e}_z\cdot z\hat{e}_z\Big|_{z=-3}dA+\int_{\text{side}}\hat{e}_r\cancel{\cdot z}\hat{e}_z\,dA$$

$$=6A+3A=9\pi(2)^2=36\pi.\ ✓$$

(c) $\underset{\sim}{v}=rz^2\hat{e}_\theta$, so $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=0+\frac{1}{r}\frac{\partial}{\partial\theta}(rz^2)+0=0,$

hence $\int_V\underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV=\int_V 0\,dV=0.$

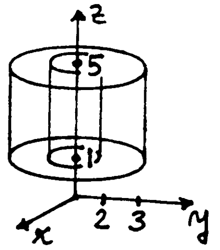


In evaluating the flat face ($x=0$ plane) we need to consider separately the two parts: $\theta=\pi/2$ and $\theta=3\pi/2$. Thus,

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^3\int_0^4 (-\hat{e}_\theta)\cdot rz^2\hat{e}_\theta\,dr\,dz + \int_0^3\int_0^4 (+\hat{e}_\theta)\cdot rz^2\hat{e}_\theta\,dr\,dz$$

$$+\iint_{top} + \iint_{bottom} + \iint_{cylind.}$$

$$= -64+64+\underbrace{0+0+0}_{\text{In each case } \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}=0} \qquad = 0 \checkmark$$

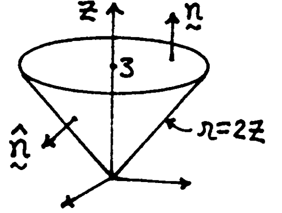


(d) $\underset{\sim}{v} = z^2\hat{e}_r + 6\hat{e}_z$; so $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_1^5\int_0^{2\pi}\int_2^3 \frac{z^2}{r}\,r\,dr\,d\theta\,dz = \frac{248\pi}{3}$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \iint_{Top} \hat{e}_z\cdot(z^2\hat{e}_r+6\hat{e}_z)\Big|_{z=5} r\,dr\,d\theta + \iint_{bottom} -\hat{e}_z\cdot(z^2\hat{e}_r+6\hat{e}_z)\Big|_{z=1} r\,dr\,d\theta$$

$$+\iint_{\substack{\text{outside}\\ \text{cylinder}}} \hat{e}_r\cdot(z^2\hat{e}_r+6\hat{e}_z)\,3\,d\theta\,dz + \iint_{\substack{\text{inside}\\ \text{cylinder}}} -\hat{e}_r\cdot(z^2\hat{e}_r+6\hat{e}_z)\,2\,d\theta\,dz$$

$$= \iint_{top} 6\cancel{rdr}d\theta - \iint_{bottom} 6\cancel{rdr}d\theta + \int_1^5\int_0^{2\pi} 3z^2\,d\theta\,dz - \int_1^5\int_0^{2\pi} 2z^2\,d\theta\,dz = \frac{248\pi}{3} \checkmark$$



(e) $\underset{\sim}{v} = z^2\hat{e}_z$ so $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^3\int_0^{2\pi}\int_0^{2z} 2z\,r\,dr\,d\theta\,dz$

$$= 2(2\pi)\int_0^3 z\frac{(2z)^2}{2}\,dz = 8\pi\frac{3^4}{4} = 162\pi$$

On the top, $\hat{n}=\hat{e}_z$ and $dA = r\,dr\,d\theta\,dz$.
On the cone, $\underset{\sim}{n} = \underset{\sim}{\nabla}(r-2z) = \hat{e}_r - 2\hat{e}_z$ so $\underset{\sim}{\hat{n}} = (\hat{e}_r - 2\hat{e}_z)/\sqrt{5}$. To obtain an expression for dA (which is harder inasmuch as the cone is not a constant-coordinate surface) let us parametrize the cone:

$$\left.\begin{aligned} x &= r\cos\theta &&= u\cos v\\ y &= r\sin\theta &&= u\sin v\\ z &= r/2 &&= u/2 \end{aligned}\right\} \begin{aligned} &u: 0\to 6\\ &v: 0\to 2\pi \end{aligned}$$

Then $E = x_u^2+y_u^2+z_u^2 = 5/4$
$F = x_u x_v + y_u y_v + z_u z_v = 0$
$G = x_v^2 + y_v^2 + z_v^2 = u^2$, so $dA = \sqrt{EG-F^2}\,du\,dv = \frac{\sqrt{5}}{2}u\,du\,dv = \frac{\sqrt{5}}{2}r\,dr\,d\theta$

so

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^{2\pi}\int_0^6 z^2\Big|_{z=3} r\,dr\,d\theta + \int_0^{2\pi}\int_0^6 -\frac{2z^2}{\sqrt{5}}\Big|_{z=r/2} \frac{\sqrt{5}}{2} r\,dr\,d\theta = 324\pi - 162\pi = 162\pi. \checkmark$$

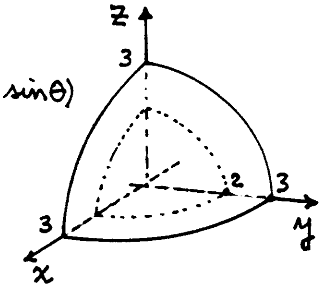
(f) $\underset{\sim}{v} = \rho^2\hat{e}_\rho$ so $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^a 4\rho\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\pi a^4.$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^{2\pi}\int_0^{\pi} \rho^2\,\rho^2\sin\phi\,d\phi\,d\theta\Big|_{\rho=a} = 4\pi a^4. \checkmark$$

(g) $\underset{\sim}{v} = \rho^3 \hat{e}_\rho$ so $\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\rho^3) = 5\rho^2$, $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a 5\rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi a^5.$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^{2\pi}\int_0^{\pi/2} \underset{\sim}{\hat{e}}_\rho \cdot \rho^3 \underset{\sim}{\hat{e}}_\rho\ \rho^2\sin\phi\,d\phi\,d\theta\Big|_{\rho=a} + \int_0^{2\pi}\int_0^{a} \cancel{\underset{\sim}{\hat{e}}_\phi \cdot \rho^3 \underset{\sim}{\hat{e}}_\rho}\ \rho\sin\phi\,d\rho\,d\theta\Big|_{\phi=\frac{\pi}{2}}$$

$$= a^5\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta = 2\pi a^5. \checkmark$$



(h) $\underset{\sim}{v} = \rho^2\sin\phi\sin\theta\ \hat{e}_\theta$, $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{1}{\rho^2}\cancel{\frac{\partial}{\partial\rho}}(0) + \frac{1}{\rho\sin\phi}\cancel{\frac{\partial}{\partial\phi}}(0) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}(\rho^2\sin\phi\sin\theta)$

$$= \rho\cos\theta$$

so $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_2^3 \rho\cos\theta\ \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3^4-2^4}{4} = \frac{65}{4}.$

For $\int_S$ we have 5 parts of S: $x=0$ face, $y=0$ face, $z=0$ face, $\rho=2$ and $\rho=3$,

$\hat{n}=\hat{e}_\theta$, $\theta=\pi/2$; $\hat{n}=-\hat{e}_\theta$, $\theta=0$; $\hat{n}=\hat{e}_\phi$, $\phi=\pi/2$; $\hat{n}=-\hat{e}_\rho$, $\rho=2$; $\hat{n}=\hat{e}_\rho$, $\rho=3$

so

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^{\pi/2}\int_2^3 \rho^2\sin\phi\sin\theta\ \rho\,d\rho\,d\phi\Big|_{\theta=\pi/2} + \int_0^{\pi/2}\int_2^3 -\rho^2\sin\phi\sin\theta\ \rho\,d\rho\,d\phi\Big|_{\theta=0}$$

$$+ \int_0^{\pi/2}\int_2^3 0\ \cancel{\rho\sin\phi\,d\rho\,d\theta}\Big|_{\phi=\pi/2} + \int_0^{\pi/2}\int_0^{\pi/2} 0\cdot\cancel{\rho^2\sin\phi\,d\phi\,d\theta}\Big|_{\rho=2} + \int_0^{\pi/2}\int_0^{\pi/2} 0\cdot\cancel{\rho^2\sin\phi\,d\phi\,d\theta}\Big|_{\rho=3}$$

$$= \int_0^{\pi/2}\int_2^3 \rho^3\sin\phi\,d\rho\,d\phi - \int_0^{\pi/2}\int_2^3 \cancel{0\,d\rho}\,d\phi = \frac{3^4-2^4}{4} = \frac{65}{4}\ \checkmark$$

(i) $\underset{\sim}{v} = \frac{1}{\rho}\hat{e}_\rho + \rho^3\hat{e}_\phi - \rho^2\sin^2\phi\ \hat{e}_\theta$, $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\rho^3\sin\phi) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}(-\rho^2\sin^2\phi)$

$$= \frac{1}{\rho^2} + \rho^2\cot\phi$$

so $\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{v}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_2^5\left(\frac{1}{\rho^2}+\rho^2\cot\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\int_2^5(\sin\phi+\rho^4\cos\phi)\,d\rho\,d\phi$

$$= 12\pi.$$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{v}\,dA = \int_0^{2\pi}\int_0^{\pi}\frac{1}{\rho}\ \rho^2\sin\phi\,d\phi\,d\theta\Big|_{\rho=5} + \int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{\rho}\right)\rho^2\sin\phi\,d\phi\,d\theta\Big|_{\rho=2} = 12\pi.\ \checkmark$$

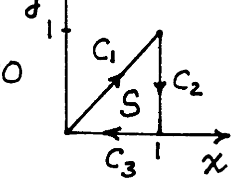
Section 16.9

1. $x = 3+(2-3)\tau = 3-\tau$, $y = 4+(1-4)\tau = 4-3\tau$, $z = 5+(3-5)\tau = 5-2\tau$

$$\int_{DA} = \int_0^1[(3-\tau)(5-2\tau)^2(-1) - 3(-3) + 2(4-3\tau)(-2)]\,d\tau = \int_0^1(4\tau^3-32\tau^2+97\tau-82]\,d\tau$$

$$= -259/6$$

2. (a) $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2+C_3} xy\,dx - (2x-y)\,\cancelto{0}{dz}$

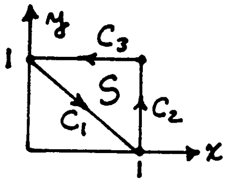


C_1: $x=\tau, y=\tau, z=0$; C_2: $x=1, y=1-\tau, z=0$; C_3: $x=1-\tau, y=0, z=0$

so $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \int_0^1 \tau^2\,d\tau + \int_0^1 0\,d\tau = 1/3.$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S -\underset{\sim}{\hat{k}}\cdot\left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy & 0 & y-2x\end{matrix}\right|\Bigg|_{z=0} dx\,dy = \int_0^1\int_y^1 x\,dx\,dy = 1/3.\ \checkmark$$

(b) C_1: $x=\tau, y=1-\tau, z=0$; C_2: $x=1, y=\tau, z=0$; C_3: $x=1-\tau, y=1, z=0$,



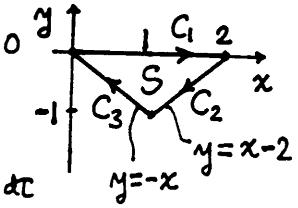
so $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2+C_3} xy^2dx - y^3dy + 4xyz\,\cancel{dz}^{\to 0}$ $= \int_{C_1+C_2+C_3}[xy^2dx/d\tau - y^3dy/d\tau]\,d\tau$

$= \int_0^1 [\tau(1-\tau)^2(1) - (1-\tau)^3(-1)]\,d\tau$

$+ \int_0^1 [(1)\tau^2(0) - \tau^3(1)]\,d\tau + \int_0^1 [(1-\tau)(1)^2(-1) - (1)^3(0)]\,d\tau = -5/12$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S \hat{k}\cdot\left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy^2 & -y^3 & 4xyz\end{matrix}\right|\Bigg|_{z=0} dx\,dy = \int_0^1\int_{1-y}^1 -2xy\,dx\,dy = -5/12\ \checkmark$$

(c) C_1: $x=2\tau, y=0, z=0$; C_2: $x=2-\tau, y=-\tau, z=0$; C_3: $x=1-\tau, y=-1+\tau, z=0$



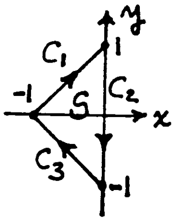
so $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2+C_3} y^2dx - x^2dy - \cancel{\sin(x^2y^2z^2)\,dz}^{\to 0}$

$= \int_{C_1}\left[y^2\frac{dx}{d\tau} - x^2\frac{dy}{d\tau}\right]d\tau + \int_{C_2}\left[y^2\frac{dx}{d\tau} - x^2\frac{dy}{d\tau}\right]d\tau + \int_{C_3}\left[y^2\frac{dx}{d\tau} - x^2\frac{dy}{d\tau}\right]d\tau$

$= \int_0^1 [(0)(2) - 4\tau^2(0) + (-\tau)^2(-1) - (2-\tau)^2(-1) + (-1+\tau)^2(-1) - (1-\tau)^2(1)]\,d\tau = 4/3.$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S -\hat{k}\cdot\left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y^2 & -x^2 & -\sin x^2y^2z^2\end{matrix}\right|\Bigg|_{z=0} dx\,dy = \int_{-1}^{0}\int_{-y}^{2+y} 2(x+y)\,dx\,dy = 4/3\ \checkmark$$

(d) C_1: $x=\tau-1, y=\tau, z=0$; C_2: $x=0, y=1-2\tau, z=0$; C_3: $x=-\tau, y=\tau-1, z=0$



so $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2+C_3}\left(xe^y\frac{dx}{d\tau} + (x+z)\frac{dy}{d\tau} - \cancel{\frac{dz}{d\tau}}^{\to 0}\right)d\tau$

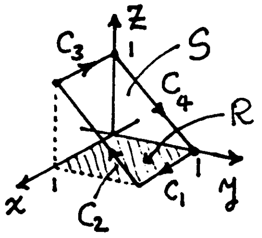
$= \int_0^1 [(\tau-1)e^{\tau}(1) + (\tau-1)(1) + 0 + 0 + (-\tau)e^{\tau-1}(-1) + (-\tau+0)(1)]\,d\tau$

$= 1-e+e^{-1}$

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S -\hat{k}\cdot\left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xe^y & x+z & -1\end{matrix}\right|\Bigg|_{z=0} dx\,dy = \int_{-1}^{0}\int_{-1-x}^{1+x}(xe^y-1)\underbrace{dy}_{\text{easier on } y \text{ first}}dx = 1-e+e^{-1}.\ \checkmark$$

(e) C_1: $x=\tau, y=1, z=0$; C_2: $x=1, y=1-\tau, z=\tau$;

C_3: $x=1-\tau, y=0, z=1$; C_4: $x=0, y=\tau, z=1-\tau$



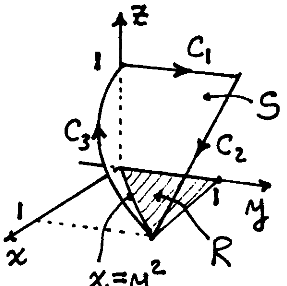
so $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+\cdots+C_4} x^2yz\,dy$

$= \int_0^1 [0 + (1)^2(1-\tau)\tau(-1) + 0 + 0]\,d\tau = -1/6$

On S, $y+z=1$ so $z=1-y$, so $\underset{\sim}{n} = \underset{\sim}{\nabla}(y+z) = \hat{\jmath}+\hat{k}$, $\underset{\sim}{\hat{n}} = \overset{\text{by right-hand rule for } \hat{n} \text{ and } C}{-}(\hat{\jmath}+\hat{k})/\sqrt{2}$, and $dA = \sqrt{1+z_x^2+z_y^2}\,dx\,dy = \sqrt{1+0+(-1)^2}\,dx\,dy = \sqrt{2}\,dx\,dy$, so

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_R -\frac{\hat{\jmath}+\hat{k}}{\sqrt{2}}\cdot\left|\begin{matrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 0&x^2yz&0\end{matrix}\right|\Bigg|_{z=1-y}\sqrt{2}\,dx\,dy = \int_0^1\int_0^1 -2xy(1-y)\,dx\,dy = -1/6.\checkmark$$

3. (a) Although it isn't asked for, perhaps we should begin by assuring ourselves that C_3 really is a plane curve, as stated, and that it lies in the same plane as C_1 and C_2, so that S is plane, as stated. Since C_3 is given by



$$C_3:\ x=\tau^2,\ y=\tau,\ z=1-\tau^2 \quad (\tau: 1\to 0)$$

both questions are answered affirmatively since we see from the parametrization that $x+z=1$, which is surely the plane of C_1 and C_2. The normal to that plane is $\underset{\sim}{n}=\underset{\sim}{\nabla}(x+z)=\hat{\imath}+\hat{k}$ so

$$\underset{\sim}{\hat{n}} = -(\hat{\imath}+\hat{k})/\sqrt{2},$$

the minus sign being needed so that C and $\underset{\sim}{\hat{n}}$ are in accordance with the right-hand rule. Also, note that the "shadow" of S onto the x,y plane is the region R bounded by $x=0$, $y=1$, $x=y^2$. Finally,

C_1: $x=0, y=\tau, z=1$ $(\tau: 0\to 1)$; C_2: $x=\tau, y=1, z=1-\tau$ $(\tau: 0\to 1)$

$$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1}(x^2dx+xzdy)+\int_{C_2}(x^2dx+xzdy)+\int_{C_3}(x^2dx+xzdy)$$

$$= \int_0^1 0\,d\tau + \int_0^1 \tau^2\,d\tau + \int_1^0[\tau^4 2\tau + \tau^2(1-\tau^2)]\,d\tau = -2/15.$$

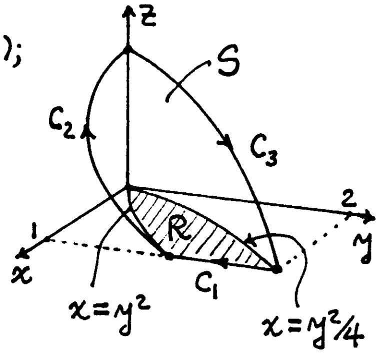
Next, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left|\begin{matrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ x^2&xz&0\end{matrix}\right| = -x\hat{\imath}+z\hat{k}$, $\underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v} = (x-z)/\sqrt{2}$, and

$dA=\sqrt{1+z_x^2+z_y^2}\,dx\,dy = \sqrt{1+(-1)^2+0}\,dx\,dy = \sqrt{2}\,dx\,dy$ (because $z=1-x$ on S), so

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \iint_R \frac{x-z}{\sqrt{2}}\bigg|_{z=1-x}\sqrt{2}\,dx\,dy = \int_0^1\int_0^{y^2}(2x-1)\,dx\,dy = -2/15.\ \checkmark$$

(b) C_1: $x=1, y=2-\tau, z=0$ $(\tau: 0\to 1)$; C_2: $x=\tau^2, y=\tau, z=1-\tau^2$ $(\tau: 1\to 0)$;
C_3: $x=\tau^2, y=2\tau, z=1-\tau^2$ $(\tau: 0\to 1)$.

NOTE: I find that when students have trouble with problems like this it is often because they do not begin by drawing a 3-dimensional sketch, as we have at the right. Often, the sketch itself is a stumbling block.



Observe from the parametrizations that $x+z=1$ for both C_2 and C_3, so S is the plane

$$x+z=1, \quad \text{or} \quad z=1-x,$$

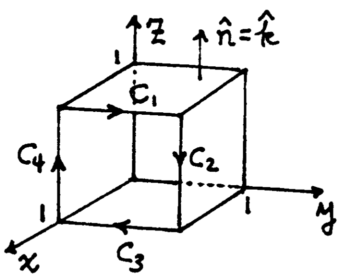
so $\mathbf{n} = \nabla(x+z) = \hat{\mathbf{i}} + \hat{\mathbf{k}} \rightarrow \hat{\mathbf{n}} = -(\hat{\mathbf{i}}+\hat{\mathbf{k}})/\sqrt{2}$, and $dA = \sqrt{1+z_x^2+z_y^2}\,dx\,dy = \sqrt{1+(-1)^2+0}\,dx\,dy$ $= \sqrt{2}\,dx\,dy$. Thus,

$$\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA = \int_{R} -\frac{\hat{\mathbf{i}}+\hat{\mathbf{k}}}{\sqrt{2}} \cdot \left.\left|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & -xz & 0 \end{matrix}\right|\right|_{z=1-x} \sqrt{2}\,dx\,dy = \int_0^1 \int_{\sqrt{x}}^{2\sqrt{x}} (1-2x)\,dy\,dx = -2/15.$$

Next, $\oint_C \mathbf{v}\cdot d\mathbf{R} = \int_{C_1+C_2+C_3} x^2dx - xz\,dy$

$$= \int_0^1 [(1)^2(0)-0]\,d\tau + \int_1^0 [\tau^4 2\tau - \tau^2(1-\tau^2)(1)]\,d\tau + \int_0^1 [\tau^4 2\tau - \tau^2(1-\tau^2)2]\,d\tau = -2/15. ✓$$

(c) The direction of $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$ on $z=1$ implies the orientation of the boundary curve $C = C_1+C_2+C_3+C_4$, as shown. These line integrals are so simple that we don't need to bother formally parametrizing them:



$$\oint_C \mathbf{v}\cdot d\mathbf{R} = \oint_C y^2dy - xy^2z\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \left\{\int_0^1 y^2dy - \int xy^2z\,\cancelto{0}{dz}\right\} + \left\{\int y^2\cancelto{0}{dy} - \int_1^0 (1)(1)^2z\,dz\right\} + \left\{\int_1^0 y^2dy - \int 0\,\cancelto{0}{dz}\right\}$$
$$+ \left\{\int y^2\cancelto{0}{dy} - \int_0^1 (1)\cancelto{}{(0)}z\,dz\right\}$$

$$= \tfrac{1}{3} - 0 + 0 + \tfrac{1}{2} - \tfrac{1}{3} - 0 + 0 + 0 = 1/2.$$

Now for $\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA$. $\nabla\times\mathbf{v} = \left|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & y^2 & -xy^2z \end{matrix}\right| = -2xyz\hat{\mathbf{i}} + y^2z\hat{\mathbf{j}}$

$\int_S = \int_{\text{Top}} + \int_{\text{bottom}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{back}}$.

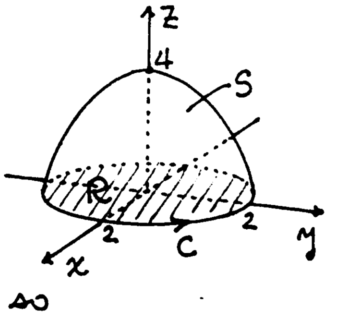
On Top $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ so $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{v} = 0$, so $\int_{\text{top}} = 0$.
On bottom $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ so $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{v} = 0$, so $\int_{\text{bottom}} = 0$.
On left $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$ so $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{v} = -y^2z = 0$ since $y=0$ there. Thus, $\int_{\text{left}} = 0$.
On right $\hat{\mathbf{n}} = \hat{\mathbf{j}}$ so $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{v} = y^2z = z$ so $\int_{\text{right}} = \int_0^1\int_0^1 z\,dx\,dz = 1/2$.
On back $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$ so $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{v} = 2xyz = 0$ since $x=0$ there. Thus $\int_{\text{back}} = 0$.
Summing these five values,

$$\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA = 0+0+0+1/2+0 = 1/2. ✓$$

(d) $\oint_C \mathbf{v}\cdot d\mathbf{R} = \int_C -y\,dx + x\,dy + 3\cancelto{0}{dz}$. Can parametrize C using the usual plane polar coordinate system

$x = 2\cos\theta,\ y = 2\sin\theta,\ z = 0 \quad \theta: 0 \to 2\pi$

(i.e., θ is our "τ"). Then



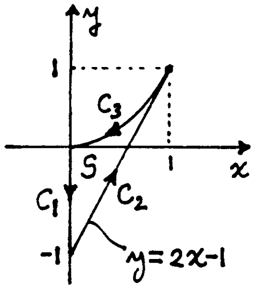
$$\oint_C = \int_0^{2\pi} [-2\sin\theta(-2\sin\theta) + 2\cos\theta(2\cos\theta)]\,d\theta$$
$$= 4\int_0^{2\pi} d\theta = 8\pi.$$

Turn to $\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA$. S is $z = 4-x^2-y^2$ or $x^2+y^2+z = 4$, so

$\underset{\sim}{n} = \nabla(x^2+y^2+z) = 2x\hat{\imath}+2y\hat{\jmath}+\hat{k}$, $\hat{n} = (2x\hat{\imath}+2y\hat{\jmath}+\hat{k})/\sqrt{4x^2+4y^2+1}$, and
$dA = \sqrt{1+z_x^2+z_y^2}\,dx\,dy = \sqrt{1+(-2x)^2+(-2y)^2}\,dx\,dy$, and $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath}-0\hat{\jmath}+2\hat{k}$

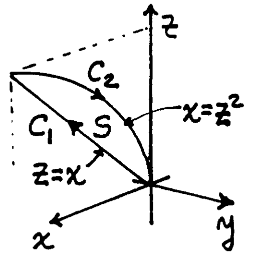
$$\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_R \frac{2x\hat{\imath}+2y\hat{\jmath}+\hat{k}}{\cancel{\sqrt{4x^2+4y^2+1}}}\cdot 2\hat{k}\,\cancel{\sqrt{1+4x^2+4y^2}}\,dx\,dy = 2\int_R dx\,dy = 2\pi 2^2 = 8\pi. ✓$$

(e) $C_1: x=0, y=-\tau, z=0$; $C_2: x=\tau, y=-1+2\tau, z=0$, for $\tau: 0\to 1$,
$C_3: x=\tau,\ y=\tau^2,\ z=0$ for $\tau: 1\to 0$. Then



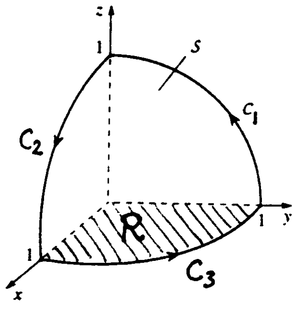
$$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2+C_3} x^2 y\,dy$$
$$= \int_{C_1}\cancelto{0}{x^2}y\,dy + \int_0^1 \tau^2(-1+2\tau)2d\tau + \int_1^0 \tau^2\tau^2\,2\tau\,d\tau = 0$$
$$\int_S \hat{n}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S \hat{k}\cdot 2xy\,\hat{k}\,dx\,dy = 2\int_0^1\int_{2x-1}^{x^2} xy\,dy\,dx = 0. ✓$$

(f) Let us use z (or x) as our parameter:
$C_1: x=z, y=0\ (z: 0\to 1)$; $C_2: x=z^2, y=0\ (z: 1\to 0)$



$$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1+C_2} x\,dx - xz\,dz$$
$$= \int_0^1 (z-z^2)\,dz + \int_1^0 (z^2 2z - z^3)\,dz = \frac{1}{2}-\frac{1}{3}-\frac{1}{4} = -1/12$$
$$\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_S -\hat{\jmath}\cdot\left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & 0 & -xz\end{matrix}\right|\Bigg|_{y=0} dx\,dz = -\int_0^1\int_{z^2}^{z} z\,dx\,dz = -1/12. ✓$$

(g) $S: z=1-x^2-y^2$ is a paraboloid of revolution, not a sphere.

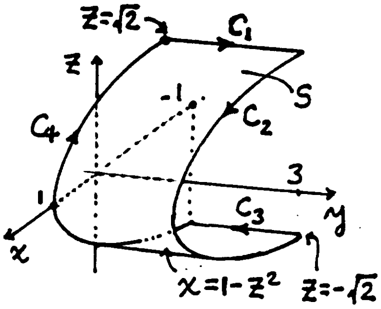


$$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} yz\,dz + \int_{C_2}\cancelto{0}{y}z\,dz + \int_{C_3} y\cancelto{0}{z}\,\cancelto{0}{dz}$$
$$= \int_{C_1} yz\,dz,$$ on which $z=1-y^2$. Use y as the parameter for C_1.
$$= \int_1^0 y(1-y^2)(-2y\,dy) = 4/15$$

Surface integral: $\underset{\sim}{n} = \nabla(x^2+y^2+z) = 2x\hat{\imath}+2y\hat{\jmath}+\hat{k}$, $\hat{n} = (2x\hat{\imath}+2y\hat{\jmath}+\hat{k})/\sqrt{4x^2+4y^2+1}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \left|\begin{matrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & 0 & yz\end{matrix}\right| = z\hat{\imath}$$

so $$\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_R \frac{2x\hat{\imath}+2y\hat{\jmath}+\hat{k}}{\cancel{\sqrt{4x^2+4y^2+1}}}\cdot z\hat{\imath}\,\underbrace{\cancel{\sqrt{1+(-2x)^2+(-2y)^2}}\,dx\,dy}_{dA}\Bigg|_{z=1-x^2-y^2}$$
$$= \iint_R 2x(1-x^2-y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 2r\cos\theta(1-r^2)\,r\,dr\,d\theta = 4/15. ✓$$



(h) $\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1}(x^2z\,\cancelto{0}{dx} - 3xy\,\cancelto{0}{dz}) + \int_{C_2}(x^2z\,dx - 3xy\,dz)$

$+\int_{C_3}(x^2z\,\cancelto{0}{dx} - 3xy\,\cancelto{0}{dz}) + \int_{C_4}(x^2z\,dx - 3x\cancelto{0}{y}\,dz)$

$= \int_{\sqrt{2}}^{-\sqrt{2}} \left[(1-z^2)^2 z(-2z) - 3(1-z^2)(3)\right] dz + \int_{-\sqrt{2}}^{\sqrt{2}} (1-z^2)^2 z(-2z)\,dz$

$= 6\sqrt{2}$. NOTE: It is simpler to use z as the parameter, for C_2 and C_4, because then each integral is a single integral (from $\sqrt{2}$ to $-\sqrt{2}$ for C_2 and from $-\sqrt{2}$ to $\sqrt{2}$ for C_4), whereas if we use x then each integral needs to be split into two parts, one for the upper sheet $z=+\sqrt{1-x}$ and one for the lower sheet $-\sqrt{1-x}$.

Surface integral: $\underset{\sim}{n} = \nabla(x+z^2) = \hat{\imath} + 2z\hat{k}$ but we need to reverse it since we are given that $\underset{\sim}{\hat{n}} = -\hat{\imath}$ at $(1,1,0)$. Thus, $\hat{n} = -(\hat{\imath}+2z\hat{k})/\sqrt{1+4z^2}$.

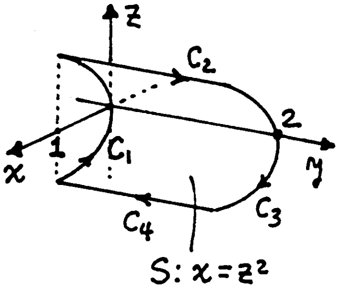
$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & 0 & -3xy \end{vmatrix} = -3x\hat{\imath} + (3y+x^2)\hat{\jmath}$. For dA, generally we use

$dA = \sqrt{1+z_x^2+z_y^2}\,dx\,dy$ but this formula would force us to treat the upper $(z>0)$ and lower $(z<0)$ parts of S separately. Rather, let us turn our head 90° and use $x = x(y,z)$, $dA = \sqrt{1+x_y^2+x_z^2}\,dy\,dz = \sqrt{1+(-2z)^2}\,dy\,dz$. Then

$$\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_R -\frac{\hat{\imath}+2z\hat{k}}{\cancel{\sqrt{1+4z^2}}}\cdot\left(-3x\hat{\imath}+(3y+x^2)\hat{\jmath}\right)\cancel{\sqrt{1+4z^2}}\,dy\,dz\Big|_{x=1-z^2}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_0^3 \left[3(1-z^2) - 2z(3y+(1-z^2)^2)\right] dy\,dz = 6\sqrt{2}. \checkmark$$

(i) Direction of C implied by the (given) fact that $\underset{\sim}{\hat{n}} = -\hat{\imath}$ at $(0,1,0)$.



$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1}(\cancelto{0}{y}z\,dx + x\cancelto{0}{y}\,dz) + \int_{C_2}(yz\,\cancelto{0}{dx} + xy\,\cancelto{0}{dz})$

$+\int_{C_3}(yz\,dx + xy\,dz) + \int_{C_4}(yz\,\cancelto{0}{dx} + xy\,\cancelto{0}{dz})$

$= \int_{C_3}(yz\,dx + xy\,dz) = \int_1^{-1}\left[2z(2z) + z^2(2)\right] dz = -4.$

Surface integral: $\underset{\sim}{n} = \underset{\sim}{\nabla}(x-z^2) = \hat{\imath} - 2z\hat{k} \rightarrow \underset{\sim}{\hat{n}} = -(\hat{\imath}-2z\hat{k})/\sqrt{1+4z^2}$

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 0 & xy \end{vmatrix} = x\hat{\imath} - z\hat{k}$. For dA, $x = x(y,z)$ is more convenient than $z = z(x,y)$, so use $dA = \sqrt{1+x_y^2+x_z^2}\,dy\,dz$
$= \sqrt{1+0+(2z)^2}\,dy\,dz$

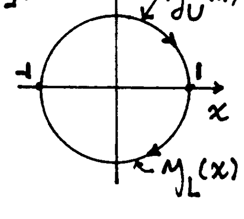
so $\int_S \underset{\sim}{\hat{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_R \frac{-\hat{\imath}+2z\hat{k}}{\cancel{\sqrt{1+4z^2}}}\cdot(x\hat{\imath}-z\hat{k})\cancel{\sqrt{1+4z^2}}\,dy\,dz\Big|_{x=z^2}$

$$= \int_{-1}^{1}\int_0^2 (-z^2-2z^2)\,dy\,dz = -4. \checkmark$$

4. As C closes up to a point the surface S passes from being open to being closed. Once it is closed we can use the divergence theorem: then,

$$\int_S \hat{\underset{\sim}{n}}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{v})\,dA = \int_V \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{v})\,dA = \int_V 0\,dA = 0$$

since the divergence of every curl is zero [see (12) in Section 16.6].



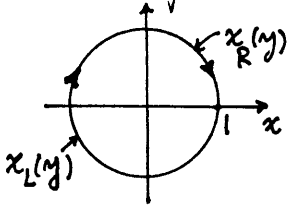
5. y is not a function (i.e., a single-valued function) on C since it is double-valued. Denoting the upper and lower branches as $y_U(x)$ and $y_L(x)$, which are different functions, then

$$\oint_C f(x,y)\,dx = \int_{-1}^{1} f(x,y_U(x))\,dx + \int_{1}^{-1} f(x,y_L(x))\,dx$$

$$= \int_{-1}^{1} [f(x,y_U(x)) - f(x,y_L(x))]\,dx$$

and there's no reason why the latter must be zero. For instance, if $f(x,y)=y$, then $\int_{-1}^{1}[f(x,y_U(x))-f(x,y_L(x))]\,dx = \int_{-1}^{1}[\sqrt{1-x^2}-(-\sqrt{1-x^2})]\,dx$
$= 2\int_{-1}^{1}\sqrt{1-x^2}\,dx \neq 0$.

Similarly for $\oint g\,dy$ but this time instead of breaking C into a top and a bottom break it into a left and a right:

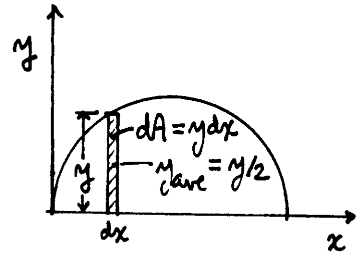


$$\oint_C g(x,y)\,dy = \int_{-1}^{1} g(x_L(y),y)\,dy + \int_{1}^{-1} g(x_R(y),y)\,dy$$

$$= \int_{-1}^{1}[g(x_L(y),y) - g(x_R(y),y)]\,dy,$$

which is not, in general, zero.

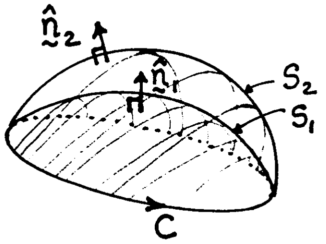
6. The first calculation is correct, the second is not. The error is the step $dA = y\,dx$ for if we take dA to be the vertical "sliver" $y\,dx$, then we should use the average value of the integrand y over that sliver, namely, $y/2$. Then



$$\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{R} = \int_S 2y\,dA = 2\int_0^4 y_{ave}\,y\,dx = 2\int_0^4 \frac{y}{2}\,y\,dx$$

$$= \int_0^4 y^2\,dx = \int_0^4 [4-(x-2)^2]\,dx = \frac{32}{3}. \checkmark$$

7. Together, S_1 and S_2 define a closed surface, with unit outward normal $\hat{\underset{\sim}{n}}_2$ on S_2 and $-\hat{\underset{\sim}{n}}_1$ on S_1. Then

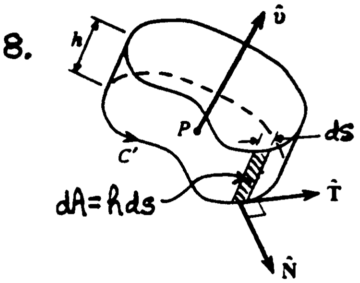


$$\int_{S=S_1+S_2} \hat{\underset{\sim}{n}}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{v})\,dA = \int_{S_1} -\hat{\underset{\sim}{n}}_1\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA + \int_{S_2}\hat{\underset{\sim}{n}}_2\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA,$$

but, by the divergence theorem, it also $= \int_V \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{v})\,dV = 0$ for every field $\underset{\sim}{v}$ [by (12) in Section 16.6]. Thus,

$$\int_{S_2}\hat{\underset{\sim}{n}}_2\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_{S_1}\hat{\underset{\sim}{n}}_1\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA. \checkmark$$

8.



$$\hat{\boldsymbol\nu}\cdot\operatorname{curl}\mathbf{v}(P)=\lim_{B\to0}\left\{\frac{\int_{S'}\hat{\boldsymbol\nu}\cdot\hat{\mathbf n}\times\mathbf v\,dA}{V}\right\}.$$

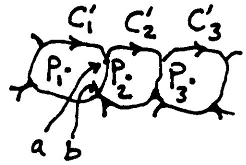
Observe that S', here, has a flat top, a flat bottom, and a curved side. On the top, $\hat{\mathbf n}=\hat{\boldsymbol\nu}$, so $\hat{\boldsymbol\nu}\cdot\hat{\mathbf n}\times\mathbf v=\hat{\boldsymbol\nu}\cdot\hat{\boldsymbol\nu}\times\mathbf v=0$ there. On the bottom, $\hat{\mathbf n}=-\hat{\boldsymbol\nu}$, so $\hat{\boldsymbol\nu}\cdot\hat{\mathbf n}\times\mathbf v=0$ there too. Thus, we need consider only the side of B. Thus,

$$\hat{\boldsymbol\nu}\cdot\operatorname{curl}\mathbf v(P)=\lim_{B\to0}\left\{\frac{\int_{side}\hat{\boldsymbol\nu}\cdot\hat{\mathbf N}\times\mathbf v\,(h\,ds)}{Ah}\right\}$$

$$=\lim_{A\to0}\frac{\oint_{C'}\hat{\boldsymbol\nu}\times\hat{\mathbf N}\cdot\mathbf v\,ds}{A}=\lim_{A\to0}\frac{\oint_{C'}\hat{\mathbf T}\cdot\mathbf v\,ds}{A}$$

$$=\lim_{A\to0}\frac{\oint_{C'}\mathbf v\cdot\hat{\mathbf T}\,ds}{A}=\lim_{A\to0}\frac{\oint_{C'}\mathbf v\cdot d\mathbf R}{A}$$

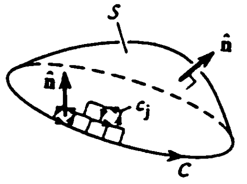
so $\quad \hat{\boldsymbol\nu}\cdot\operatorname{curl}\mathbf v(P)\,dA\sim\oint_{C'}\mathbf v\cdot d\mathbf R$



Such a statement holds for each little "fish scale" on S:

$$\hat{\boldsymbol\nu}_1\cdot\operatorname{curl}\mathbf v(P_1)\,dA_1\sim\oint_{C_1'}\mathbf v\cdot d\mathbf R$$
$$\hat{\boldsymbol\nu}_2\cdot\operatorname{curl}\mathbf v(P_2)\,dA_2\sim\oint_{C_2'}\mathbf v\cdot d\mathbf R$$
$$\vdots\qquad\qquad\vdots$$

Summing these, the left-hand side gives $\int_S\hat{\boldsymbol\nu}\cdot\operatorname{curl}\mathbf v\,dA$ (← i.e., the unit normal to S). By internal cancellation (i.e., the contribution of the segment ab to $\oint_{C_1'}$ is equal and opposite to its contribution to $\oint_{C_2'}$ so these contributions cancel; similarly all along the contours $C_1', C_2', \dots$, except along the boundary curve C of S) the right-hand sides sum to $\oint_C\mathbf v\cdot d\mathbf R$, which step gives (if only heuristically), Stokes's theorem.

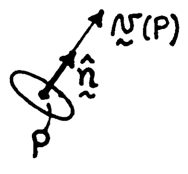


9. $\oint_C\mathbf E\cdot d\mathbf R=\int_S\hat{\mathbf n}\cdot\nabla\times\mathbf E\,dA$ by Stokes's theorem, so

$$\int_S\hat{\mathbf n}\cdot\nabla\times\mathbf E\,dA=-\int_S\frac{\partial\mathbf B}{\partial t}\cdot\hat{\mathbf n}\,dA,$$

$$\text{or,}\quad\int_S\hat{\mathbf n}\cdot\underbrace{\left(\nabla\times\mathbf E+\frac{\partial\mathbf B}{\partial t}\right)}_{\text{"}\mathbf v\text{", say}}dA=0.$$

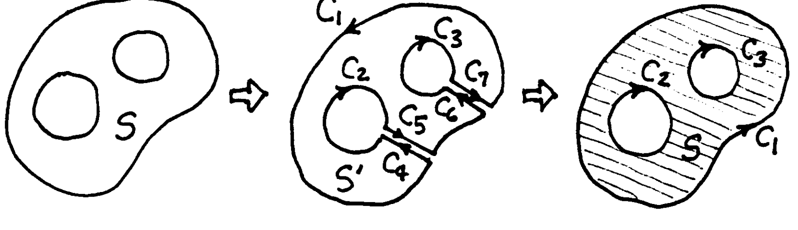
Now, if $\int_S\hat{\mathbf n}\cdot\mathbf v\,dA=0$ for every possible S in the region, then $\mathbf v$ must be identically $\mathbf 0$ in the region — in which case we obtain $\nabla\times\mathbf E+\partial\mathbf B/\partial t=\mathbf 0$ or $\nabla\times\mathbf E=-\partial\mathbf B/\partial t$. Here is a heuristic argument to support that claim. Suppose there is a point P in the region such that $\mathbf v(P)\neq\mathbf 0$. If we assume that $\mathbf v$ is a continuous function of x, y, z, then there



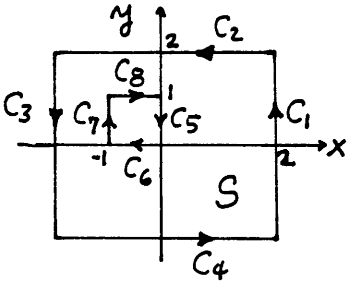
must exist a disk of radius ϵ ($\epsilon > 0$) such that $\underset{\sim}{v}\cdot\hat{\underset{\sim}{n}} > 0$ everywhere on the disk, where the disk is oriented so that its normal $\hat{\underset{\sim}{n}}$ is aligned with $\underset{\sim}{v}(P)$ and its center is at P. Let S be that disk! Then $\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{v}\,dA > 0$ (since $\hat{\underset{\sim}{n}}\cdot\underset{\sim}{v} > 0$ everywhere on S), which result contradicts the equation $\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{v}\,dA = 0$. Hence, the supposition $\underset{\sim}{v}(P) \neq \underset{\sim}{0}$ must have been false; $\underset{\sim}{v}$ cannot be nonzero anywhere in the region.

10. (a) The key point is simply that $\int_{C_3} = -\int_{C_4}$ so $\int_{C_1} + \int_{C_2} + \cancel{\int_{C_3}} + \cancel{\int_{C_4}} = \int_{C_1} + \int_{C_2}$.

(b)



so $\oint_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \oint_{C_1}\underset{\sim}{v}\cdot d\underset{\sim}{R} + \oint_{C_2}\underset{\sim}{v}\cdot d\underset{\sim}{R} + \oint_{C_3}\underset{\sim}{v}\cdot d\underset{\sim}{R}$, where $\hat{\underset{\sim}{n}}$ is perpendicular to the paper and toward the viewer.



11. (a) With $\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{k}}$, $\int_{S_1} \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_{C_1}\underset{\sim}{v}\cdot d\underset{\sim}{R} + \int_{C_2}\underset{\sim}{v}\cdot d\underset{\sim}{R} + \cdots + \int_{C_8}\underset{\sim}{v}\cdot d\underset{\sim}{R}$

Since $\int \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int y^3 dx$, $\int_{C_1} = \int_{C_3} = \int_{C_5} = \int_{C_7} = \int_{C_6} = 0$,

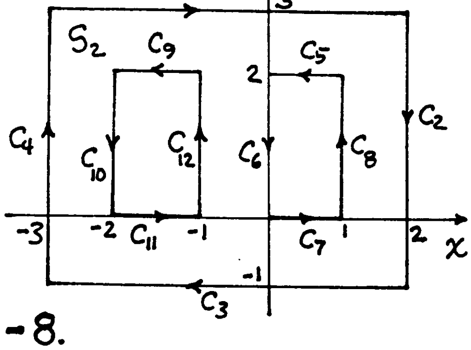
so the right-hand side $= \int_{C_2} + \int_{C_4} + \int_{C_8}$

$$= \int_2^{-2} 2^3 dx + \int_{-2}^{2} (-2)^3 dx + \int_{-1}^{0} (1)^3 dx = -32 - 32 + 1 = -63.$$

Meanwhile, the surface integral gives

$$\int_S \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \iint_S \hat{k}\cdot(-3y^2\hat{k})\,dx\,dy = \int_{-2}^{2}\int_{-2}^{2} -3y^2\,dx\,dy - \int_{0}^{1}\int_{-1}^{0} -3y^2\,dx\,dy = -64 + 1 = -63. ✓$$

(b) With $\hat{\underset{\sim}{n}} = -\hat{k}$, $\int_{S_2} \hat{\underset{\sim}{n}}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{v}\,dA = \int_{C_1}\underset{\sim}{v}\cdot d\underset{\sim}{R} + \cdots + \int_{C_{12}}\underset{\sim}{v}\cdot d\underset{\sim}{R}$



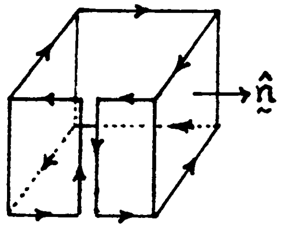
Since $\int \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int xy\,dx$, $\int_{C_2} = \int_{C_4} = \int_{C_6} = \int_{C_7} = \int_{C_8}$ $= \int_{C_{10}} = \int_{C_{11}} = \int_{C_{12}} = 0$, so the RHS is

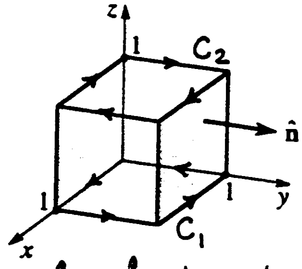
$$\text{RHS} = \int_{C_1} xy\,dx + \int_{C_3} xy\,dx + \int_{C_5} xy\,dx + \int_{C_9} xy\,dx$$
$$= \int_{-3}^{2} 3x\,dx + \int_{2}^{-3} -1x\,dx + \int_{1}^{0} 2x\,dx + \int_{-1}^{-2} 2x\,dx = -8.$$

$$\text{LHS} = \int_{S_2} \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{v}\, dA = \int_{S_2} -\hat{k} \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 0 & 0 \end{vmatrix} dA = \int_{S_2} x\, dA$$

$$= \int_{-1}^{3}\int_{-3}^{2} x\, dx\, dy - \int_{0}^{2}\int_{0}^{1} x\, dx\, dy - \int_{0}^{2}\int_{-2}^{-1} x\, dx\, dy = -8. \checkmark$$

(c) Slitting the surface gives: The adjacent slit edges give contributions that cancel, so we can rejoin the two edges and omit those two contours.





The result is the arrangement shown above right, where the closed rectangular edge curves at bottom ($z=0$) and top ($z=1$) are C_1 and C_2, respectively. Thus,

$$\int_S \hat{\underset{\sim}{n}} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{v}\, dA = \int_{C_1} \underset{\sim}{v} \cdot d\underset{\sim}{R} + \int_{C_2} \underset{\sim}{v} \cdot d\underset{\sim}{R}$$

$$\text{LHS} = \int_S \hat{\underset{\sim}{n}} \cdot \{ [6x \sin(x^2+z^2) + x^3 y]\hat{\jmath} - x^3(1+z)\hat{k} \}\, dA$$

$$= \int_0^1\int_0^1 \hat{\jmath} \cdot \{ \quad \text{"} \quad \}\Big|_{y=1} dx\,dz$$

$$+ \int_0^1\int_0^1 (-\hat{\jmath}) \cdot \{ \quad \text{"} \quad \}\Big|_{y=0} dx\,dz$$

$$+ \int_0^1\int_0^1 \hat{\imath} \cdot \{ \quad \text{"} \quad \}\Big|_{x=1} dy\,dz$$

$$+ \int_0^1\int_0^1 (-\hat{\imath}) \cdot \{ \quad \text{"} \quad \}\Big|_{x=0} dy\,dz$$

$$= \int_0^1\int_0^1 [\cancel{6x \sin(x^2+z^2)} + x^3]\, dx\,dz - \int_0^1\int_0^1 \cancel{6x \sin(x^2+z^2)}\, dx\,dz + 0 + 0$$

$$= \int_0^1\int_0^1 x^3\, dx\,dz = 1/4$$

$$\text{RHS} = \int_{C_1} [x^3 y(1+\cancelto{0}{z})\, dx + 3\cos(x^2+\cancelto{0}{z^2})\, \cancelto{0}{dz}] + \int_{C_2} [x^3 y(1+\cancelto{1}{z})\, dx + 3\cos(x^2+\cancelto{1}{z^2})\, \cancelto{0}{dz}]$$

$$= \int_0^1 \cancel{x^3 y}\Big|_{y=0} dx + \int_1^0 x^3 y\Big|_{y=1} dx + \int_1^0 \cancel{2x^3 y}\Big|_{y=0} dx + \int_0^1 2x^3 y\Big|_{y=1} dx$$

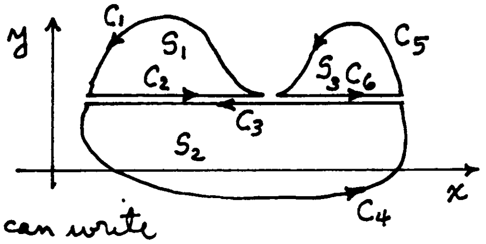
$$= -\tfrac{1}{4} + \tfrac{2}{4} = 1/4. \checkmark$$

12. (a) $\int_S \frac{\partial Q}{\partial x}\, dA = \int_{y_B}^{y_T}\int_{x_L(y)}^{x_R(y)} \frac{\partial Q}{\partial x}\, dx\,dy = \int_{y_B}^{y_T} [Q(x_R(y), y) - Q(x_L(y), y)]\, dy$

$= \int_{y_B}^{y_T} Q(x_R(y), y)\, dy + \int_{y_T}^{y_B} Q(x_L(y), y)\, dy = \oint_C Q\, dy$, where C is counterclockwise.

(b) Same idea as in (a)

(c) Partition S as shown: Then each subregion, S_1, S_2, S_3, is convex in x and y so, as proved in (a) and (b), we can write



$$\int_{S_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \int_{C_1}(P\,dx+Q\,dy) + \int_{C_2}(P\,dx+Q\,dy)$$

$$\int_{S_2}(\quad '' \quad)\,dA = \int_{C_3}(P\,dx+Q\,dy) + \int_{C_4}(P\,dx+Q\,dy)$$

$$\int_{S_3}(\quad '' \quad)\,dA = \int_{C_5}(P\,dx+Q\,dy) + \int_{C_6}(P\,dx+Q\,dy)$$

Adding these equations and cancelling $\int_{C_2}+\int_{C_6}$ with $\int_{C_3}$ gives

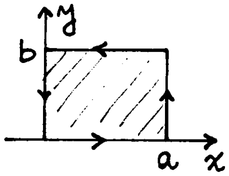
$$\int_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx = \int_{C_1}+\int_{C_4}+\int_{C_5}$$

$$= \oint_C(P\,dx+Q\,dy). \checkmark$$

13. Using Green's theorem, with $P=-y$ and $Q=x$,

$$\oint_C -y\,dx+x\,dy = \int_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \int_S(1+1)\,dA = 2A. \checkmark$$

14. (a)

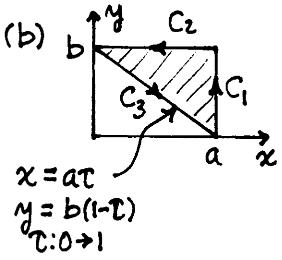


$$\text{Area} = \frac{1}{2}\oint_C x\,dy-y\,dx = \frac{1}{2}\Big\{\overset{\text{bottom}}{0} + \overset{\text{right}}{\int_0^b a\,dy} - \overset{\text{top}}{\int_a^0 b\,dx} + \overset{\text{left}}{0}\Big\}$$

$$= \frac{1}{2}2ab$$

$$= ab. \checkmark$$

(b)



$$\text{Area} = \frac{1}{2}\int_{C_1}x\,dy-y\cancelto{0}{dx} + \frac{1}{2}\int_{C_2}x\cancelto{0}{dy}-y\,dx + \frac{1}{2}\int_{C_3}x\,dy-y\,dx$$

$$= \frac{1}{2}\int_0^b a\,dy \quad + \frac{1}{2}\int_a^0 -b\,dx + \frac{1}{2}\int_0^1[a\tau(-b)-b(1-\tau)a]\,d\tau$$

$$= \frac{1}{2}ab + \cancel{\frac{1}{2}ab} + \frac{1}{2}(\cancel{-ab}) = \frac{1}{2}ab. \checkmark$$

(c) $\text{Area} = \frac{1}{2}\oint_C x\,dy-y\,dx = \frac{1}{2}\int_0^{2\pi}[(3\cos\tau)(3\cos\tau)-(3\sin\tau)(-3\sin\tau)]\,d\tau = \frac{9}{2}2\pi = 9\pi. \checkmark$

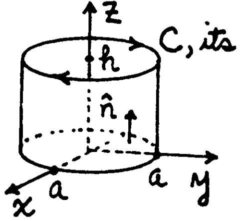
15. $\oint_C(P\,dx+Q\,dy) = \oint_C \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_0^{2\pi}(\sin^2\tau+\cos^2\tau)\,d\tau = 2\pi$, using $\begin{matrix}x=\cos\tau\\ y=\sin\tau\\ \tau: 0\to 2\pi\end{matrix}$.

However, $\int_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \int_S\left(\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}\right)dA = \int_S 0\,dA = 0.$

The next-to-last equality is not so clear because the integrand is indeterminate at the origin: 0/0. In any case, the point is that $\underset{\sim}{v} = P(x,y)\hat{\imath}+Q(x,y)\hat{\jmath}$ is not C^1, as called for in Theorem 16.9.2, so we have no guarantee that (33) will hold. For recall that for $\mathbf{v}$ to be C^1 we need P, Q, P_x, P_y, Q_x, Q_y to all be continuous in the region. In fact, none of these are continuous at the origin. For example, consider P:

Approaching the origin along $y = mx$, $P = -mx/(x^2 + m^2x^2)$, which does not approach a limit that is both finite and unique as $x \to 0$. Similarly for Q, P_x, P_y, Q_x, Q_y.

16. (a)

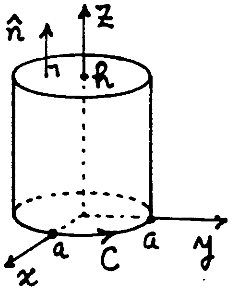


C, its direction consistent with $\hat{n} = \hat{k}$ on the base $z = 0$

$$\oint_C \mathbf{v} \cdot d\mathbf{R} = \oint_C \omega r \hat{e}_\theta \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z)$$
$$= \oint_C \omega r^2 d\theta \Big|_{r=a} = 2\pi\omega a^2.$$

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, dA = \int_S \hat{\mathbf{n}} \cdot 2\omega \hat{\mathbf{e}}_z\, dA = \int_{side} \cancelto{0}{-\hat{e}_r \cdot 2\omega \hat{e}_z} \Big|_{r=a} dA$$
$$+ \int_{base} \hat{e}_z \cdot 2\omega \hat{e}_z \Big|_{z=0} dA$$
$$= 0 + 2\omega \int_{base} dA = 2\pi\omega a^2. \checkmark$$

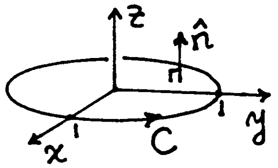
(b)



$$\oint_C \mathbf{v} \cdot d\mathbf{R} = \oint_C (3r\hat{e}_r - r\cancelto{0}{z^2}\hat{e}_\theta + 5r^2\hat{e}_z) \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + \cancelto{0}{dz}\,\hat{e}_z)$$
$$= \oint_C 3r\,dr = 3r^2/2 \Big|_a^a = 0.$$

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, dA = \int_{Top} \hat{\mathbf{e}}_z \cdot \nabla \times \mathbf{v}\, dA + \int_{side} \hat{\mathbf{e}}_r \cdot \nabla \times \mathbf{v}\, dA$$
$$= \int_{top} \frac{1}{r}\left[\frac{\partial}{\partial r}(-r^2z^2) - \frac{\partial}{\partial \theta}(3r)\right] dA$$
$$+ \int_{Side} \left[\frac{1}{r}\frac{\partial}{\partial \theta}(5r^2) - \frac{\partial}{\partial z}(-rz^2)\right] dA$$
$$= -2\int_0^{2\pi}\int_0^a z^2 r\,dr\,d\theta \Big|_{z=h} + 2\int_0^h \int_0^{2\pi} z r\, r\,d\theta\,dz \Big|_{r=a}$$
$$= -2h^2 \frac{a^2}{2} 2\pi + 2\frac{h^2}{2} a^2 2\pi = 0. \checkmark$$

(c)

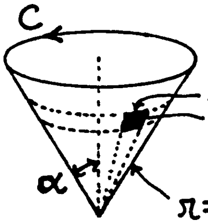


$$\oint_C \mathbf{v} \cdot d\mathbf{R} = \oint_C (50 - 2y)\,dx = \cancelto{0}{50x}\Big|_1^1 - 2\oint_C y\,dx.$$ Setting

$x = \cos\tau, y = \sin\tau$ $(\tau: 0 \to 2\pi)$, this $= -2\int_0^{2\pi} \sin\tau(-\sin\tau)\,d\tau$
$= 2\pi.$

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, dA = \int_S \hat{k} \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 50-2y & 0 & 0 \end{vmatrix} dA = 2\int_S dA = 2\pi(1)^2 = 2\pi. \checkmark$$

(d)



$$\left.\begin{matrix} r\,d\theta = z\tan\alpha\,d\theta \\ \dfrac{dz}{\cos\alpha} \end{matrix}\right\} \text{so } dA = \frac{\tan\alpha}{\cos\alpha} z\,dz\,d\theta$$

$r = z\tan\alpha$ so $\mathbf{n} = \nabla(r - z\tan\alpha) = \hat{e}_r - \tan\alpha\,\hat{e}_z,$

but (due to the orientation of C) we need to reverse the direction of $\mathbf{n}$ (and normalize it), so $\hat{\mathbf{n}} = -\cos\alpha\,\hat{e}_r + \sin\alpha\,\hat{e}_z$.

Then,

$$\oint_C \mathbf{v}\cdot d\mathbf{R} = \oint_C \omega r\,\hat{e}_\theta \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z) = \int_0^{2\pi} \omega r^2\,d\theta\Big|_{r=h\tan\alpha} = 2\pi\omega h^2\tan^2\alpha$$

$$\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA = \int_S (-\cos\alpha\,\hat{e}_r + \sin\alpha\,\hat{e}_z)\cdot\frac{1}{r}\frac{\partial}{\partial r}(\omega r^2)\,\hat{e}_z\,dA\Big|_{r=z\tan\alpha}$$

$$= 2\omega\sin\alpha\int_0^h\int_0^{2\pi} z\tan\alpha\,d\theta\,\frac{dz}{\cos\alpha} = 2\pi\omega h^2\tan^2\alpha.\ \checkmark$$

NOTE: We derived $dA = \frac{\tan\alpha}{\cos\alpha}z\,dz\,d\theta$ from a simple sketch, but we could have used the $dA = \sqrt{EG-F^2}\,du\,dv$ formula from Chapter 15.

(e) $$\oint_C \mathbf{v}\cdot d\mathbf{R} = \oint_C (3r\,\hat{e}_r - r^4\,\hat{e}_\theta)\cdot(\underset{0 \text{ on } C}{\underbrace{dr}}\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + \underset{0 \text{ on } C}{\underbrace{dz}}\,\hat{e}_z) = \int_0^{2\pi} -r^5\,d\theta\Big|_{r=h\tan\alpha}$$

$$= -2\pi h^5\tan^5\alpha$$

$\hat{n}$ and dA are the same as in (d). Using those expressions,

$$\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA = \int_0^{2\pi}\int_0^h (-\cos\alpha\,\hat{e}_r + \sin\alpha\,\hat{e}_z)\cdot(-5r^3\,\hat{e}_z)\,\frac{\tan\alpha}{\cos\alpha}\,z\,dz\,d\theta\Big|_{r=z\tan\alpha} = -2\pi h^5\tan^5\alpha\ \checkmark$$

(f) This time we are given $\mathbf{v} = \rho^2\sin\phi\,\hat{e}_\theta$ in spherical coordinates so $\hat{n} = -\hat{\mathbf{e}}_\phi$ and $dA = \rho\sin\phi\,d\rho\,d\theta = \rho\sin\alpha\,d\rho\,d\theta$, and $\nabla\times\mathbf{v} = \frac{1}{\rho\sin\alpha}\frac{\partial}{\partial\phi}(\rho^2\sin^2\phi)\,\hat{e}_\rho - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\sin\phi)\,\hat{e}_\phi$. Thus,

$$\int_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}\,dA = \int_0^{2\pi}\int^{h/\cos\alpha} (-\hat{e}_\phi)\cdot[(\text{etc})\,\hat{e}_\rho - 3\rho\sin\alpha\,\hat{e}_\phi]\,\rho\sin\alpha\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\int_0^{h/\cos\alpha} 3\rho^2\sin^2\alpha\,d\rho\,d\theta = 2\pi h^3\,\frac{\tan^2\alpha}{\cos\alpha}$$

$$\oint_C \mathbf{v}\cdot d\mathbf{R} = \oint_C (\rho^2\sin\phi\,\hat{e}_\theta)\cdot(d\rho\,\hat{e}_\rho + \rho\,d\phi\,\hat{e}_\phi + \rho\sin\phi\,d\theta\,\hat{e}_\theta)$$

$$= \int_0^{2\pi}\rho^3\sin^2\phi\,d\theta\Big|_{\rho=h/\cos\alpha,\ \phi=\alpha} = 2\pi h^3\,\frac{\tan^2\alpha}{\cos\alpha}\ \checkmark$$

Section 16.10

1\. (b) $$\nabla\times\mathbf{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & -3y & z^3 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \mathbf{0}$$ everywhere; also, $\mathbf{v}$ is C^1 everywhere.

(c) $\nabla\times\mathbf{v} = \text{etc.} = \mathbf{0}$ everywhere except for $y=0$. $\mathbf{v}$ is C^1 and irrotational in the half-space $y>0$ and also in the half-space $y<0$.

(d) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} = \frac{-2yz+2yz}{(\ )^3}\hat{\imath} - \frac{-2xz+2xz}{(\ )^3}\hat{\jmath} + \frac{-2xy+2xy}{(\ )^3}\hat{k}$

$= \underset{\sim}{0}$, and $\underset{\sim}{v}$ is C', everywhere except at $x=y=z=0$.

(f) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & -\frac{z}{y^2+z^2} & \frac{y}{y^2+z^2} \end{vmatrix} = \frac{z^2-y^2+y^2-z^2}{(y^2+z^2)^2}\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$ and $\underset{\sim}{v}$ is C'

everywhere except where $y^2+z^2=0$; i.e., everywhere except all along the x axis.

(g) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & y & -x \end{vmatrix} = 0\hat{\imath} - (-1-1)\hat{\jmath} + 0\hat{k} = 2\hat{\jmath}$ so although $\underset{\sim}{v}$ is C' everywhere it is irrotational nowhere.

2.(b) We don't need to check first to see if $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{0}$; if we can find a Φ such that $\underset{\sim}{v} = \underset{\sim}{\nabla}\Phi$, then $\underset{\sim}{v}$ is irrotational and Φ is its potential.

$\underset{\sim}{v} = x^3\hat{\imath} - y\hat{\jmath} + \sin z\,\hat{k} = \underset{\sim}{\nabla}\Phi = \partial\Phi/\partial x\,\hat{\imath} + \partial\Phi/\partial y\,\hat{\jmath} + \partial\Phi/\partial z\,\hat{k}$, so

$\partial\Phi/\partial x = x^3 \Rightarrow \Phi = x^4/4 + A(y,z)$

$\partial\Phi/\partial y = -y = 0 + A_y(y,z)$, $A(y,z) = -y^2/2 + B(z)$, $\Phi = x^4/4 - y^2/2 + B(z)$

$\partial\Phi/\partial z = \sin z = 0 - 0 + B'(z)$, $B(z) = -\cos z + C$, $\Phi = x^4/4 - y^2/2 - \cos z + C$, everywhere.

(c) $\partial\Phi/\partial x = z \Rightarrow \Phi = zx + A(y,z)$

$\partial\Phi/\partial y = y^3 = 0 + A_y(y,z)$, $A(y,z) = y^4/4 + B(z)$, $\Phi = zx + y^4/4 + B(z)$

$\partial\Phi/\partial z = x = x + B'(z)$, $B(z) = C$, $\Phi = zx + y^4/4 + C$, everywhere.

(d) $\partial\Phi/\partial x = x(2z+y)$, $\Phi = x^2z + x^2y/2 + A(y,z)$

$\partial\Phi/\partial y = x^2/2 = x^2/2 + A_y(y,z)$, $A(y,z) = 0 + B(z)$, $\Phi = x^2z + x^2y/2 + B(z)$

$\partial\Phi/\partial z = x^2 = x^2 + B'(z)$, $B(z) = C$, $\Phi = x^2z + x^2y/2 + C$, everywhere.

(e) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath} - 0\hat{\jmath} + (\frac{\partial}{\partial x}e^{-|x|} - e^{-x})\hat{k}$.

For $x>0$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\partial/\partial x\, e^{-x} - e^{-x})\hat{k} = -2e^{-x}\hat{k} \neq \underset{\sim}{0}$;

for $x<0$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\partial/\partial x\, e^{x} - e^{-x})\hat{k} = (e^{x} - e^{-x})\hat{k} \neq \underset{\sim}{0}$;

for $x=0$, $\underset{\sim}{\nabla}\times\underset{\sim}{v}$ doesn't exist because $\partial/\partial x\, e^{-|x|}$ doesn't exist at $x=0$.

Thus, $\underset{\sim}{v}$ is irrotational nowhere.

(f) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2ye^{x} & 2e^{-|x|} & -3 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + (\frac{\partial}{\partial x}(2e^{-|x|}) - 2e^{x})\hat{k}$

For $x>0$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\frac{\partial}{\partial x}(2e^{-x}) - 2e^{x})\hat{k} = (-2e^{-x} - 2e^{x})\hat{k} \neq \underset{\sim}{0}$

For $x<0$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\frac{\partial}{\partial x}(2e^{x}) - 2e^{x})\hat{k} = \underset{\sim}{0}$, so $\underset{\sim}{v}$ is irrotational only in the half-space $x<0$. Let us find its potential Φ there.

$\partial\Phi/\partial x = 2ye^{x} \Rightarrow \Phi = 2ye^{x} + A(y,z)$

$\partial\Phi/\partial y = 2e^{-|x|} = 2e^{x}$ (in $x<0$) $= 2e^{x} + A_y(y,z)$, $A(y,z) = 0 + B(z)$, $\Phi = 2ye^{x} + B(z)$

$\partial\Phi/\partial z = -3 = 0 + B'(z)$, $B(z) = -3z + C$, $\Phi = 2ye^{x} - 3z + C$ in $x<0$ half-space

(g) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz & 3y & x^2 \end{vmatrix} = 0\hat{\imath} - (2x-2x)\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$ everywhere.

$\partial\Phi/\partial x = 2xz \Rightarrow \Phi = x^2z + A(y,z)$

$\partial\Phi/\partial y = 3y = 0 + A_y(y,z)$, $A(y,z) = 3y^2/2 + B(z)$, $\Phi = x^2z + 3y^2/2 + B(z)$

$\partial\Phi/\partial z = x^2 = x^2 + B'(z)$, $B(z) = C$, $\Phi = x^2z + 3y^2/2 + C$ everywhere.

(h) $\partial\Phi/\partial x = z \Rightarrow \Phi = zx + A(y,z)$

$\partial\Phi/\partial y = z^2 = 0 + A_y(y,z)$, $A = z^2y$, $\Phi = zx + z^2y + B(z)$

$\partial\Phi/\partial z = x + 2yz = x + 2yz + B'(z)$, $B(z) = C$, $\Phi = zx + z^2y + C$ everywhere

(i) $\partial\Phi/\partial x = 0 \Rightarrow \Phi = A(y,z)$

$\partial\Phi/\partial y = 2ze^{2y} = A_y(y,z)$, $A(y,z) = ze^{2y} + B(z)$, $\Phi = ze^{2y} + B(z)$

$\partial\Phi/\partial z = e^{2y} = e^{2y} + B'(z)$, $B(z) = C$, $\Phi = ze^{2y} + C$ everywhere

(j) $\partial\Phi/\partial x = 2xy \Rightarrow \Phi = x^2y + A(y,z)$

$\partial\Phi/\partial y = 2ze^{2y} + x^2 = x^2 + A_y(y,z)$, $A(y,z) = ze^{2y} + B(z)$, $\Phi = x^2y + ze^{2y} + B(z)$

$\partial\Phi/\partial z = e^{2y} = e^{2y} + B'(z)$, $B(z) = C$, $\Phi = x^2y + ze^{2y} + C$ everywhere

(k) $\partial\Phi/\partial x = xy \Rightarrow \Phi = x^2y/2 + A(y,z)$

$\partial\Phi/\partial y = 0 = x^2/2 + A_y(y,z)$ which is impossible. Indeed,

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 0 & -y \end{vmatrix} = -\hat{\imath} - 0\hat{\jmath} - x\hat{k} \neq \underset{\sim}{0}$ so $\underset{\sim}{v}$ is not irrotational anywhere.

(l) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = xz\hat{\imath} - yz\hat{\jmath} \neq \underset{\sim}{0}$ so $\underset{\sim}{v}$ is not irrotational anywhere. (actually, $\underset{\sim}{\nabla}\times\underset{\sim}{v}$ is $\underset{\sim}{0}$ on the planes $y=x$ and $z=0$, but a plane in 3-space is not an open region — i.e., a domain.)

(m) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath} - (2x-2x)\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$ so $\underset{\sim}{v}$ is irrotational everywhere.

$\partial\Phi/\partial x = 2xz \Rightarrow \Phi = x^2z + A(y,z)$

$\partial\Phi/\partial y = 0 = A_y(y,z)$, $A(y,z) = B(z)$, $\Phi = x^2z + B(z)$

$\partial\Phi/\partial z = x^2 = x^2 + B'(z)$, $B(z) = C$, $\Phi = x^2z + C$ everywhere.

(n) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (2y+2-2y-2)\hat{\imath} - (x-x)\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$ everywhere

$\partial\Phi/\partial x = xz \Rightarrow \Phi = x^2z/2 + A(y,z)$

$\partial\Phi/\partial y = 2(y+1)z = A_y(y,z)$, $A(y,z) = y^2z + 2yz + B(z)$, $\Phi = x^2z/2 + y^2z + 2yz + B(z)$

$\partial\Phi/\partial z = \frac{x^2}{2} + y^2 + 2y = \frac{x^2}{2} + y^2 + 2y + B'(z)$, $B(z) = C$, $\Phi = x^2z/2 + y^2z + 2yz + C$ everywhere

(o) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (x-x)\hat{\imath} - (y-y)\hat{\jmath} + (z-z)\hat{k} = \underset{\sim}{0}$ everywhere

$\partial\Phi/\partial x = yz \Rightarrow \Phi = xyz + A(y,z)$,

$\partial\Phi/\partial y = xz = xz + A_y(y,z)$, $A(y,z) = B(z)$, $\Phi = xyz + B(z)$

$\partial\Phi/\partial z = xy = xy + B'(z)$, $B(z) = C$, $\Phi = xyz + C$ everywhere.

3. (a) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (f_y+1)\hat{\imath} - (f_x-0)\hat{\jmath} + (0-x)\hat{k}$ cannot be made to vanish, for any choice of f, because of the $-x\hat{k}$ term.

(b) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (f_y-0)\hat{\imath} - (f_x-0)\hat{\jmath} + (2x-2x)\hat{k} = \underset{\sim}{0}$ if we chose $f(x,y,z) =$ any function of z only, say $f(z)$.

(c) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (g_y - 2xy)\hat{\imath} - (g_x - f_z)\hat{\jmath} + (2yz - f_y)\hat{k} = \underset{\sim}{0}$ if we choose f and g so that

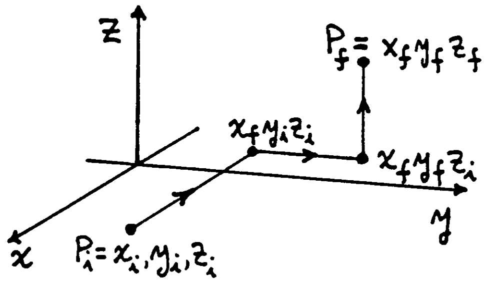
$$g_y = 2xy,\quad g_x = f_z,\quad f_y = 2yz.$$

There is an infinite set of f's and g's satisfying these relations – for example, $g = xy^2$ and $f = y^2 z$.

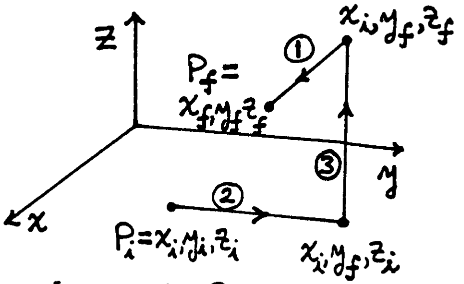
(d) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (g_y - f_z)\hat{\imath} - (g_x - 2z)\hat{\jmath} + (f_x - 0)\hat{k} = \underset{\sim}{0}$ if we chose f and g so that $g_y = f_z$, $g_x = 2z$ so $g = 2xz + A(y,z)$, $f_x = 0$ so $f = B(y,z)$.

There is an infinite set of f's and g's satisfying these relations – for example, $f = 0$, $g = 2xz$.

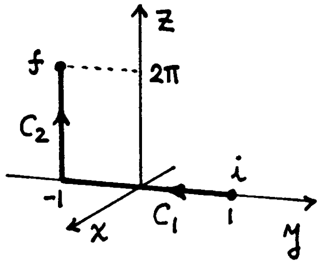
4. (a) Since the conditions of Theorem 16.10.1 are presumed to be met, any path can be taken from P_i to P_f. The question, therefore, is whether the integrals given amount to integration along some path from P_i to P_f. The answer is yes, with the three integrals corresponding to the segments labeled ①, ②, ③, respectively.



(b) Same idea as in (a), but this time the path sequence is as shown at the right.



(c) This time our answer is no, the formula is incorrect. Reasoning: the first integral must come first because it starts at x_i, y_i, z_i. That integral gets us to x_f, y_i, z_i so one of the other two integrals must then carry us from x_f, y_i, z_i to either x_f, y_f, z_i or x_f, y_i, z_f. Neither of those integrals fits that requirement since the second does not start at x_f, y_i, z_i, nor does the third.



5. (a) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath} - 0\hat{\jmath} + (1-1)\hat{k} = \underset{\sim}{0}$, so we can use path simplification or the potential method.

Path simplification: $\tau=0$ gives $(x,y,z)=(0,1,0)$ and $\tau=\pi$ gives $(x,y,z)=(0,-1,2\pi)$. Deform the path to be as shown at the right. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} y\,\overset{0}{\cancel{dx}} + x\,dy + \int_{C_2} y\,\overset{0}{\cancel{dx}} + x\,\overset{0}{\cancel{dy}} = \int_1^{-1} 0\,dy = 0$$

Potential method: Find $\Phi = xy + \overset{0}{\cancel{\text{const}}}$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = xy\Big|_{0,1,0}^{0,-1,2\pi} = 0-0 = 0. \checkmark$$

(b) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath} - 0\hat{\jmath} + (-6x^2\sin 2y + 6x^2\sin 2y)\hat{k} = \underset{\sim}{0}$. That is, it $= \underset{\sim}{0}$ everywhere, and hence within any domain within which we wish to deform the contour or find a scalar potential.

Path simplification: Use the same deformed path as in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} v_x\,\overset{0}{\cancel{dx}} + \underset{0}{\cancel{v_y}}\,dy + \int_{C_2} v_x\,\underset{0}{\cancel{dx}} + v_y\,\overset{0}{\cancel{dy}} = 0.$$

Potential method: Find $\Phi = x^3\cos 2y + \overset{0}{\cancel{\text{const}}}$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = x^3\cos 2y\Big|_{0,1,0}^{0,-1,2\pi} = 0-0 = 0. \checkmark$$

(c) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (10x^{3/2}e^{2y} - 10x^{3/2}e^{2y})\hat{k} = 0\hat{k} = \underset{\sim}{0}$.

Path simplification: Use the same deformed path as in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 5x^{3/2}e^{2y}\,\underset{0}{\cancel{dx}} + (\underline{5y^2} + 4\underset{0}{\cancel{x}}^{5/2}e^{2y})\,dy + \int_{C_2} (\quad)\,\underset{0}{\cancel{dx}} + (\quad)\,\underset{0}{\cancel{dy}}$$

$$= \int_1^{-1} 5y^2\,dy = -10/3.$$

Potential method: Find $\Phi = 2x^{5/2}e^{2y} + \frac{5}{3}y^3$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(2x^{5/2}e^{2y} + \frac{5}{3}y^3\right)\Big|_{(0,1,0)}^{(0,-1,2\pi)} = -10/3. \checkmark$$

(d) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (15y^2z^{3/2} - 15y^2z^{3/2})\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$.

Path simplification: Use the path in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 3\,\underset{0}{\cancel{dx}} + 6y^2\underset{0}{\cancel{z}}^{5/2}\,dy + (5y^3z^{3/2}+2)\,\underset{0}{\cancel{dz}}$$

$$+ \int_{C_2} 3\,\underset{0}{\cancel{dx}} + 6y^2z^{5/2}\,\underset{0}{\cancel{dy}} + (5y^3z^{3/2}+2)\,dz$$

$$= \int_0^{2\pi}\left[5(-1)^3z^{3/2}+2\right]dz = 4\pi - 2(2\pi)^{5/2}.$$

Potential method: Find $\Phi = 2y^3z^{5/2} + 2z + 3x$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = (2y^3z^{5/2} + 2z + 3x)\Big|_{(0,1,0)}^{(0,-1,2\pi)} = -2(2\pi)^{5/2} + 4\pi. \checkmark$$

(e) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\sin z - \sin z)\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$.

Path simplification: Use the same deformed path as in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} (y^3-\cos z)\,dy + (y\sin z+2)\,\underset{0}{\cancel{dz}} + \int_{C_2} (y^3-\cos z)\,\underset{0}{\cancel{dy}} + (y\sin z+2)\,dz$$

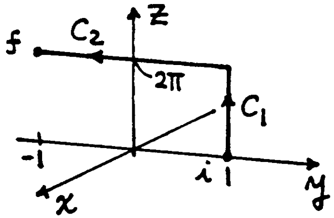
$$= \int_1^{-1} (y^3-1)\,dy + \int_0^{2\pi} [(-1)\sin z+2]\,dz = 2+4\pi$$

Potential method: Find $\Phi = -y\cos z + 2z + y^4/4$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(-y\cos z + 2z + y^4/4\right)\Big|_{(0,1,0)}^{(0,-1,2\pi)} = (1+4\pi+\tfrac{1}{4}) - (-1+0+\tfrac{1}{4}) = 2+4\pi. \checkmark$$

(f) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (-2y+2y)\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$.

Path simplification: Use the deformed path shown at the right, say. Then



$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 2x^2\,\overset{0}{\cancel{dx}} - 2yz\,\overset{0}{\cancel{dy}} - (y^2+3)\,dz$$
$$+ \int_{C_2} 2x^2\,\underset{0}{\cancel{dx}} - 2yz\,dy - (y^2+3)\,\underset{0}{\cancel{dz}}$$

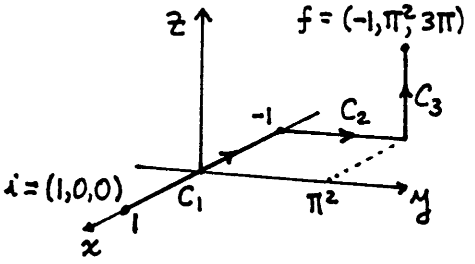
$$= \int_0^{2\pi} -(1+3)\,dz + \int_1^{-1} -2y(2\pi)\,dy = -8\pi.$$

Potential Method: Find $\Phi = \frac{2}{3}x^2 - y^2z - 3z$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(\tfrac{2}{3}x^2 - y^2z - 3z\right)\Big|_{(0,1,0)}^{(0,-1,2\pi)} = -2\pi - 6\pi = -8\pi. \checkmark$$

6. (a) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ so we can use path deformation <u>or</u> the potential method. This time the initial and final points are $(1,0,0)$ and $(-1,\pi^2,3\pi)$, respectively.

Path deformation: Use the path shown at the right, say. Then



$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} x\,dx + y\,\overset{0}{\cancel{dy}} + z\,\overset{0}{\cancel{dz}} + \int_{C_2} x\,\overset{0}{\cancel{dx}} + y\,dy + z\,\overset{0}{\cancel{dz}} + \int_{C_3} x\,\overset{0}{\cancel{dx}} + y\,\overset{0}{\cancel{dy}} + z\,dz$$

$$= \int_1^{-1} x\,dx + \int_0^{\pi^2} y\,dy + \int_0^{3\pi} z\,dz = \frac{x^2}{2}\Big|_1^{-1} + \frac{y^2}{2}\Big|_0^{\pi^2} + \frac{z^2}{2}\Big|_0^{3\pi} = \frac{\pi^2}{2}(9+\pi^2)$$

Potential method: Find $\Phi = (x^2+y^2+z^2)/2$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = (x^2+y^2+z^2)/2\Big|_{(1,0,0)}^{(-1,\pi^2,3\pi)} = \frac{\pi^2}{2}(9+\pi^2). \checkmark$$

(b) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (2z-2z)\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0}$.

Path deformation: Use the deformed path shown in (a). Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 8\,dx + (z^2-3y)\,\underset{0}{\cancel{dy}} + (2zy-\sqrt{z})\,\underset{0}{\cancel{dz}} + \int_{C_2} 8\,\underset{0}{\cancel{dx}} + (z^2-3y)\,dy + (2zy-\sqrt{z})\,\underset{0}{\cancel{dz}}$$

$$+\int_{C_3} 8\underset{0}{\cancel{dx}} + (z^2-3y)\underset{0}{\cancel{dy}} + (2zy-\sqrt{z})dz$$

$$= \int_1^{-1} 8\,dx + \int_0^{\pi^2} (0-3y)\,dy + \int_0^{3\pi} (2\pi^2 z-\sqrt{z})dz = -16+\frac{15}{2}\pi^4-\frac{2}{3}(3\pi)^{3/2}.$$

Potential method: Find $\Phi = 8x+z^2y-\frac{3y^2}{2}-\frac{2}{3}z^{3/2}$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(8x+z^2y-\frac{3y^2}{2}-\frac{2}{3}z^{3/2}\right)\Big|_{(1,0,0)}^{(-1,\pi^2,3\pi)} = -16+\frac{15}{2}\pi^4-\frac{2}{3}(3\pi)^{3/2}. \checkmark$$

(c) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{\imath}-0\hat{\jmath}+(-3\sqrt{y}+3\sqrt{y})\hat{k} = \underset{\sim}{0}.$

Path deformation: Use the deformed path shown in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1}(x^2-2y^{3/2})dx+(\,)\underset{0}{\cancel{dy}}+(\,)\underset{0}{\cancel{dz}}+\int_{C_2}(\,)\underset{0}{\cancel{dx}}-3x\sqrt{y}\,dy+(\,)\underset{0}{\cancel{dz}}+\int_{C_3}(\,)\underset{0}{\cancel{dx}}+(\,)\underset{0}{\cancel{dy}}+e^z dz$$

$$= \int_1^{-1}(x^2-0)dx - 3\int_0^{\pi^2}(-1)\sqrt{y}\,dy + \int_0^{3\pi} e^z dz = 2\pi^3+e^{3\pi}-5/3$$

Potential method: Find $\Phi = \frac{x^3}{3}-2xy^{3/2}+e^z$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(\frac{x^3}{3}-2xy^{3/2}+e^z\right)\Big|_{(1,0,0)}^{(-1,\pi^2,3\pi)} = -\frac{1}{3}+2\pi^3+e^{3\pi}-\frac{1}{3}-1 = 2\pi^3+e^{3\pi}-5/3 \checkmark$$

(d) $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (4xyz-4xyz)\hat{\imath}-(2y^2z-2y^2z)\hat{\jmath}+(2yz^2-2yz^2)\hat{k} = \underset{\sim}{0}.$

Path Deformation: Use the deformed path shown in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} y^2z^2dx+(\,)\underset{0}{\cancel{dy}}+(\,)\underset{0}{\cancel{dz}}+\int_{C_2}(\,)\underset{0}{\cancel{dx}}+2xyz^2dy+(\,)\underset{0}{\cancel{dz}}+\int_{C_3}(\,)\underset{0}{\cancel{dx}}+(\,)\underset{0}{\cancel{dy}}+2xy^2z\,dz$$

$$= \int_1^{-1} 0\,dx + \int_0^{\pi^2} 0\,dy + \int_0^{3\pi} 2(-1)(\pi^2)^2 z\,dz = -9\pi^6$$

Potential method: Find $\Phi = xy^2z^2$ so, by (11),

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = xy^2z^2\Big|_{(1,0,0)}^{(-1,\pi^2,3\pi)} = -9\pi^6-0 = -9\pi^6 \checkmark$$

7. If $\underset{\sim}{\nabla}\times\overbrace{(P\hat{\imath}+Q\hat{\jmath}+R\hat{k})}^{\underset{\sim}{v},\text{ say}} = \underset{\sim}{0}$ then $\underset{\sim}{v}$ is irrotational and there exists a scalar potential f, say, such that $\underset{\sim}{\nabla} f = \underset{\sim}{v}$. Then $Pdx+Qdy+Rdz = \frac{\partial f}{\partial x} i$

$$Pdx+Qdy+Rdz = \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz = df$$

is an exact differential. Conversely, suppose $Pdx+Qdy+Rdz$ is an exact differential, so there is an f such that

$$Pdx+Qdy+Rdz = df$$

and $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_C Pdx+Qdy+Rdz = \int_C df = f\Big|_{\text{initial}}^{\text{final}}$

is independent of path. Hence, (d) holds in Theorem 16.10.1 and hence (by the theorem) so does (b); i.e., $\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{\nabla}\times(P\hat{\imath}+Q\hat{\jmath}+R\hat{k}) = \underset{\sim}{0}$.

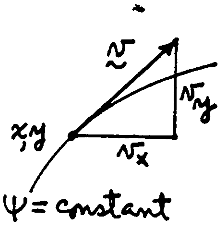
8. Q is the flow rate crossing OP from left to right, in meter^3/sec — per unit z depth.

(a) Consider the Ψ = constant curve through x,y. Along that curve

$$d\Psi = \Psi_x dx + \Psi_y dy = 0,$$

or, from (8.3), $\quad -v_y dx + v_x dy = 0,$

$$dy/dx = v_y/v_x$$



so that (see figure at right) at each point on the Ψ=constant curve the tangent line and the velocity vector $\underset{\sim}{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ are aligned. Thus, the Ψ= constant curve is a streamline.

(b) $\underset{\sim}{\nabla}\cdot(2y\hat{\imath}) = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(0) = 0+0=0.$

$\partial\Psi/\partial y = v_x = 2y \to \Psi = \int 2y\,\partial y = y^2 + A(x)$

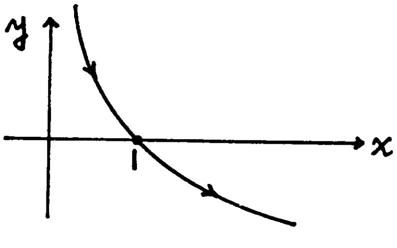
$\partial\Psi/\partial x = -v_y = 0 = 0 + A'(x)$ so $A(x)$ = constant and $\Psi(x,y) = y^2$. Thus, the streamlines are the $\Psi = y^2$ = constant curves, i.e., the y= constant curves.

(c) $\underset{\sim}{\nabla}\cdot(2xe^{2y}\hat{\imath} - e^{2y}\hat{\jmath}) = \frac{\partial}{\partial x}(2xe^{2y}) + \frac{\partial}{\partial y}(-e^{2y}) = 2e^{2y} - 2e^{2y} = 0.$

$\partial\Psi/\partial y = v_x = 2xe^{2y}$, $\Psi = \int 2xe^{2y}\,\partial y = xe^{2y} + A(x)$

$\partial\Psi/\partial x = -v_y = e^{2y} = e^{2y} + A'(x)$, $A'(x)=0$, $A(x) = \overset{0,\text{ say}}{\text{constant}}$, so

$\Psi(x,y) = xe^{2y}$



$\Psi(1,0) = 1e^0 = 1$, so the streamline through $(1,0)$ is given by

$$xe^{2y} = 1.$$

As $x \to 0+$, $y = -\frac{1}{2}\ln x \to +\infty$; as $x\to\infty$, $y \to -\infty$.

$y' = -\frac{1}{2x} < 0$ for $x > 0$, and $\to 0$ as $x \to \infty$.

$y'' = \frac{1}{2x^2} > 0$ for $x>0$, so the curvature is positive:  ; hence, the streamline is somewhat as we have sketched it, above.

(d)

$$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \Omega(x,y)\hat{k} \text{ says } \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x(x,y) & v_y(x,y) & 0 \end{vmatrix} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k} = \Omega\hat{k}$$

or,

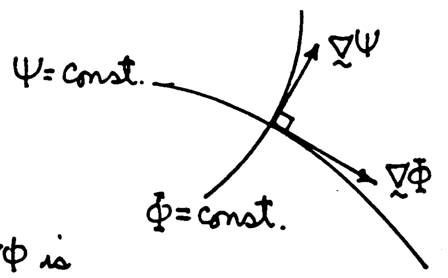
$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \Omega,$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial\Psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\Psi}{\partial y}\right) = \Omega,$$

$$\nabla^2\Psi = -\Omega(x,y).$$

(e) $\underset{\sim}{\nabla}\Psi\cdot\underset{\sim}{\nabla}\Phi = \Psi_x\Phi_x + \Psi_y\Phi_y$

$= (-v_y)(v_x) + (v_x)(v_y) = 0,$



so $\underset{\sim}{\nabla}\Psi$ is perpendicular to $\underset{\sim}{\nabla}\Phi$ (wherever they are not $\underset{\sim}{0}$). But $\underset{\sim}{\nabla}\Psi$ is perpendicular to the Ψ=const. curve and $\underset{\sim}{\nabla}\Phi$ is perpendicular to the Φ=const. curve, so it follows that the Ψ= const.

and ϕ=const. curves are perpendicular to each other at each point in the field at which $\underset{\sim}{\nabla}\Psi$ and $\underset{\sim}{\nabla}\phi$ are nonzero.

9. $dq = p(v,T)\,dv + C_v(T)\,dT$, where $p(v,T) = RT/v$.
This expression involves only two independent variables, v and T, whereas to do curls and cross products we need 3-space. Thus, let us add a third variable "z" so that we can construct a Cartesian v, T, z space, with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. Thus, let us re-express dq as

$$dq = p(v,T)\,dv + C_v(T)\,dT + 0\,dz. \qquad ♀$$

Then

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial v & \partial/\partial T & \partial/\partial z \\ p(v,T) & C_v(T) & 0 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + \left(0 - \frac{\partial p}{\partial T}\right)\hat{k} = -\frac{R}{v}\hat{k} \neq \underset{\sim}{0},$$

so ♀ is not an exact differential. However, if we multiply ♀ through by $1/T$ as

$$\frac{dq}{T} = \frac{p(v,T)}{T}\,dv + \frac{C_v(T)}{T}\,dT + 0\,dz = \frac{R}{v}\,dv + \frac{C_v(T)}{T}\,dT + 0\,dz \qquad ☆$$

then

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial v & \partial/\partial T & \partial/\partial z \\ R/v & C_v(T)/T & 0 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \underset{\sim}{0},$$

so ☆ is an exact differential, say ds. NOTE the importance of adding the z variable; the nature of z is immaterial, it merely serves to render our space 3-dimensional.

10. Equating $\hat{\imath}, \hat{\jmath}, \hat{k}$ components of $\underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{w}$ gives

$$\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} = v_x, \quad \frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} = v_y, \quad \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = v_z. \qquad (1a,b,c)$$

There is enough leeway in these to set one component of $\underset{\sim}{w}$ to zero, say $w_x = 0$, so the latter two equations become

$$\frac{\partial w_y}{\partial x} = v_z \quad \text{and} \quad \frac{\partial w_z}{\partial x} = -v_y, \qquad (2a,b)$$

integration of which gives $w_y = \int_{x_0}^{x} v_z(\xi,y,z)\,\partial\xi + A(y,z)$, (3a)

$$w_z = -\int_{x_0}^{x} v_y(\xi,y,z)\,\partial\xi + B(y,z). \qquad (3b)$$

If we now put (3a) and (3b) into (1a) we will have one equation on the two functions A, B. Thus, we can probably survive setting one of them equal to zero, let's say $A(y,z)=0$. Then (1a) gives this equation on $B(y,z)$:

$$-\frac{\partial}{\partial y}\int_{x_0}^{x} v_y(\xi,y,z)\,\partial\xi + \frac{\partial B}{\partial y} - \frac{\partial}{\partial z}\int_{x_0}^{x} v_z(\xi,y,z)\,\partial\xi = v_x(x,y,z). \qquad (4)$$

Since $\underset{\sim}{v}$ is C^1 by assumption we can use the Leibniz rule (Theorem 13.8.1) to re-express (4) as

$$-\int_{x_0}^{x}\left[\frac{\partial v_y}{\partial y}(\xi,y,z)+\frac{\partial v_z}{\partial z}(\xi,y,z)\right]d\xi+\frac{\partial B}{\partial y}=v_x(x,y,z). \qquad (5)$$

Since $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\partial v_x/\partial x+\partial v_y/\partial y+\partial v_z/\partial z=0$ by assumption, (5) becomes

$$\int_{x_0}^{x}\frac{\partial v_x}{\partial \xi}(\xi,y,z)\,d\xi+\frac{\partial B}{\partial y}=v_x(x,y,z) \qquad (6)$$

or,

$$\cancel{v_x(x,y,z)}-v_x(x_0,y,z)+\frac{\partial B}{\partial y}=\cancel{v_x(x,y,z)}, \qquad (7)$$

so

$$B(y,z)=\int_{y_0}^{y}v_x(x_0,\eta,z)\,\partial\eta \qquad (8)$$

where y_0 is arbitrary such that $y_1\le y_0\le y_2$. Thus, we have shown that a suitable $\underset{\sim}{w}$ is

$$\underset{\sim}{w}(x,y,z)=\underset{\substack{\uparrow\\ w_x}}{0}\hat{\imath}+\underbrace{\left[\int_{x_0}^{x}v_z(\xi,y,z)\,\partial\xi\right]}_{w_y}\hat{\jmath}+\underbrace{\left[\int_{y_0}^{y}v_x(x_0,\eta,z)\,\partial\eta-\int_{x_0}^{x}v_y(\xi,y,z)\,\partial\xi\right]}_{w_z}\hat{k}, \qquad (9)$$

to which we can add the gradient of an arbitrary C^2 function f since $\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}f=\underset{\sim}{0}$ by equation (13) of Section 16.6, so $\underset{\sim}{v}=\underset{\sim}{\nabla}\times(\underset{\sim}{w}+\underset{\sim}{\nabla}f)=\underset{\sim}{\nabla}\times\underset{\sim}{w}+\underset{\sim}{0}$ $=\underset{\sim}{\nabla}\times\underset{\sim}{w}$. Although we don't <u>need</u> to, let us <u>verify</u> (9) using the Leibniz rule:

$$\begin{aligned}
\underset{\sim}{\nabla}\times\underset{\sim}{w}&=\left(\frac{\partial}{\partial y}w_z-\frac{\partial}{\partial z}w_y\right)\hat{\imath}-\left(\frac{\partial}{\partial x}w_z-\frac{\partial}{\partial z}w_x\right)\hat{\jmath}+\left(\frac{\partial}{\partial x}w_y-\frac{\partial w_x}{\partial y}\right)\hat{k}\\
&=\left(v_x(x_0,y,z)-\int_{x_0}^{x}\frac{\partial v_y}{\partial y}(\xi,y,z)\,\partial\xi-\int_{x_0}^{x}\frac{\partial v_z}{\partial z}(\xi,y,z)\,\partial\xi\right)\hat{\imath}\\
&\quad-\left(0-v_y(x,y,z)\right)\hat{\jmath}+\left(v_z(x,y,z)\right)\hat{k}\\
&=\left(v_x(x_0,y,z)+\int_{x_0}^{x}\frac{\partial v_x}{\partial \xi}(\xi,y,z)\,\partial\xi\right)\hat{\imath}+v_y(x,y,z)\hat{\jmath}+v_z(x,y,z)\hat{k}\\
&=\cancel{v_x(x_0,y,z)}+v_x(x,y,z)-\cancel{v_x(x_0,y,z)}+v_y(x,y,z)\hat{\jmath}+v_z(x,y,z)\hat{k}=\underset{\sim}{v}(x,y,z).\ \checkmark
\end{aligned}$$

11. (a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\partial(a)/\partial x+\partial(b)/\partial y+\partial(c)/\partial z=0+0+0=0.$ ✓

With $x_0=y_0=0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z)=\int_0^x c\,\partial\xi\ \hat{\jmath}+\left(\int_0^y a\,\partial\eta-\int_0^x b\,\partial\xi\right)\hat{k}=0\hat{\imath}+cx\hat{\jmath}+(ay-bx)\hat{k}$$

(b) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\partial(y)/\partial x+\partial(x)/\partial y=0+0=0.$ ✓

With $x_0=y_0=0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z)=0\hat{\imath}+0\hat{\jmath}+\left[\int_0^y\eta\,\partial\eta-\int_0^x\xi\,\partial\xi\right]\hat{k}=\tfrac{1}{2}(y^2-x^2)\hat{k}$$

(c) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=\partial(3)/\partial x+\partial(x^2)/\partial y+\partial(4y)/\partial z=0+0+0=0.$ ✓

With $x_0=y_0=0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z) = 0\hat{\imath} + \left(\int_0^x 4y\,\partial\xi\right)\hat{\jmath} + \left(\int_0^y 3\,\partial\eta - \int_0^x \xi^2\,\partial\xi\right)\hat{k} = 4xy\hat{\jmath} + (3y - x^3/3)\hat{k}$$

(d) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \partial(xy)/\partial x + \partial(-yz)/\partial z = y - y = 0.$ ✓
With $x_0 = y_0 = 0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z) = 0\hat{\imath} + \left(\int_0^x -yz\,\partial\xi\right)\hat{\jmath} + \left(\int_0^y (0)\eta\,\partial\eta - \int_0^x 0\,\partial\xi\right)\hat{k} = -xyz\,\hat{\jmath}$$

(e) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \partial(2y)/\partial x + \partial(x^2y^2)/\partial y + \partial(-2x^2yz)/\partial z = 2xy^2 - 2xy^2 = 0.$ ✓
With $x_0 = y_0 = 0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z) = 0\hat{\imath} + \left(\int_0^x -2\xi^2 yz\,\partial\xi\right)\hat{\jmath} + \left(\int_0^y 2\eta\,\partial\eta - \int_0^x \xi^2y^2\,\partial\xi\right)\hat{k} = -\frac{2}{3}x^3yz\hat{\jmath} + \left(y^2 - \frac{x^3y^2}{3}\right)\hat{k}$$

(f) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \partial(x^3y)/\partial x + \partial(-z)/\partial y + \partial[-3x^2y(z+1)]/\partial z = 3x^2y - 3x^2y = 0.$ ✓
With $x_0 = y_0 = 0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z) = 0\hat{\imath} + \left(\int_0^x -3\xi^2y(z+1)\,\partial\xi\right)\hat{\jmath} + \left(\int_0^y 0\,\partial\eta - \int_0^x (-z)\,\partial\xi\right)\hat{k} = -x^3y(z+1)\hat{\jmath} + xz\hat{k}.$$

(g) $\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = \partial(x\sin y)/\partial x + \partial(\cos y + 2z)/\partial y + \partial(x^2y)/\partial z = \sin y - \sin y = 0.$ ✓
With $x_0 = y_0 = 0$, say, (10.3) gives

$$\underset{\sim}{w}(x,y,z) = 0\hat{\imath} + \left(\int_0^x \xi^2y\,\partial\xi\right)\hat{\jmath} + \left(\int_0^y 0\,\partial\eta - \int_0^x (\cos y + 2z)\,\partial\xi\right)\hat{k} = \frac{x^3y}{3}\hat{\jmath} - x(\cos y + 2z)\hat{k}$$

12. Let both "$\underset{\sim}{u}$" and "$\underset{\sim}{v}$" in equation (8) of Section 16.6 be the velocity field $\underset{\sim}{v}$. Then it becomes

$$\underset{\sim}{\nabla}(\underset{\sim}{v}\cdot\underset{\sim}{v}) = (\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} + (\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} + \underset{\sim}{v}\times\underset{0}{(\cancel{\underset{\sim}{\nabla}\times\underset{\sim}{v}})} + \underset{\sim}{v}\times\underset{0}{(\cancel{\underset{\sim}{\nabla}\times\underset{\sim}{v}})}$$
$$= 2(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v},$$

so $(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} = \underset{\sim}{\nabla}(\frac{1}{2}v^2)$ and, since $\underset{\sim}{v}$ is irrotational (and hence $= \underset{\sim}{\nabla}\Phi$), (12.2) becomes

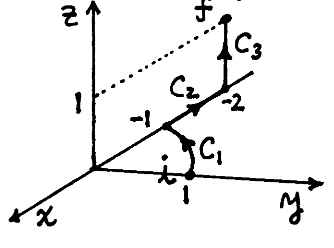
$$\frac{\partial}{\partial t}\underset{\sim}{\nabla}\Phi + \underset{\sim}{\nabla}\left(\frac{1}{2}v^2\right) + \underset{\sim}{\nabla}\left(\frac{p}{\sigma}\right) + \underset{\sim}{\nabla}(gz) = \underset{\sim}{0}$$

or, since $\frac{\partial}{\partial t}\underset{\sim}{\nabla}\Phi = \frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial x}\hat{\imath} + \frac{\partial\Phi}{\partial y}\hat{\jmath}\right) = \frac{\partial^2\Phi}{\partial t\,\partial x}\hat{\imath} + \frac{\partial^2\Phi}{\partial t\,\partial y}\hat{\jmath} = \frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial t}\right)\hat{\imath} + \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial t}\right)\hat{\jmath} = \underset{\sim}{\nabla}\left(\frac{\partial\Phi}{\partial t}\right)$,

$$\underset{\sim}{\nabla}\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\sigma} + gz\right) = \underset{\sim}{0}. \qquad ☆$$

It follows from the latter that $\partial\Phi/\partial t + \frac{1}{2}v^2 + p/\sigma + gz = \text{constant}$. Since the $\underset{\sim}{\nabla}$ in ☆ contains only spatial derivatives, the "constant" can be an arbitrary function of <u>time</u>, say $F(t)$.

13. (a) $\underset{\sim}{v} = z\hat{e}_r + r\hat{e}_z$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{e}_r + (1-1)\hat{e}_\theta + 0\hat{e}_z = \underset{\sim}{0}$ so we can use path simplification or the potential method.
We find from the given parametrization that the initial and final points are as shown at the right. We don't care what the <u>actual</u> path is since we are going to



deform it anyway.

Path simplification: Let us use the deformed path C_1, C_2, C_3 shown on the preceding page. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_C (z\hat{e}_r + r\hat{e}_z)\cdot(dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z)$$

$$= \int_{C_1} z\,\overset{0}{\cancel{dr}} + r\,\overset{0}{\cancel{dz}} + \int_{C_2} \overset{0}{\cancel{z}}\,dr + r\,\overset{0}{\cancel{dz}} + \int_{C_3} z\,\overset{0}{\cancel{dr}} + r\,dz = \int_0^1 2\,dz = 2.$$

(Of course the $z\,dr + r\,dz$ is "just begging" to be written as $d(rz)$, so $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_C d(rz) = rz\big|_i^f = (2)(1)-(1)(0) = 2$, but this is really the potential method, where the potential is $\Phi = zr$, to which method we turn ...)

Potential method: $$\left.\begin{array}{l}\partial\Phi/\partial r = z\\ \frac{1}{r}\partial\Phi/\partial\theta = 0\\ \partial\Phi/\partial z = r\end{array}\right\}\text{ gives } \Phi = rz + \overset{0}{\cancel{\text{const}}}.$$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = rz\big|_i^f = (2)(1)-(1)(0) = 2.$ ✓

(b) $\underset{\sim}{v} = z\sin\theta\,\hat{e}_r + z\cos\theta\,\hat{e}_\theta + r\sin\theta\,\hat{e}_z$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = (\cos\theta-\cos\theta)\hat{e}_r + (\sin\theta-\sin\theta)\hat{e}_\theta + \frac{1}{r}(z\cos\theta - z\cos\theta)\,\hat{e}_z = \underset{\sim}{0}$, so we can use path simplification or the potential method.

Path Simplification: Use the deformed path shown in (a), say.

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} z\sin\theta\,\overset{0}{\cancel{dr}} + \overset{0}{\cancel{z}}\cos\theta\,r\,d\theta + r\sin\theta\,\overset{0}{\cancel{dz}}$$

$$+\int_{C_2}\overset{0}{\cancel{z}}\sin\theta\,dr + z\cos\theta\,r\,\overset{0}{\cancel{d\theta}} + r\sin\theta\,\overset{0}{\cancel{dz}} + \int_{C_3} z\sin\theta\,\overset{0}{\cancel{dr}} + z\cos\theta\,r\,\overset{0}{\cancel{d\theta}} + r\sin\theta\,dz$$

$$= \int_0^1 (2)\,\overset{0}{\cancel{\sin\pi}}\,dz = 0.$$

Potential method: $\partial\Phi/\partial r = z\sin\theta \to \Phi = rz\sin\theta + A(\theta,z)$

$\frac{1}{r}\partial\Phi/\partial\theta = z\cos\theta = z\cos\theta + \frac{1}{r}\frac{\partial A}{\partial\theta}$, $\partial A/\partial\theta = 0$, $A(\theta,z) = B(z)$, $\Phi = rz\sin\theta + B$

$\partial\Phi/\partial z = r\sin\theta = r\sin\theta + B'(z)$, $B(z) = \cancel{\text{const}}$, $\Phi = rz\sin\theta$,

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = rz\sin\theta\big|_i^f = (2)(1)\sin\pi - (1)(0)\sin\frac{\pi}{2} = 0.$ ✓

(c) $\underset{\sim}{v} = \cos 5\theta\,\hat{e}_r - 5\sin 5\theta\,\hat{e}_\theta + z^2\,\hat{e}_z$,

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\,\hat{e}_r + 0\,\hat{e}_\theta + \frac{1}{r}(-5\sin 5\theta + 5\sin 5\theta)\,\hat{e}_z = \underset{\sim}{0}$ so we can use path simplification or the potential method.

Path Simplification: Use the deformed path shown in (a), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} \cos 5\theta\,\overset{0}{\cancel{dr}} - 5\sin 5\theta\,r\,d\theta + z^2\,\overset{0}{\cancel{dz}}$$

$$+\int_{C_2}\cos 5\theta\,dr - 5\sin 5\theta\,r\,\overset{0}{\cancel{d\theta}} + z^2\,\overset{0}{\cancel{dz}} + \int_{C_3}\cos 5\theta\,\overset{0}{\cancel{dr}} - 5\sin 5\theta\,r\,\overset{0}{\cancel{d\theta}} + z^2\,dz$$

$$= \int_{\pi/2}^{\pi} -5(\sin 5\theta)(1)d\theta + \int_1^2 \cos 5\pi \, dr + \int_0^1 z^2 dz = -1-1+\frac{1}{3} = -5/3$$

Potential Method: $\left.\begin{array}{r} \partial\Phi/\partial r = \cos 5\theta \\ \frac{1}{r}\partial\Phi/\partial\theta = -5\sin 5\theta \\ \partial\Phi/\partial z = z^2 \end{array}\right\}$ By inspection, $\Phi = r\cos 5\theta + \frac{z^3}{3} + \cancelto{0}{\text{const}}$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \left(r\cos 5\theta + \frac{z^3}{3}\right)\Big|_i^f = 2\cos 5\pi + \frac{1}{3} - (1)\cos\frac{5\pi}{2} = -5/3.$ ✓

(d) $\underset{\sim}{v} = 2(rz - \cos\theta)\hat{e}_r + 2\sin\theta\,\hat{e}_\theta + r^2\hat{e}_z$,
$\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{e}_r + (2r-2r)\hat{e}_\theta + \frac{1}{r}(2\sin\theta - 2\sin\theta)\,\hat{e}_z = \underset{\sim}{0}$ so we can use path simplification or the potential method.
Path Simplification: Use the deformed path shown in (a), say. Then

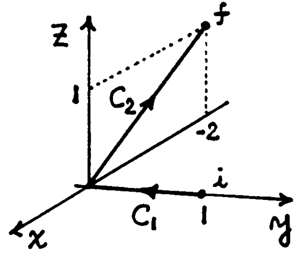
$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} (\;)\cancelto{0}{dr} + 2\sin\theta \, r d\theta + (\;)\cancelto{0}{dz} + \int_{C_2} 2(rz - \cos\theta)dr + (\;)r\,\cancelto{0}{d\theta} + (\;)\cancelto{0}{dz}$$
$$+ \int_{C_3} (\;)\cancelto{0}{dr} + (\;)r\cancelto{0}{d\theta} + r^2 dz$$

$$= \int_{\pi/2}^{\pi} 2\sin\theta\,(1)d\theta + \int_1^2 2(0-\cos\pi)dr + \int_0^1 4dz = 2+2+4 = 8.$$

Potential Method: $\partial\Phi/\partial r = 2(rz - \cos\theta)$, $\Phi = r^2 z - 2r\cos\theta + A(\theta,z)$
$\frac{1}{r}\partial\Phi/\partial\theta = 2\cancel{\sin\theta} = \frac{1}{r}(2r\cancel{\sin\theta} + A_\theta)$, $A_\theta = 0$, $A(\theta,z) = B(z)$, $\Phi = r^2 z - 2r\cos\theta + B(z)$
$\partial\Phi/\partial z = r^2 = r^2 + B'(z)$, $B(z) = \cancelto{0}{\text{const}}$, $\Phi = r^2 z - 2r\cos\theta$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = (r^2 z - 2r\cos\theta)\Big|_i^f = (4)(1) - 2(2)\cos\pi - 0 + 0 = 8.$ ✓

(e) $\underset{\sim}{v} = \rho^5\hat{e}_\rho$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{e}_\rho + 0\hat{e}_\phi + 0\hat{e}_\theta = \underset{\sim}{0}$ so we can use path simplification or the potential method.



Path Simplification: This time we are working with spherical polars so a "simplest path" will be different from the one used in (a)-(d), where we used cylindrical coordinates. For spherical polars we can use two straight lines, from i to the origin (C_1) and then from the origin to f (C_2), as shown above, on each of which only ρ varies. Thus,

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} \rho^5 d\rho + (\;)\cancelto{0}{d\phi} + (\;)\cancelto{0}{d\theta} + \int_{C_2} \rho^5 d\rho + (\;)\cancelto{0}{d\phi} + (\;)\cancelto{0}{d\theta}$$
$$= \int_1^0 \rho^5 d\rho + \int_0^{\sqrt{5}} \rho^5 d\rho = -\frac{1}{6} + \frac{(\sqrt{5})^6}{6} = \frac{62}{3}.$$

Potential Method:
$$\left.\begin{array}{l} \partial\Phi/\partial\rho=\rho^5 \\ \frac{1}{\rho}\partial\Phi/\partial\phi=0 \\ \frac{1}{\rho\sin\phi}\partial\Phi/\partial\theta=0 \end{array}\right\} \text{by inspection, } \Phi=\rho^6/6$$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \frac{\rho^6}{6}\Big|_i^f = \frac{(\sqrt{5})^6-1^6}{6} = \frac{124}{6} = \frac{62}{3}$. ✓

(f) $\underset{\sim}{v} = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{e}_\rho + 0\hat{e}_\phi + \frac{1}{\rho}[\cos\phi-\cos\phi]\hat{e}_\theta = \underset{\sim}{0}$ so we can use path simplification or the potential method.

Path Simplification: Use the path shown in (e), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} \sin\phi\, d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta} + \int_{C_2} \sin\phi\, d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta}$$
$$= \int_1^0 \sin\tfrac{\pi}{2}\, d\rho + \int_0^{\sqrt{5}} \frac{2}{\sqrt{5}}\, d\rho = -1+2=1.$$

Potential Method:
$$\left.\begin{array}{l} \partial\Phi/\partial\rho=\sin\phi \\ \frac{1}{\rho}\partial\Phi/\partial\phi=\cos\phi \\ \frac{1}{\rho\sin\phi}\partial\Phi/\partial\theta=0 \end{array}\right\} \text{gives } \Phi=\rho\sin\phi$$

so (11) gives $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \rho\sin\phi\Big|_i^f = \sqrt{5}\,\frac{2}{\sqrt{5}} - (1)\sin\frac{\pi}{2} = 1$. ✓

(g) $\underset{\sim}{v} = 3\rho^2\cos\theta\,\hat{e}_\rho - \rho^2\frac{\cos\theta}{\sin\phi}\hat{e}_\theta$, $\underset{\sim}{\nabla}\times\underset{\sim}{v} = 0\hat{e}_\rho + \frac{1}{\rho}\left[\frac{1}{\sin\phi}(-3\rho^2\sin\theta) + \frac{3\rho^2\sin\theta}{\sin\phi}\right]\hat{e}_\phi = \underset{\sim}{0}$

so we can use path simplification or the potential method.

Path Simplification: Use the path shown in (e), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 3\rho^2\cos\theta\, d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta} + \int_{C_2} 3\rho^2\cos\theta\, d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta}$$
$$= \int_1^0 3\rho^2\cos\tfrac{\pi}{2}\, d\rho + \int_0^{\sqrt{5}} 3\rho^2\cos\pi\, d\rho = -5\sqrt{5}.$$

Potential Method:
$$\left.\begin{array}{l} \partial\Phi/\partial\rho=3\rho^2\cos\theta \\ \frac{1}{\rho}\partial\Phi/\partial\phi=0 \\ \frac{1}{\rho\sin\phi}\partial\Phi/\partial\theta=-\frac{\rho^2\cos\theta}{\sin\phi} \end{array}\right\} \Phi=\rho^3\cos\theta+\cancel{\text{const}}.$$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \rho^3\cos\theta\Big|_i^f = (\sqrt{5})^3\cos\pi - (1)^3\cos\frac{\pi}{2} = -5\sqrt{5}$. ✓

(h) $\underset{\sim}{v} = \frac{\cos\phi}{\rho}\hat{e}_\phi - \frac{\sin\theta}{\rho\sin\phi}\hat{e}_\theta$, $\nabla \times \underset{\sim}{v} = 0\hat{e}_\rho + 0\hat{e}_\phi + 0\hat{e}_\theta = \underset{\sim}{0}$ so we can use path simplification or the potential method.

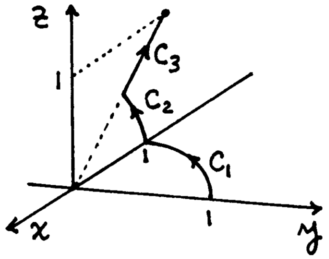
Path Simplification: Use the path shown in (d), say. Then

$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} 0\,d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta} + \int_{C_2} 0\,d\rho + (\ )\cancelto{0}{d\phi} + (\ )\cancelto{0}{d\theta} = 0.$$

Potential Method:
$$\left.\begin{array}{l} \partial\Phi/\partial\rho = 0 \\ \frac{1}{\rho}\partial\Phi/\partial\phi = \frac{\cos\phi}{\rho} \\ \frac{1}{\rho\sin\phi}\partial\Phi/\partial\theta = -\frac{\sin\theta}{\rho\sin\phi} \end{array}\right\} \text{gives } \Phi = \sin\phi + \cos\theta + \cancelto{0}{\text{const}}.$$

so, by (11), $\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = (\sin\phi + \cos\theta)\Big|_i^f = \frac{2}{\sqrt5} + \cos\pi - \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{2}{\sqrt5} - 2$

These two results (0 and $\frac{2}{\sqrt5} - 2$) do not agree. The error is in the path deformation for $\underset{\sim}{v}$ is not C^1 in any region containing the origin (because of the $1/\rho$'s in $\underset{\sim}{v}$) so Theorem 16.10.1 cannot be used to validate our proposed deformation — which was shown in part (d). Staying away from the origin, let us use, instead, the contour C_1, C_2, C_3 shown at the right: on C_1 only θ changes, on C_2 only ϕ changes, and on C_3 only ρ changes. Then



$$\int_C \underset{\sim}{v}\cdot d\underset{\sim}{R} = \int_{C_1} (\ )\cancelto{0}{d\rho} - \frac{\sin\theta}{\rho\sin\phi}\rho\sin\phi\,d\theta + (\ )\cancelto{0}{d\phi}$$
$$+ \int_{C_2} (\ )\cancelto{0}{d\rho} + (\ )\cancelto{0}{d\theta} + \frac{\cos\phi}{\rho}\rho\,d\phi + \int_{C_3} 0\,d\rho + (\ )\cancelto{0}{d\theta} + (\ )\cancelto{0}{d\phi}$$
$$= -\int_{\pi/2}^{\pi}\sin\theta\,d\theta + \int_{\phi=\pi/2}^{\sin\phi = 2/\sqrt5}\cos\phi\,d\phi = \cos\theta\Big|_{\pi/2}^{\pi} + \sin\phi\Big|_{\phi=\pi/2}^{\sin\phi=2/\sqrt5} = -1 - 0 + \frac{2}{\sqrt5} - 1 = \frac{2}{\sqrt5} - 2. \checkmark$$

14. Cylindrical: $\int_i^f \underset{\sim}{\nabla}\Phi\cdot d\underset{\sim}{R} = \int\left(\frac{\partial\Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\Phi}{\partial z}\hat{e}_z\right)\cdot(dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z)$

$$= \int_i^f \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi = \int_i^f d\Phi = \Phi\Big|_i^f$$

Spherical: $\int_i^f \underset{\sim}{\nabla}\Phi\cdot d\underset{\sim}{R} = \int_i^f\left(\frac{\partial\Phi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{e}_\phi + \frac{1}{\rho\sin\phi}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta\right)\cdot(d\rho\,\hat{e}_\rho + \rho\,d\phi\,\hat{e}_\phi + \rho\sin\phi\,d\theta\,\hat{e}_\theta)$

$$= \int_i^f \frac{\partial\Phi}{\partial\rho}d\rho + \frac{\partial\Phi}{\partial\phi}d\phi + \frac{\partial\Phi}{\partial\theta}d\theta = \int_i^f d\Phi = \Phi\Big|_i^f$$

CHAPTER 17

Section 17.2

1. (a) Define $f(x)=g(x)+h(x)$ where g,h are even.

Then $f(-x)=g(-x)+h(-x)$

$= g(x)+h(x)$ since g,h are even

$= f(x)$,

so f is <u>even</u> too.

(d) Define $f(x)=g(x)h(x)$ where g,h are odd.

Then $f(-x)=g(-x)h(-x)$

$=[-g(x)][-h(x)]$ since g,h are odd

$= g(x)h(x)$

$= f(x)$,

so f is <u>even</u>.

2. $$\int_{-A}^{A} f(x)\,dx = \int_{-A}^{0} f(x)\,dx + \int_{0}^{A} f(x)\,dx = \int_{A}^{0} f(-t)(-dt) + \int_{0}^{A} f(x)\,dx$$
$$= \int_{0}^{A} f(t)\,dt + \int_{0}^{A} f(x)\,dx = 2\int_{0}^{A} f(x)\,dx.$$

3.(a) $f(-x)=f(x)$ since f is even

$f(-x)=-f(x)$ " " " odd.

Subtracting these gives $2f(x)=0$ or $f(x)=0$.

(b) $F(x)=\int_0^x f(t)\,dt$

$F(-x)=\int_0^{-x} f(t)\,dt = \int_0^x f(-\tau)(-d\tau) = -\int_0^x f(\tau)\,d\tau = -F(x)$ so F is odd.

(d) $F(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

$$F(-x)=\lim_{\Delta x\to 0}\frac{f(-x+\Delta x)-f(-x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x-\Delta x)-f(x)}{\Delta x} = -\lim_{\Delta x\to 0}\frac{f(x)-f(x-\Delta x)}{\Delta x} = -f'(x)$$
$$= -F(-x)$$

so F is odd.

4. $f(x)=f_e(x)+f_o(x)$ ①

$f(-x)=f_e(-x)+f_o(-x) = f_e(x)-f_o(x)$ ②

adding ① and ② gives $f_e(x)=[f(x)+f(-x)]/2$

and subtracting them gives $f_o(x)=[f(x)-f(-x)]/2$

5. (a) $f_e(x)=(2-5x+2+5x)/2 = 2$,

$f_o(x)=(2-5x-(2+5x))/2 = -5x$, so f is neither odd nor even.

(b) $\sin(x+2) = \underbrace{\sin 2\cos x}_{f_e(x)} + \underbrace{\cos 2\sin x}_{f_o(x)}$ by the trigonometric identity $\sin(A+B) = \sin A\cos B + \sin B\cos A$. Or, using (6),

$$\sin(x+2) = \frac{\sin(x+2) + \sin(-x+2)}{2} + \frac{\sin(x+2) - \sin(-x+2)}{2}.$$

To simplify this, we still need the trig. identity given above:

$$= \tfrac{1}{2}[\cancel{\sin x\cos 2} + \sin 2\cos x - \cancel{\sin x\cos 2} + \sin 2\cos x]$$
$$+ \tfrac{1}{2}[\sin x\cos 2 + \cancel{\sin 2\cos x} + \sin x\cos 2 - \cancel{\sin 2\cos x}]$$
$$= \underbrace{\sin 2\cos x}_{f_e} + \underbrace{\cos 2\sin x}_{f_o}.$$

Both f_e and f_o are nonzero, so f is neither odd nor even.

(d) $$xe^{-x} = \tfrac{1}{2}(xe^{-x} + (-x)e^{x}) + \tfrac{1}{2}(xe^{-x} - (-x)e^{x})$$
$$= -x\left(\frac{e^{x} - e^{-x}}{2}\right) + x\left(\frac{e^{x} + e^{-x}}{2}\right) = \underbrace{-x\sinh x}_{f_e} + \underbrace{x\cosh x}_{f_o}$$

Since both f_e, f_o are nonzero f is neither odd nor even.

(f) $$x^2\cos x^3 - 8 = \tfrac{1}{2}[x^2\cos x^3 - 8 + (-x)^2\cos(-x)^3 - 8] + \tfrac{1}{2}[x^2\cos x^3 - 8 - (-x)^2\cos(-x)^3 + 8]$$
$$= \underbrace{x^2\cos x^3 - 8}_{f_e} + \underbrace{0}_{f_o}$$

Since $f_o = 0$, f is even.

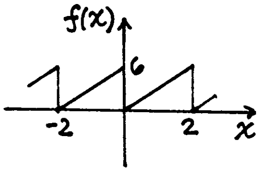
6. (c) $F(G(-x)) \underset{\substack{\uparrow \\ \text{Since } G \text{ odd}}}{=} F(-G(x)) \underset{\substack{\uparrow \\ \text{Since } F \text{ even}}}{=} F(G(x))$ so $F(G(x)$ is even

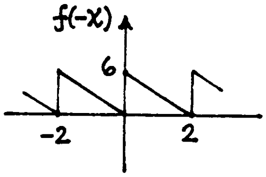
7. $f(-x) = -f(x)$ so, for $x = 0$, $f(0) = -f(0)$, $2f(0) = 0$, $f(0) = 0$

8. $c_1 f(x) + c_2 g(x) = 0$, ①
so $c_1 f(-x) + c_2 g(-x) = 0$ too, or, since f is even and g is odd,
$c_1 f(x) - c_2 g(x) = 0$.
Adding the first and third equations gives $2c_1 f(x) = 0$ so, since $f(x)$ is not identically zero, $c_1 = 0$. Subtracting, instead, gives $2c_2 g(x) = 0$ so, since $g(x)$ is not identically zero, $c_2 = 0$. Since ① implies $c_1 = c_2 = 0$, f and g are LI.

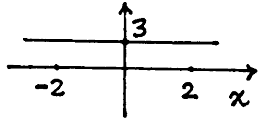
10. $f(x) + g(x) = 0$, so, changing x to $-x$,
$f(-x) + g(-x) = 0$ or, since f is even and g is odd,
$f(x) - g(x) = 0$. Adding the first and third equations shows that $f(x) = 0$ and subtracting them shows that $g(x) = 0$.

11. (a) $T = 2$. To find f_e and f_o we can use (6) and proceed graphically:

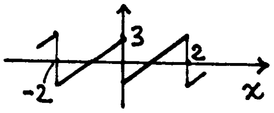




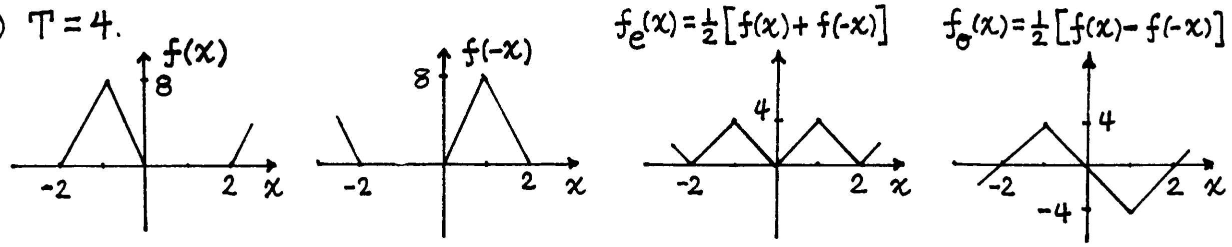
$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$



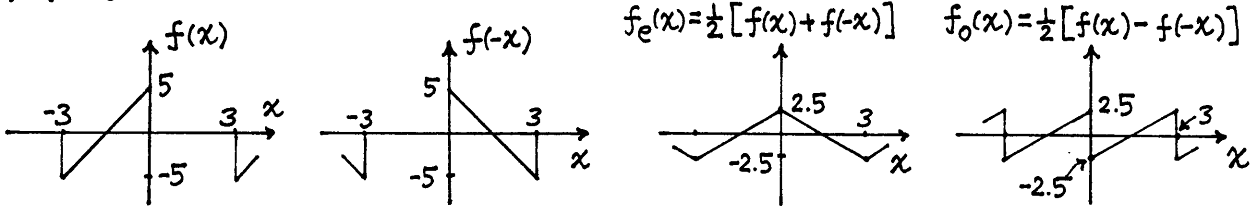
$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$



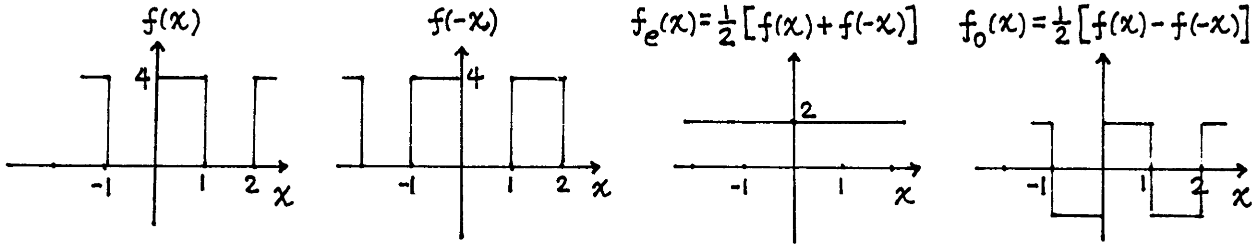
(b) T = 4.



(c) T = 6.



(d) T = 2.



12. (b) Obviously e^x is not periodic, but let us depend on the defining property $f(x+T)=f(x)$ instead: $e^{x+T}=e^x$ so $e^T=1$ so $T=0$. Hence, e^x is not periodic.

(c) No

(d) Obviously it is periodic, with T found from $\omega(x+T)+\phi=\omega x+\phi+2\pi$, or, $\omega T=2\pi$, $T=2\pi/\omega$.

(e) $T=2\pi/6=\pi/3$.

(g) This one is tricky. Since $\tan x=\sin x/\cos x$ and $\sin x$ and $\cos x$ are each periodic with fundamental period 2π, we might expect that $\tan x$, likewise, is periodic with fundamental period 2π. Let us see: Set

$$\tan(x+T)=\frac{\sin(x+T)}{\cos(x+T)}=\frac{\sin x\cos T+\sin T\cos x}{\cos x\cos T-\sin x\sin T}=\frac{\sin x}{\cos x}$$

so $\cos x(\cancel{\sin x\cos T}+\sin T\cos x)=\sin x(\cancel{\cos x\cos T}-\sin x\sin T)$

so $(\cos^2 x+\sin^2 x)\sin T=0$

so $\sin T=0$, the smallest (nonzero) root of which gives

$T=\pi$,

as can be seen (more easily) from the graph of $\tan x$.

(h) No (i) No

(j) It's simplest to use the trig. identity $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$. The 1/2 is periodic with T arbitrary and the $\frac{1}{2}\cos 2x$ is periodic with period π, so

$\cos^2 x$ has fundamental period $T=\pi$.

(k) $\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 2x$ so, as in (j), $T=\pi$.

(l) $\sin x \cos 2x = \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$.

$\sin 3x$ has periods $2\pi/3,\ 4\pi/3,\ \boxed{6\pi/3}, \ldots$

$\sin x$ " " $\boxed{2\pi},\ 4\pi, \ldots$

The fundamental period of $\sin x \cos 2x$ is the smallest period common to both terms ($\sin 3x$ and $\sin x$), namely, 2π. Thus, $T=2\pi$.

(m) $e^{\sin x}$ repeats every time $\sin 3x$ does, and $\sin 3x$ is periodic with fundamental period $2\pi/3$. Thus, $e^{\sin x}$ has $T=2\pi/3$.

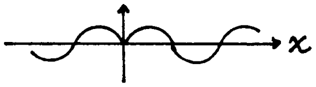
(n) $T=\pi$

(o) $T=2\pi$

(p) $T=2\pi$

(q) $T=2\pi$

(r) $T=\pi/2$

(s) $\sin|x|$ is not periodic since its graph is 

(t) $T=2\pi$

(u) No.

13. (b) a_0 is periodic with arbitrary period

$a_1\cos x$ is periodic with periods $\boxed{2\pi},\ 4\pi,\ 6\pi, \ldots$

$a_2\cos 2x$ " " " $\pi,\ \boxed{2\pi},\ 3\pi, \ldots$

Their sum is periodic and has as its fundamental period the smallest period found in each of the three lists, namely, $T=2\pi$.

(c) $6\cos x$ is periodic with periods $\boxed{2\pi},\ 4\pi,\ 6\pi, \ldots$

$-4\sin 3x$ " " " $2\pi/3,\ 4\pi/3,\ \boxed{6\pi/3}, \ldots$

so $6\cos x - 4\sin 3x$ is periodic with fundamental period $T=2\pi$.

(d) $T=2\pi/5$

(e) $T=2\pi$

(f) $T=2\pi$

(g) a_0 is periodic with arbitrary period

$a_1\cos\frac{\pi x}{l}$ and $b_1\sin\frac{\pi x}{l}$ are periodic with periods $\boxed{2l},\ 4l,\ 6l, \ldots$

$a_2\cos\frac{2\pi x}{l}$ and $b_2\sin\frac{2\pi x}{l}$ " " " $l,\ \boxed{2l},\ 3l,\ 4l, \ldots$

$a_3\cos\frac{3\pi x}{l}$ and $b_3\sin\frac{3\pi x}{l}$ " " " $\frac{2}{3}l,\ \frac{4}{3}l,\ \boxed{\frac{6}{3}l},\ \frac{8}{3}l, \ldots$

and so on. The smallest period in each of these lists is seen to be $2l$. Thus, $T=2l$.

14. $\int_A^{A+T} f(x)\,dx \overset{\substack{x=t+A\\ \downarrow}}{=} \int_0^T f(t+A)\,dt$. No help, so proceed differently:

$$\int_A^{A+T} f(x)\,dx = \int_A^T f(x)\,dx + \underbrace{\int_T^{A+T} f(x)\,dx}_{\swarrow\text{ Set } x=\xi+A \text{ here.}} = \int_A^T f(x)\,dx + \int_0^A f(\xi+T)\,d\xi$$

$$= \int_0^A f(\xi)\,d\xi + \int_A^T f(x)\,dx = \int_0^T f(x)\,dx. \checkmark$$

15.(a) $f'(x+T) = \lim_{\Delta x\to 0} \frac{f(x+T+\Delta x)-f(x+T)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$. For example, $\sin x$ is periodic with periods $2\pi, 4\pi, \dots$ and $\frac{d}{dx}\sin x = \cos x$ is too.

(b)
$$\int_0^{x+T} f(t)\,dt = \int_0^x f(t)\,dt + \int_x^{x+T} f(t)\,dt$$
$$\Big\downarrow \text{ Obtained as in Exercise 14}$$
$$= \int_0^x f(t)\,dt + \int_0^T f(t)\,dt,$$

so $\int_0^{x+T} f(t)\,dt = \int_0^x f(t)\,dt$ if and only if $\int_0^T f(t)\,dt = 0$ or, equivalently, if the average value $\frac{1}{T}\int_0^T f(t)\,dt = 0$.

16. Let T be any positive rational number.
If x is rational then $f(x+T) = f(\text{rational } \#) = 1 = f(x)$.
If x is irrational then $f(x+T) = f(\text{irrational } \#) = 0 = f(x)$.
$\uparrow$ $x+T$ = irrational # + rational # = irrat.#
In both cases $f(x+T) = f(x)$.
Now let T be any positive irrational number.
If x is rational then $f(x+T) = f(\text{rat.}\# + \text{irrat.}\#) = f(\text{irrat.}\#) = 0 \neq f(x)$, so f is not periodic with any rational period.

17. $g(f(x+T)) = g(f(x))$ so g is periodic with period T too. For ex., $\sin x$ is periodic with periods $2\pi, 4\pi, 6\pi, \dots$ and $e^{\sin x}$ is too; likewise, $\cos(\sin x)$ is too (although it is also periodic with periods $\pi, 3\pi, 5\pi, \dots$ since

$$\cos(\sin(x+n\pi)) = \cos(\sin x \cos n\pi + \overset{0}{\cancel{\sin n\pi}} \cos x)$$
$$= \cos(\sin x \cos n\pi) = \cos(\sin x)$$

for $n = 1, 2, 3, \dots$.)

Section 17.3

1. (a) Yes; indeed it is continuous on $[0, \pi]$.
(b) No; it is continuous on $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$, but the limit of $\tan x$ as $x \to \pi/2-$, and as $x \to \pi/2-$, does not exist (since it is infinite).
(c) No; it is continuous on $0 < x \le \pi$, but $\lim_{x\to 0+} \sin\frac{1}{x}$ does not exist.
(d) Same as (c).
(e) Yes; indeed it is continuous on $[0, \pi]$.
(f) No; it is continuous on $0 \le x < 1$ and on $1 < x \le \pi$, but the limit of $1/(1-x)$ as $x \to 1-$, and as $x \to 1+$, does not exist.

(g) No; it is continuous on $0< x \le \pi$, but its limit as $x \to 0+$ does not exist.

(h) Yes; indeed it is continuous on $[0,\pi]$.

(i) Yes; it is continuous on $0 \le x < 2$ and on $2 < x \le \pi$, and has finite limits (namely, 100) as $x \to 2-$ and as $x \to 2+$.

(j) Yes; it is continuous on $0 \le x < 2$ and on $2 < x \le \pi$, and has the finite limits 2 and 1 as $x \to 2-$ and as $x \to 2+$, respectively.

2.(a) $\int_{-l}^{l} \cos\frac{m\pi x}{l} \cos\frac{n\pi x}{l}\, dx = \frac{1}{2}\int_{-l}^{l} \cos\frac{(m+n)\pi x}{l}\, dx + \frac{1}{2}\int_{-l}^{l} \cos\frac{(m-n)\pi x}{l}\, dx$

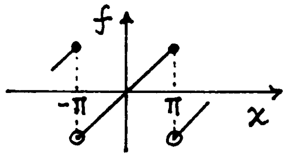
If $m \ne n$ this $= \frac{l}{2(m+n)\pi} \sin\frac{(m+n)\pi x}{l}\Big|_{-l}^{l} + \frac{l}{2(m-n)\pi} \cos\frac{(m-n)\pi x}{l}\Big|_{-l}^{l} = 0-0+0-0 = 0.$

If $m = n \ne 0$ this $= \frac{l}{4m\pi} \sin\frac{2m\pi x}{l}\Big|_{-l}^{l} + \frac{1}{2} 2l = 0-0+l = l.$

If $m=n=0$ this $= \int_{-l}^{l} dx = 2l.$

3. Yes.

4. (a)



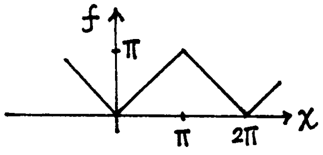
a_0 and a_n's $= 0$ since f is odd.

period $= 2l = 2\pi$ so $l = \pi$.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$
$= \frac{2}{n^2\pi}(\sin nx - nx\cos nx)\Big|_0^{\pi} = -\frac{2n\pi}{n^2\pi}\cos n\pi = \frac{2(-1)^{n+1}}{n}$

so $f(x) = 2\sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$ for all x except at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ where $f(x) = \pi$ but the series converges to 0.

(b)



period $= 2l = 2\pi$ so $l = \pi$

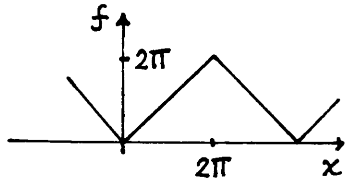
$a_0 =$ average value $= \pi/2$ by inspection of the graph

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$

$= \frac{2}{n^2\pi}(\cos nx + nx\sin nx)\Big|_0^{\pi} = \frac{2}{n^2\pi}(\cos n\pi - 1) = \begin{cases} -4/n^2\pi, & n \text{ odd} \\ 0 & , n \text{ even} \end{cases}$

so $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2}\cos nx$, which series converges to $f(x)$ for all x.

(c)



period $= 2l = 4\pi$ so $l = 2\pi$

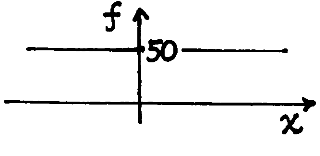
$a_0 =$ average value $= \pi$

$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} |x| \cos\frac{n\pi x}{n\pi}\, dx = \frac{4}{n^2\pi}\int_0^{n\pi} t\cos t\, dt = \begin{cases} -\frac{8}{n^2\pi}, & \text{odd} \\ 0, & \text{even} \end{cases}$

b_n's $= 0$ since $f(x)$ is even, so

$f(x) = \pi - \frac{8}{\pi}\sum_{1,3,\ldots}^{\infty} \frac{1}{n^2}\cos\frac{nx}{2}$, which series converges to $f(x)$ for all x.

(d)

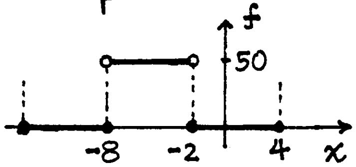


$a_0 = 50$, a_n's $= b_n$'s $= 0$

$f(x) = 50$ (is a very short Fourier series!) for all x.

(e) As in (d), the Fourier series of is simply $f(x) = 50$. The latter converges to $f(x)$ at all x except at $x=0, \pm 2, \pm 4, \ldots$ where it converges to 50 whereas $f(x) = 100$ at those points.

(f)



period $= 2l = 12$ so $l = 6$

$a_0 =$ average value $= 25$, by inspection

$a_n = \frac{1}{6}\int_{-8}^{4} f(x)\cos\frac{n\pi x}{6}\,dx$ (Note: $\int_{-8}^{4}$ is more convenient than $\int_{-6}^{6}$. We can integrate on any $2l$-interval due to the $2l$-periodicity of the integrand.)

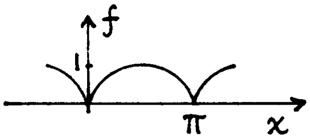
$$= \frac{1}{6}\int_{-8}^{-2} 50\cos\frac{n\pi x}{6}\,dx = \frac{50}{6}\left.\frac{\sin\frac{n\pi x}{6}}{\frac{n\pi}{6}}\right|_{-8}^{-2}$$

$$= \frac{50}{n\pi}\left(\sin\frac{4n\pi}{3} - \sin\frac{n\pi}{3}\right)$$

$$b_n = \frac{1}{6}\int_{-8}^{4} f(x)\sin\frac{n\pi x}{6}\,dx = \frac{1}{6}\int_{-8}^{-2} 50\sin\frac{n\pi x}{6}\,dx = -\frac{50}{6}\left.\frac{\cos\frac{n\pi x}{6}}{\frac{n\pi}{6}}\right|_{-8}^{-2} = \frac{50}{n\pi}\left(\cos\frac{4n\pi}{3} - \cos\frac{n\pi}{3}\right)$$

so $f(x) = 25 + \sum_1^\infty \frac{50}{n\pi}\left[\left(\sin\frac{4n\pi}{3} - \sin\frac{n\pi}{3}\right)\cos\frac{n\pi x}{6} + \left(\cos\frac{4n\pi}{3} - \cos\frac{n\pi}{3}\right)\sin\frac{n\pi x}{6}\right]$, which converges to $f(x)$ at all points where f is continuous; at the discontinuities it converges to the average value 25.

(g)



period $= 2l = \pi$ so $l = \pi/2$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} |\sin x|\cos 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2}\sin x\cos 2nx\,dx$$

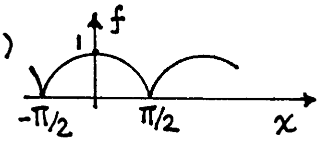
$$= \frac{2}{\pi}\int_0^{\pi/2}[\sin(2n+1)x - \sin(2n-1)x]\,dx$$

$$= \frac{2}{\pi}\left[\frac{\cos(2n-1)x}{2n-1} - \frac{\cos(2n+1)x}{2n+1}\right]\Bigg|_0^{\pi/2} = \frac{2}{\pi}\left(0 - 0 - \frac{1}{2n-1} + \frac{1}{2n+1}\right) = -\frac{4}{\pi}\frac{1}{4n^2-1},$$

b_n's $= 0$ since f is even,

so $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{1}{4n^2-1}\cos 2nx$, which converges to $f(x)$ for all x.

(h)



period $= 2l = \pi$ so $l = \pi/2$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} |\cos x|\cos 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2}\cos x\cos 2nx\,dx$$

$$= \frac{4}{\pi}\frac{1}{2}\int_0^{\pi/2}[\cos(2n+1)x + \cos(2n-1)x]\,dx = \frac{2}{\pi}\left(\frac{\sin(2n+1)\pi/2}{2n+1} + \frac{\sin(2n-1)\pi/2}{2n-1}\right)$$

$= \frac{2}{\pi}(-1)^n\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{4}{\pi}\frac{(-1)^{n+1}}{4n^2-1}$, $b_n = 0$ since f is even,

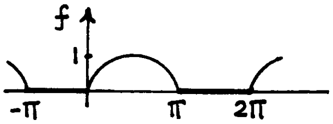
so $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1}\cos 2nx$, which converges to $f(x)$ for all x.

Check: We should also be able to obtain this series by shifting x by $\pi/2$ in part (g). That step gives

$$\frac{2}{\pi}-\frac{4}{\pi}\sum_1^\infty \frac{1}{4n^2-1}\cos 2n(x+\tfrac{\pi}{2}) = \frac{2}{\pi}-\frac{4}{\pi}\sum_1^\infty \frac{1}{4n^2-1}(\cos 2nx \cos n\pi - \sin 2nx \overset{0}{\cancel{\sin n\pi}})$$

$$= \frac{2}{\pi}+\frac{4}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1}\cos 2nx. \checkmark$$

(i)



period $= 2\ell = 2\pi$ so $\ell = \pi$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx = \frac{1}{2\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^\pi \sin x \cos nx\,dx$$

$= \frac{1}{2\pi}\int_0^\pi [\sin(n+1)x - \sin(n-1)x]\,dx = 0$ if $n=1$; otherwise it

$$= \frac{1}{2\pi}\left(-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right)\Big|_0^\pi = \frac{1}{2\pi}\left(-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right) = \frac{1}{2\pi}\Big[(-1)^{n-1}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$-\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\Big] = \frac{1}{2\pi}\frac{2}{2n^2-1}[(-1)^n-1] = \begin{cases} 0, & n \text{ odd} \\ -\frac{2}{\pi(n^2-1)}, & n \text{ even}\end{cases}$$

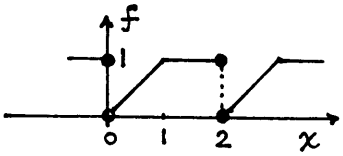
$b_n = \frac{1}{\pi}\int_0^\pi \sin x \sin nx\,dx = \frac{1}{2\pi}\int_0^\pi [\cos(n-1)x - \cos(n+1)x]\,dx = \frac{1}{2}$ for $n=1$; for $n \neq 1$ it

$$= \frac{1}{2\pi}\left(\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right)\Big|_0^\pi = 0, \text{ so}$$

$f(x) = \frac{1}{\pi}+\frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=2,4,\dots}^\infty \frac{1}{n^2-1}\cos nx = \frac{1}{\pi}+\frac{1}{2}\sin x - \frac{2}{\pi}\sum_1^\infty \frac{1}{4n^2-1}\cos 2nx$, which converges to $f(x)$ for all x. NOTE: This result could also be obtained by noting that the even and odd parts of f are $\frac{1}{2}|\sin x|$ and $\frac{1}{2}\sin x$, respectively, and using (g), above, for the $\frac{1}{2}|\sin x|$ part.

(j) $f(x) = 20 + 3\sin 4x$ is already in Fourier series form, so there is nothing further to do.

(k)



period $= 2\ell = 2$ so $\ell = 1$

$a_0 =$ average value $= 3/4$ (by inspection)

$a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 x\cos n\pi x\,dx + \int_1^2 \cos n\pi x\,dx$

$= \frac{1}{n^2\pi^2}(\cos n\pi - 1) + 0 = \frac{(-1)^n-1}{n^2\pi^2}$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 x\sin n\pi x\,dx + \int_1^2 \sin n\pi x\,dx$$

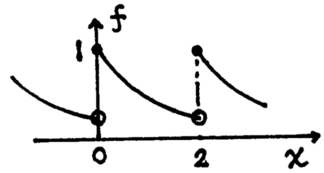
$$= -\frac{n\pi\cos n\pi}{n^2\pi^2} - \frac{\cos 2n\pi}{n\pi} + \frac{\cos n\pi}{n\pi} = -\frac{1}{n\pi}$$

so $f(x) = \frac{3}{4}+\sum_1^\infty\left(\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right)$, which converges to $f(x)$ for all x except at the jumps ($x = 0, \pm 2, \pm 4, \dots$), where it converges to the average value 1/2, whereas $f(x) = 1$ at those points.

(l) $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ is the Fourier series of $\cos^2 x$; there's nothing more to do. If this result is not clear, note, analogously, that the Taylor series of $(1+x)^2$ about $x=0$ is simply $1+2x+x^2$.

(m) $\sin^2 x = 1-\cos^2 x = 1-(\frac{1}{2}+\frac{1}{2}\cos 2x) = \frac{1}{2}-\frac{1}{2}\cos 2x$.

(n)



period $= 2l = 2$ so $l = 1$

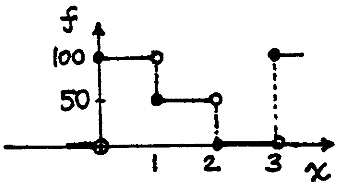
$$a_0 = \frac{1}{2}\int_0^2 e^{-x}\,dx = (1-e^{-2})/2$$

$$a_n = \frac{1}{1}\int_0^2 e^{-x}\cos n\pi x\,dx = \frac{1-e^{-2}}{n^2\pi^2+1}$$

$$b_n = \frac{1}{1}\int_0^2 e^{-x}\sin n\pi x\,dx = \frac{-n\pi(e^{-2}-1)}{n^2\pi^2+1}$$

so $f(x) = \frac{1-e^{-2}}{2} + (1-e^{-2})\sum_1^\infty \frac{1}{n^2\pi^2+1}(\cos n\pi x + n\pi \sin n\pi x)$, which converges to $f(x)$ for all x except at the discontinuities, where it converges to the average value $(1+e^{-2})/2$, whereas $f(x)=1$ at those points.

(o)



period $= 2l = 3$ so $l = 3/2$

$a_0 = 50$

$$a_n = \frac{1}{3/2}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left\{\int_0^1 100\cos\frac{2n\pi x}{3}\,dx + \int_1^2 50\cos\frac{2n\pi x}{3}\,dx + 0\right\}$$

$$= \frac{50}{n\pi}\left(2\sin\frac{2n\pi}{3} + \sin\frac{4n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

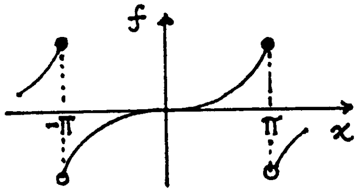
$$b_n = \frac{1}{3/2}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx = \frac{2}{3}\left\{\int_0^1 100\sin\frac{2n\pi x}{3}\,dx + \int_1^2 50\sin\frac{2n\pi x}{3}\,dx\right\}$$

$$= \frac{50}{n\pi}\left(2-\cos\frac{2n\pi}{3} - \cos\frac{4n\pi}{3}\right)$$

so $f(x) = 50 + \frac{50}{\pi}\sum_1^\infty \frac{1}{n}\left[\left(\sin\frac{2n\pi}{3} + \sin\frac{4n\pi}{3}\right)\cos\frac{2n\pi x}{3} + \left(2-\cos\frac{2n\pi}{3}-\cos\frac{4n\pi}{3}\right)\sin\frac{2n\pi x}{3}\right]$,

which converges to $f(x)$ everywhere except at the discontinuities, where it converges to the average value: at $x=0, \pm 3, \pm 6, \ldots$ it converges to 50 whereas $f(x)=100$ there; at $x=1,4,-2,7,-5,\ldots$ it converges to 75 whereas $f(x)=75$ there; and at $x=2,5,-1,8,-4,\ldots$ it converges to 25 whereas $f(x)=0$ there.

5. (b)



period $= 2l = 2\pi$ so $l = \pi$

Odd function, so $a_0 = 0$, a_n's $= 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\,dx$$

The Maple command `int((x^3/Pi)*sin(n*x), x=-Pi..Pi);` gives

$$b_n = -2\,\frac{n^3\cos(n\pi)\pi^3 - 3n^2\sin(n\pi)\pi^2 - 6n\cos(n\pi)\pi + 6\sin(n\pi)}{n^4\pi}$$

which we can simplify to $b_n = \frac{2}{n^3}(6-n^2\pi^2)(-1)^n$. Thus,

$$f(x) = 2\sum_1^\infty \frac{6-n^2\pi^2}{n^3}(-1)^n \sin nx.$$

(c) f is even so $b_n=0$. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos^2 x\,dx$. The Maple command int((1/Pi)*(cos(x))^2, x=0..Pi); gives $a_0 = 1/2$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\cos^2 x\cos nx\,dx$$

and the Maple command int((2/Pi)*(cos(x))^2*cos(n*x), x=0..Pi); gives a_n = many terms. Next, the command simplify("); reduces that result to

$$a_n = 2\,\frac{\sin(n\pi)(n^2-2)}{\pi n(n^2-4)}$$

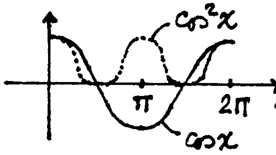
so $a_n = 0$ except possibly for $n=2$ in which case we have the indeterminate form 0/0. Applying l'Hôpital's rule for that case we need merely focus on the indeterminate part, the $\sin n\pi/(n^2-4)$ ratio. L'Hôpital gives

$$\lim_{n\to 2}\frac{\sin n\pi}{n^2-4} = \lim_{n\to 2}\frac{\pi\cos n\pi}{2n} = \frac{\pi}{2}\frac{\cos 2\pi}{2} = \frac{\pi}{4},$$

so $a_2 = 2\,\frac{\pi}{4}\,\frac{(2^2-2)}{2\pi} = \frac{1}{2}$.

Thus, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ is the desired Fourier series.

NOTE: In this example we said that the period is 2π (so $l = \pi$). Actually, although 2π is a period of $\cos^2 x$ it is not the fundamental period, which is π. Thus, we should have said



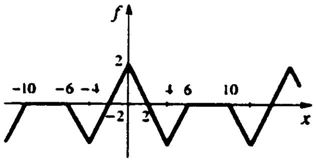
$$\text{period} = 2l = \pi \text{ so } l = \pi/2,$$
$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx,\quad a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2 x\cos 2nx\,dx.$$

However, the final result, $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, would have been the same. In working out the Fourier series of a given periodic function it is simplest to take $2l$ to be the fundamental period, but even if we take it to be some integer multiple of the fundamental period we will still arrive at the same Fourier series, which point is examined in Exercise 10.

6. We will use Maple to obtain computer plots of various partial sums of the Fourier series

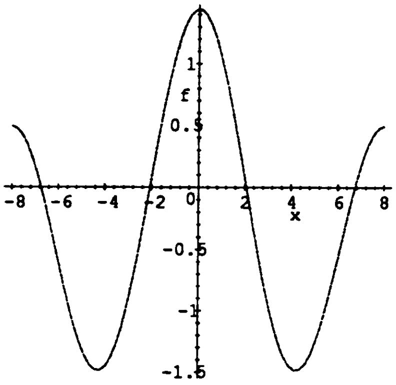
$$f(x) = -\frac{1}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - 2\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{4}}{n^2}\cos\frac{n\pi x}{8}$$

of



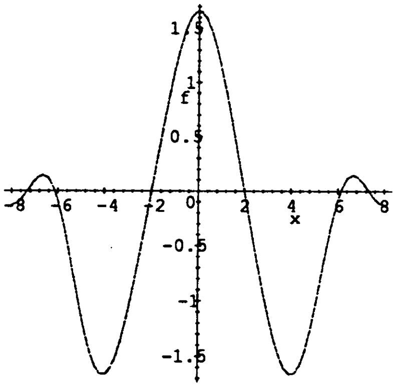
(a)

```
> p(x):=sum(((1-2*cos(i*Pi/2)+cos(3*i*Pi/4))/i^2)*cos(i*Pi*x/8),i=1.
  .2):
> q(x):=-1/4+(16/Pi^2)*p(x):
> implicitplot(f=q(x),x=-8..8,f=-3..3,numpoints=5000);
```



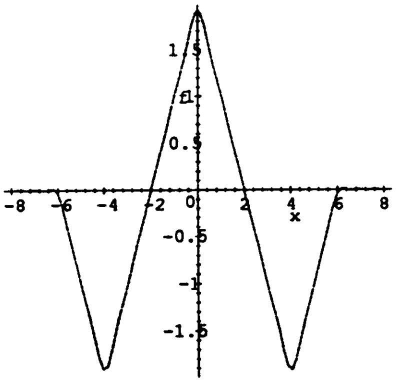
(b)

```
> p(x):=sum(((1-2*cos(i*Pi/2)+cos(3*i*Pi/4))/i^2)*cos(i*Pi*x/8),i=1.
  .5):
> q(x):=-1/4+(16/Pi^2)*p(x):
> implicitplot(f=q(x),x=-8..8,f=-3..3,numpoints=5000);
```



(c)

```
> p(x):=sum(((1-2*cos(i*Pi/2)+cos(3*i*Pi/4))/i^2)*cos(i*Pi*x/8),i=1.
  .10):
> q(x)=-1/4+(16/Pi^2)*p(x):
> implicitplot(f=q(x),x=-8..8,f=-3..3,numpoints=8000);
```



We see that even with n=10 the convergence to f is "pretty good". Note that we have used the numpoints option to increase the number of points as we have increased n since a fine subdivision of the interval is needed if we are to do a good job at plotting the highest harmonics.

7. (a) Letting $x=\pi/2$ in (11), and noting from Fig. 2 that $f(\pi/2)=4$, gives

$$4 = 2+\frac{8}{\pi}\sum_{1,3,\dots}^{\infty}\frac{\sin(n\pi/2)}{n}, \text{ or, } \frac{\pi}{4} = \sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

(b) For a convergent alternating series, the error committed by terminating the series after n terms is less, in magnitude, than the first omitted term, namely, a_{n+1}. Since $a_n = (\sin n\pi/2)/n$, choose the number terms retained, N, so that $|a_{N+1}| = |(\sin(N+1)\pi/2)/(N+1)| = 1/(N+1) \leq 10^{-6}$, i.e., $N+1=10^6$ or $N \approx 10^6$, say.

8. Period $= 2l = 2\pi$ so $l=\pi$.

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\,dx = \pi^2/4$, either by the substitution $x=\pi\cos\theta$ or the Maple int command.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\cos nx\,dx = \int_{\pi}^{0}\sqrt{1-\cos^2\theta}\,\cos(n\pi\cos\theta)(-\pi\sin\theta\,d\theta) \quad (x=\pi\cos\theta)$$

$$= \pi\int_0^{\pi}\sin^2\theta\cos(n\pi\cos\theta)\,d\theta$$

$$= \pi\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)\left[J_0(n\pi)-2J_2(n\pi)\cos2\theta+2J_4(n\pi)\cos4\theta-\cdots\right]d\theta$$

$$= \frac{\pi}{4}\int_{-\pi}^{\pi}(1-\cos2\theta)\left[J_0(n\pi)-2J_2(n\pi)\cos2\theta+2J_4(n\pi)\cos4\theta-\cdots\right]d\theta$$

By the Euler formula (24a), with l set $=\pi$, we see that the only nonzero terms are these:

$$a_n = \frac{\pi}{4}\int_{-\pi}^{\pi}(1)J_0(n\pi)\,d\theta + \frac{\pi}{4}\int_{-\pi}^{\pi}\cos2\theta\;2J_2(n\pi)\cos2\theta\,d\theta$$

$$= \frac{\pi}{4}2\pi J_0(n\pi) + \frac{\pi}{4}2J_2(n\pi)\pi,$$

so $f(x) = \frac{\pi^2}{4}+\frac{\pi^2}{2}\sum_1^{\infty}\left[J_0(n\pi)+J_2(n\pi)\right]\cos nx.$ ✓

9. If $f(x) = a_0+\sum_1^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$,
then

$$f(x-c) = a_0+\sum_1^{\infty}\left(a_n\cos\frac{n\pi(x-c)}{l}+b_n\sin\frac{n\pi(x-c)}{l}\right)$$

$$= a_0+\sum_1^{\infty}\Big(a_n\cos\frac{n\pi x}{l}\cos\frac{n\pi c}{l}+a_n\sin\frac{n\pi x}{l}\sin\frac{n\pi c}{l}$$
$$+b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}-b_n\cos\frac{n\pi x}{l}\sin\frac{n\pi c}{l}\Big)$$

$$= A_0+\sum_1^{\infty}\left(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l}\right)$$

so, by comparison, $A_0 = a_0$, $A_n = a_n \cos\frac{n\pi c}{\ell} - b_n \sin\frac{n\pi c}{\ell}$,
$B_n = a_n \sin\frac{n\pi c}{\ell} + b_n \cos\frac{n\pi c}{\ell}$.

10. The underlying idea is this. Suppose $f(x)$ is periodic with fundamental period 2ℓ. Then its F.S. is

$$f(x) = a_0 + a_1 \cos\frac{\pi x}{\ell} + b_1 \sin\frac{\pi x}{\ell} + a_2 \cos\frac{2\pi x}{\ell} + b_2 \sin\frac{2\pi x}{\ell} + \cdots$$

If its fundamental period is 2ℓ, then it also has periods $4\ell, 6\ell, \ldots$. If, for some reason, we use the period 4ℓ, say, then we will obtain a Fourier series of the form

$$f(x) = a_0' + a_1' \cos\frac{\pi x}{2\ell} + b_1' \sin\frac{\pi x}{2\ell} + a_2' \cos\frac{2\pi x}{2\ell} + b' \sin\frac{2\pi x}{2\ell} + a_3' \cos\frac{3\pi x}{2\ell} + \cdots$$

We will find that $a_0' = a_0$, $a_1' = b_1' = 0$, $a_2' = a_1$, $b_2' = b_1$, $a_3' = b_3' = 0$, $a_4' = a_2$, $b_4' = b_2$, and so on, so the resulting series will be the same, independent of whether we use, for the period, 2ℓ or 4ℓ (or, indeed, 6ℓ, 8ℓ, or any other multiple of 2ℓ). For example, consider again the square wave shown in Fig. 1, but this time take the period $= 2\ell$ to be 4π, so that "ℓ" $= 2\pi$. Then

$$a_0' = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f\,dx = \frac{1}{4\pi} 2\pi(4) = 2,$$

$$a_n' = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\frac{n\pi x}{2\pi}\,dx = \frac{4}{2\pi}\left\{\int_{-2\pi}^{-\pi}\cos\frac{nx}{2}\,dx + \int_0^{\pi}\cos\frac{nx}{2}\,dx\right\} = \text{etc.} = 0,$$

$$b_n' = \frac{4}{2\pi}\left\{\int_{-2\pi}^{-\pi}\sin\frac{nx}{2}\,dx + \int_0^{\pi}\sin\frac{nx}{2}\,dx\right\} = \text{etc.} = \begin{cases} 0, & n \neq 2, 6, 10, \ldots \\ 16/n\pi, & n = 2, 6, 10, \ldots \end{cases}$$

so $$f(x) = 2 + \frac{16}{\pi}\sum_{2,6,10,\ldots}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{2\pi} \overset{n/2=m \atop \downarrow}{=} 2 + \frac{8}{\pi}\sum_{1,3,\ldots}^{\infty}\frac{1}{m}\sin mx,$$

which is indeed the same as (11).

11. (a) $$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\underbrace{p(x)}_{U}\underbrace{\cos\frac{n\pi x}{\ell}\,dx}_{dV} = \frac{1}{\ell}p(x)\frac{\ell}{n\pi}\cancelto{0}{\sin\frac{n\pi x}{\ell}}\Big|_{-\ell}^{\ell} - \frac{1}{\ell}\int_{-\ell}^{\ell}\underbrace{p'(x)}_{U}\frac{\ell}{n\pi}\underbrace{\sin\frac{n\pi x}{\ell}\,dx}_{dV}$$

$$= -\frac{1}{n\pi}p'(x)\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Big|_{-\ell}^{\ell} + \frac{1}{n\pi}\int_{-\ell}^{\ell}\underbrace{p''(x)}_{U}\underbrace{\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)dx}_{dV}$$

$$= \frac{2\ell}{n^2\pi^2}(-1)^n p'(\ell) - \frac{\ell}{n^2\pi^2}p''(x)\left(\frac{\ell}{n\pi}\cancelto{0}{\sin\frac{n\pi x}{\ell}}\right)\Big|_{-\ell}^{\ell} + \frac{\ell}{n^2\pi^2}\int_{-\ell}^{\ell}\underbrace{p'''(x)}_{U}\underbrace{\left(\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right)dx}_{dV}$$

$$= \frac{2\ell}{n^2\pi^2}(-1)^n p'(\ell) + \frac{\ell^2}{n^3\pi^3}p'''(x)\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Big|_{-\ell}^{\ell} - \frac{\ell^2}{n^3\pi^3}\int_{-\ell}^{\ell}p'''(x)\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2\ell}{n^2\pi^2}(-1)^n p'(\ell) - \frac{2\ell^3}{n^4\pi^4}(-1)^n p'''(\ell) + \cdots$$

$$= \frac{2\ell}{n^2\pi^2}(-1)^n\left[p'(\ell) - \frac{\ell^2}{n^2\pi^2}p'''(\ell) + \cdots\right].$$

Here we have used the fact that $p, p'', p'''', \dots$ are even functions and $p', p''', \dots$ are odd (from Exercise 3 of Section 17.2).

(b) $$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \underbrace{p(x)}_{U} \underbrace{\sin\frac{n\pi x}{\ell}\,dx}_{dV} = \frac{1}{\ell}p(x)\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Big|_{-\ell}^{\ell} - \frac{1}{\ell}\int_{-\ell}^{\ell} p'(x)\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)dx$$

$$= -\frac{2}{n\pi}(-1)^n p(\ell) + \frac{\ell}{n\pi}\boxed{\frac{1}{\ell}\int_{-\ell}^{\ell} p'(x)\cos\frac{n\pi x}{\ell}\,dx}$$

↓

Note that since p is odd, p' is even, so this integral is actually "a_n" given by (11.1) where "p" is p'

↓

$$\frac{2\ell}{n^2\pi^2}(-1)^n\left[p''(\ell) - \frac{\ell^2}{n^2\pi^2}p''''(\ell) + \cdots\right]$$

so $$b_n = -\frac{2}{n\pi}(-1)^n\left[p(\ell) - \frac{\ell^2}{n^2\pi^2}p''(\ell) + \frac{\ell^4}{n^4\pi^4}p^{(iv)}(\ell) - \cdots\right]$$

12. (a) $f(x) = x$ is odd so a_n's $= 0$ and, with $\ell = 3$, (11.2) gives
$$b_n = -\frac{2}{n\pi}(-1)^n[\ell - 0 + \cdots] = -\frac{6}{n\pi}(-1)^n,$$
so
$$f(x) = x = -\frac{6}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\sin\frac{n\pi x}{3}$$

(b) $f(x) = x^2$ is even so b_n's $= 0$. $a_0 = \frac{1}{6}\int_{-3}^{3} x^2\,dx = 3$ and (11.1) gives
$$a_n = \frac{2\ell}{n^2\pi^2}(-1)^n[2\ell - 0 + \cdots] = \frac{36}{n^2\pi^2}(-1)^n$$
so
$$f(x) = x^2 = 3 + \frac{36}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{3}$$

(c) $f(x) = 6 + 2x - x^3 = (\underset{\overset{\uparrow}{\text{even}}}{6}) + (\underset{\overset{\uparrow}{\text{odd}}}{2x - x^3})$. The FS of 6 is simply 6, so focus on the odd part. (11.2) gives $b_n = -\frac{2}{n\pi}(-1)^n\left[2\ell - \ell^3 - \frac{\ell^2}{n^2\pi^2}(-6\ell) + 0 - \cdots\right]$
so
$$f(x) = 6 + 2x - x^3 = 6 + \frac{8}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\left(1 - \frac{12}{n^2\pi^2}\right)\sin\frac{n\pi x}{2}$$

(d) $f(x) = x^4 - 8x^2$ is even so b_n's $= 0$. $a_0 = \frac{1}{4}\int_{-2}^{2}(x^4 - 8x^2)\,dx = -112/15$. (11.1) gives $a_n = \frac{2\ell}{n^2\pi^2}(-1)^n\left[4\ell^3 - 16\ell - \frac{\ell^2}{n^2\pi^2}24\ell + 0 - \cdots\right] = -\frac{768}{n^4\pi^4}(-1)^n$
so
$$f(x) = x^4 - 8x^2 = -\frac{112}{15} - \frac{768}{\pi^4}\sum_1^\infty \frac{(-1)^n}{n^4}\cos\frac{n\pi x}{2}$$

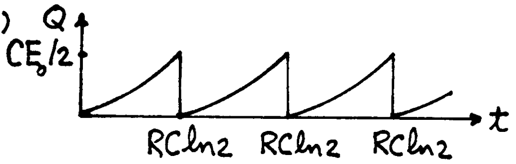
(e) $f(x) = x^3 - 3x$ is odd so a_n's $= 0$. (11.2) gives $b_n = -\frac{2}{n\pi}(-1)^n[$
$$b_n = -\frac{2}{n\pi}(-1)^n\left[\ell^3 - 3\ell - \frac{\ell^2}{n^2\pi^2}6\ell + 0 - \cdots\right] = -\frac{2}{n\pi}(-1)^n\left(-2 - \frac{6}{n^2\pi^2}\right)$$
so
$$f(x) = x^3 - 3x = 4\sum_1^\infty \frac{(-1)^n}{n\pi}\left(1 + \frac{3}{n^2\pi^2}\right)\sin n\pi x$$

13. (a) $Q(t) = CE_0 + Ae^{-t/RC}$

$Q(0) = 0 = CE_0 + A$ so $A = -CE_0$ and $Q(t) = CE_0(1 - e^{-t/RC})$

(b) Set $Q(t)/C = E_0(1 - e^{-t/RC}) = E_0/2$. Then $e^{-t/RC} = \frac{1}{2}$ so $t = RC\ln 2$.

(c)



period = RCln2 ≡ τ, say.

$$a_0 = \frac{1}{\tau}\int_0^\tau CE_0(1 - e^{-t/RC})\,dt = \frac{CE_0}{\tau}(t + RCe^{-t/RC})\Big|_0^\tau = \frac{2\ln 2 - 1}{2\ln 2}CE_0$$

$$a_n = \frac{2CE_0}{\tau}\int_0^\tau (1 - e^{-t/RC})\cos\frac{2n\pi t}{\tau}\,dt = \frac{2CE_0}{\tau}\,\mathrm{Re}\int_0^\tau (1 - e^{-t/RC})e^{i2n\pi t/\tau}\,dt$$

$$= \frac{2CE_0}{\tau}\mathrm{Re}\left\{\frac{e^{i2n\pi t/\tau}}{i2n\pi/\tau} - \frac{e^{(\frac{i2n\pi}{\tau} - \frac{1}{RC})t}}{\frac{i2n\pi}{\tau} - \frac{1}{RC}}\right\}\Bigg|_0^\tau = 2CE_0\,\mathrm{Re}\left\{-\frac{i}{2n\pi}\cancelto{0}{(e^{i2n\pi} - 1)} - \frac{e^{(2n\pi i - \ln 2)} - 1}{2n\pi i - \ln 2}\right\}$$

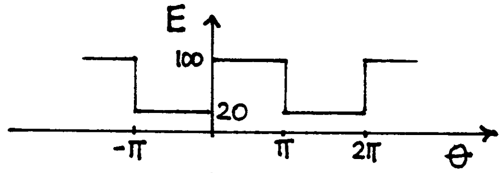
$$= \frac{2CE_0}{2}\mathrm{Re}\frac{1}{2n\pi i - \ln 2} = -\frac{CE_0\ln 2}{4n^2\pi^2 + (\ln 2)^2}$$

$$b_n = \frac{2CE_0}{\tau}\int_0^\tau (1 - e^{-t/RC})\sin\frac{2n\pi t}{\tau}\,dt = \frac{2CE_0}{\tau}\,\mathrm{Im}\int_0^\tau (1 - e^{-t/RC})\,e^{i2n\pi t/\tau}\,dt$$

$$= (\text{from above}) = \frac{2CE_0}{2}\,\mathrm{Im}\frac{1}{2n\pi i - \ln 2} = -\frac{2n\pi CE_0}{4n^2\pi^2 + (\ln 2)^2}$$

so $$Q(t) = CE_0\left\{\frac{2\ln 2 - 1}{2\ln 2} - \sum_1^\infty\left[\frac{\ln 2}{4n^2\pi^2 + (\ln 2)^2}\cos\frac{2n\pi t}{RC\ln 2} + \frac{2n\pi}{4n^2\pi^2 + (\ln 2)^2}\sin\frac{2n\pi t}{RC\ln 2}\right\}\right.$$

14.



period = 2l = 2π, l = π.

$a_0 = 60$ (by inspection of the graph)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} E(\theta)\cos\frac{n\pi\theta}{\pi}\,d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 20\cos n\theta\,d\theta + \frac{1}{\pi}\int_0^\pi 100\cos n\theta\,d\theta = 0 + 0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} E(\theta)\sin\frac{n\pi\theta}{\pi}\,d\theta = \frac{1}{\pi}\int_{-\pi}^{0} 20\sin n\theta\,d\theta + \frac{1}{\pi}\int_0^\pi 100\sin n\theta\,d\theta$$

$$= -\frac{20}{n\pi}\cos n\theta\Big|_{-\pi}^{0} - \frac{100}{n\pi}\cos n\theta\Big|_0^\pi = -\frac{20}{n\pi}[1 - \cos(-n\pi)] + \frac{100}{n\pi}[1 - \cos n\pi] = \frac{80}{n\pi}[1 - (-1)^n]$$

so $$E(\theta) = 60 + \frac{160}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin n\theta}{n}.$$

15. $$\text{Ave. Power} = \frac{R}{T}\int_{-T/2}^{T/2}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi t}{T} + b_n\sin\frac{2n\pi t}{T}\right)\right]\left[a_0 + \sum_{m=1}^{\infty}\left(a_m\cos\frac{2m\pi t}{T} + b_m\sin\frac{2m\pi t}{T}\right]dt\right.$$

and using the orthogonality relations (24), with $l = T/2$, we obtain

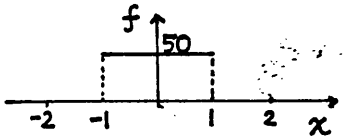
$$\text{Ave. Power} = \frac{R}{T}\left(a_0^2T + \frac{T}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right) = \left[a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]R.$$

If $i(t)$ = constant = I_{rms}, then for that case $a_0 = I_{rms}$, a_n's = b_n's = 0 so

$$\left(I_{rms}^2 + \frac{1}{2}\sum_{n=1}^{\infty} 0\right)R = \left[a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]R$$

so $$I_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{1}^{\infty}(a_n^2+b_n^2)}\ . \ \checkmark$$

16. (a)



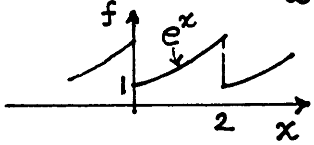
period $= 2l = 4$, so $l = 2$.

$$c_n = \frac{1}{4}\int_{-2}^{2} f(x)\, e^{-in\pi x/2}\,dx = \frac{1}{4}\int_{-1}^{1} 50\, e^{-in\pi x/2}\,dx$$

$$= \frac{25}{2}\,\frac{e^{-in\pi x/2}}{-in\pi/2}\Big|_{-1}^{1} = \frac{50}{n\pi}\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right) = \frac{50}{n\pi}\sin\frac{n\pi}{2}$$

so $$f(x) = \frac{50}{\pi}\sum_{-\infty}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\, e^{in\pi x/2}$$

(b)



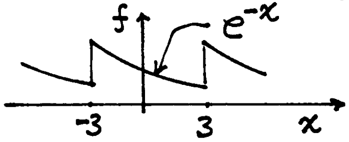
period $= 2l = 2$ so $l = 1$.

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{x} e^{-in\pi x}\,dx = \frac{1}{2}\,\frac{e^{(1-in\pi)x}}{1-in\pi}\Big|_{-1}^{1}$$

$$= \frac{1}{2}\,\frac{e^{1-n\pi i} - e^{-1+n\pi i}}{1-in\pi} = \frac{e\cos n\pi - e^{-1}\cos n\pi}{2(1-n\pi i)} = \frac{(-1)^n\sinh 1}{1-n\pi i}\,\frac{1+n\pi i}{1+n\pi i} = \frac{1+n\pi i}{1+n^2\pi^2}(-1)^n\sinh 1$$

so $$f(x) = \sinh 1 \sum_{-\infty}^{\infty}(-1)^n\,\frac{1+n\pi i}{1+n^2\pi^2}\, e^{in\pi x}$$

(c)



period $= 2l = 6$ so $l = 3$.

$$c_n = \frac{1}{6}\int_{-3}^{3} e^{-x} e^{-in\pi x/3}\,dx = \frac{1}{6}\,\frac{e^{-(3+n\pi i)x/3}}{-(3+n\pi i)/3}\Big|_{-3}^{3}$$

$$= \frac{1}{2}\,\frac{e^{3}e^{n\pi i} - e^{-3}e^{-n\pi i}}{3+n\pi i} = \frac{e^{3}\cos n\pi - e^{-3}\cos n\pi}{2(3+n\pi i)} = \frac{(-1)^n\sinh 3}{3+n\pi i}\,\frac{3-n\pi i}{3-n\pi i} = \frac{3-n\pi i}{9+n^2\pi^2}(-1)^n\sinh 3$$

so $$f(x) = \sinh 3 \sum_{-\infty}^{\infty}(-1)^n\,\frac{3-n\pi i}{9+n^2\pi^2}\, e^{in\pi x/3}$$

(d) $f(x) = 6\sin x = 6\,\dfrac{e^{ix}-e^{-ix}}{2i} = 3i(-e^{ix}+e^{-ix})$.

That is, with period $= 2l = 2\pi$ so $l = \pi$, $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ where all c_n's $= 0$ except for $c_1 = -3i$ and $c_{-1} = 3i$.

(e) $f(x) = 4 - 5\cos 2x = 4 - \frac{5}{2}(e^{i2x} + e^{-i2x})$ so all c_n's $= 0$ except for $c_0 = 4$, $c_2 = -5/2$, $c_{-2} = -5/2$.

17. (a) No, since the interval of convergence will be much too small. For example, if we expand $F(t)$ (in Fig. 8) about $t = 0$ we obtain the Taylor series $F(t) = 1/2a$, which is valid only over $0 < t < a$.

(b) This does not appear to be a good idea. On the plus side, the solution over each interval will be only a finite sum rather than an infinite series, but we will need to obtain such solutions over many intervals if we are to approach steady state and the expression of the solution will become very unwieldy as we proceed.

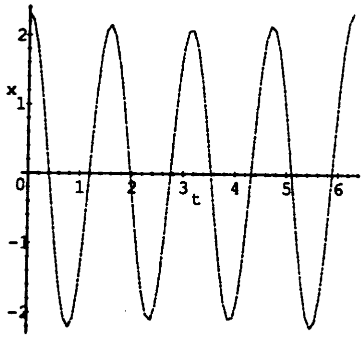
(c) No! Equation (34) is a superposition-type solution that is valid only if the ODE is linear, and $mx'' + cx' + \alpha x + \beta x^3 = F(t)$ is not linear because of the x^3 term.

(d) We need merely let $a \to 0$ in (37). Since l'Hôpital's rule gives $\sin n a / n\pi a \to 1/\pi$ as $a \to 0$, (37) gives the steady-state response

$$x(t) = \frac{1}{30\pi} + \frac{1}{\pi}\sum_{1}^{\infty}\left[\frac{15-n^2}{(15-n^2)^2+0.0016n^2}\cos nt + \frac{0.04n}{(15-n^2)^2+0.0016n^2}\sin nt\right].$$

Though it wasn't asked for, let us obtain a computer plot of that steady-state solution over one period (i.e., 2π), using Maple:

```
> with(plots):
> s(t):=sum(((15-i*2)*cos(i*t)+.04*i*sin(i*t))/((15-i^2)^2+.0016*i^2
  ),i=1..20):
> f(t):=1/(30*Pi)+(1/Pi)*s(t):
> implicitplot(x=f(t),t=0..2*Pi,x=-10..10,numpoints=5000);
```

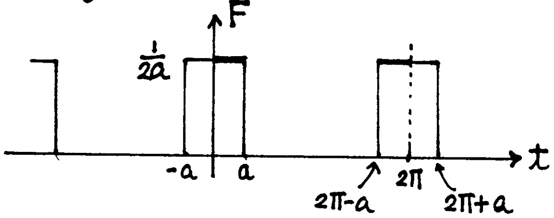


(e) I will only outline a solution method. In steady-state the solution is 2π-periodic so it suffices to consider any one 2π-interval, such as $0 \le t \le 2\pi$, say, assuming that steady state has already been achieved. Our problem is

$$x'' + 0.04x' + 15x = F(t)$$

where $F(t)$ is shown at the right, together with the periodicity boundary conditions

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi).$$

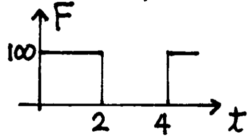


- Break the interval into the pieces $0<t<a$, $a<t<2\pi-a$, $2\pi-a<t<2\pi$.
- Solve $x''+0.04x'+15x = 1/2a$; $x(0)=x_0$, $x'(0)=x_0'$ as initial conditions.
- Let the solution of that problem give $x(a) \equiv x_1$, $x'(a) \equiv x_1'$.
- Solve $x''+0.04x'+15x=0$; $x(a)=x_1$, $x'(a)=x_1'$ as initial conditions.
- Let the solution of that problem give $x(2\pi-a) \equiv x_2$, $x'(2\pi-a) \equiv x_2'$.
- Solve $x''+0.04x'+15x=0$; $x(2\pi-a)=x_2$, $x'(2\pi-a)=x_2'$ as initial conditions.
- Impose on the solution of that last problem the periodicity conditions $x(2\pi)=x_0$

and $x'(2\pi)=x_0'$. Those steps should give the steady-state solution for each of the 3 intervals.

- Finally, let $a\to 0$ in those expressions and just retain the "middle" solution (i.e., on $a<t<2\pi-a$). NOTE: You should find that the solution is continuous but suffers jumps in slope at $t=0$ and 2π (i.e., at the point of action of each delta function)

18. (a)



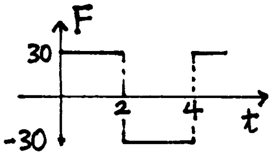
$$F(t) = 50 + \frac{200}{\pi}\sum_{1,3,\dots}^{\infty} \frac{1}{n}\sin\frac{n\pi t}{2}$$

$$x''+x = 50+\frac{200}{\pi}\sum_{1,3,\dots}^{\infty} \frac{1}{n}\sin\frac{n\pi t}{2} \qquad ♀$$

$x''+x=50$ has particular solution 50;
$x''+x=\sin\frac{n\pi t}{2}$ has particular solution $x=\frac{4}{4-n^2\pi^2}\sin\frac{n\pi t}{2}$, so ♀ has the steady-state solution

$$x(t) = 50+\sum_{1,3,\dots}^{\infty}\left(+\frac{200}{n\pi}\right)\frac{4}{4-n^2\pi^2}\sin\frac{n\pi t}{2} = 50+\frac{800}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n(4-n^2\pi^2)}\sin\frac{n\pi t}{2}.$$

(b)



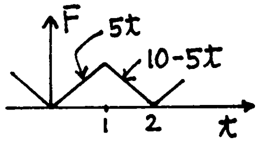
$$F(t) = \frac{120}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi t}{2}.$$

$$x''+x = \frac{120}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi t}{2}. \qquad ‡$$

$x''+x=\sin\frac{n\pi t}{2}$ has particular solution $x=\frac{4}{4-n^2\pi^2}\sin\frac{n\pi t}{2}$, so ‡ has the steady-state solution

$$x(t) = \sum_{1,3,\dots}\frac{120}{n\pi}\frac{4}{4-n^2\pi^2}\sin\frac{n\pi t}{2} = \frac{480}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n(4-n^2\pi^2)}\sin\frac{n\pi t}{2}$$

(c)



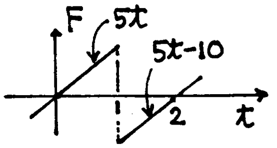
$$F(t) = \frac{5}{2}-\frac{20}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2}\cos n\pi t$$

$$x''+x = \frac{5}{2}-\frac{20}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2}\cos n\pi t \qquad ★$$

$x''+x=5/2$ has particular solution $5/2$;
$x''+x=\cos n\pi t$ " " " $\frac{1}{1-n^2\pi^2}\cos n\pi t$, so ★ has the steady-state solution

$$x(t) = \frac{5}{2}+\sum_{1,3,\dots}^{\infty}\left(\frac{-20}{n^2\pi^2}\right)\frac{1}{1-n^2\pi^2}\cos n\pi t = \frac{5}{2}-\frac{20}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2(1-n^2\pi^2)}\cos n\pi t.$$

(d)



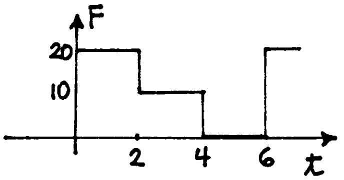
$$F(t) = -\frac{10}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin n\pi t$$

$$x''+x = -\frac{10}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin n\pi t \qquad \$$$

$x''+x=\sin n\pi t$ has particular solution $\frac{1}{1-n^2\pi^2}\sin n\pi t$, so # has the steady-state solution

$$x(t)=\sum_1^\infty \frac{-10}{n\pi}(-1)^n\frac{1}{1-n^2\pi^2}\sin n\pi t=\frac{10}{\pi}\sum_1^\infty\frac{(-1)^n}{n(n^2\pi^2-1)}\sin n\pi t$$

(e)



$$F(t)=10+\frac{10}{\pi}\sum_1^\infty\frac{1}{n}\left(2-\cos\frac{2n\pi}{3}-\cos\frac{4n\pi}{3}\right)\sin\frac{n\pi t}{3}$$

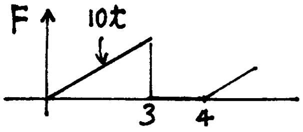
$$x''+x=10+\frac{10}{\pi}\sum_1^\infty\frac{1}{n}\left(2-\cos\frac{2n\pi}{3}-\cos\frac{4n\pi}{3}\right)\sin\frac{n\pi t}{3} \quad \#$$

$x''+x=10$ has particular solution 10;
$x''+x=\sin\frac{n\pi t}{3}$ has particular solution $\frac{9}{9-n^2\pi^2}\sin\frac{n\pi t}{3}$, so # has the steady-state solution

$$x(t)=10+\sum_1^\infty\frac{10}{n\pi}\left(2-\cos\frac{2n\pi}{3}-\cos\frac{4n\pi}{3}\right)\frac{9}{9-n^2\pi^2}\sin\frac{n\pi t}{3}$$

$$=10+\frac{90}{\pi}\sum_1^\infty\frac{2-\cos\frac{2n\pi}{3}-\cos\frac{4n\pi}{3}}{n(9-n^2\pi^2)}\sin\frac{n\pi t}{3}$$

(f)



$$F(t)=\frac{45}{4}+\sum_1^\infty\frac{20}{n^2\pi^2}\left[\left(\cos\frac{3n\pi}{2}+\frac{3n\pi}{2}\sin\frac{3n\pi}{2}-1\right)\cos\frac{n\pi t}{2}+\left(\sin\frac{3n\pi}{2}-\frac{3n\pi}{2}\cos\frac{3n\pi}{2}\right)\sin\frac{n\pi t}{2}\right]$$

$$x''+x=F(t)=\text{"} \qquad *$$

$x''+x=45/4$ has particular solution $45/4$;
$x''+x=\cos n\pi t/2$ has particular solution $\frac{4}{4-n^2\pi^2}\cos\frac{n\pi t}{2}$;

$x''+x=\sin n\pi t/2$ " " " $\frac{4}{4-n^2\pi^2}\sin\frac{n\pi t}{2}$, so the * equation has the steady-state solution

$$x(t)=\frac{45}{4}+\sum_1^\infty\frac{20}{n^2\pi^2}\frac{4}{4-n^2\pi^2}\left[\left(\cos\frac{3n\pi}{2}+\frac{3n\pi}{2}\sin\frac{3n\pi}{2}-1\right)\cos\frac{n\pi t}{2}+\left(\sin\frac{3n\pi}{2}-\frac{3n\pi}{2}\cos\frac{3n\pi}{2}\right)\sin\frac{n\pi t}{2}\right]$$

$$=\frac{45}{4}+\frac{80}{\pi^2}\sum_1^\infty\frac{1}{n^2(4-n^2\pi^2)}\left[\left(\cos\frac{3n\pi}{2}+\frac{3n\pi}{2}\sin\frac{3n\pi}{2}-1\right)\cos\frac{n\pi t}{2}+\left(\sin\frac{3n\pi}{2}-\frac{3n\pi}{2}\cos\frac{3n\pi}{2}\right)\sin\frac{n\pi t}{2}\right]$$

Section 17.4

1.(a) Since $f_{ext}(x)$ is antisymmetric about $x=0$ and periodic with period $2L$, we can expand

$$f_{ext}(x) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad (-\infty<x<\infty) \qquad ♀$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f_{ext}(x)}_{odd}\,\underbrace{\sin\frac{n\pi x}{L}}_{odd}}_{even}\,dx = \frac{2}{L}\int_0^L f_{ext}(x)\sin\frac{n\pi x}{L}\,dx \quad \text{(since integrand is even)}$$

$$= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad (\text{since } f_{ext}=f \text{ on } 0<x<L)$$

Thus, if we limit the x domain to $0<x<L$ then ♀ gives

$$f(x) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad (0<x<L)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

These two formulas coincide with (4).

(b) Since $f_{ext}(x)$ is symmetric about $x=0$ and periodic with period $4L$, we can expand

$$f_{ext}(x) = \sum_{1}^{\infty} a_n \cos\frac{n\pi x}{2L} \quad (-\infty<x<\infty)$$

where

$$b_n = \frac{1}{2L}\int_{-2L}^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx = \frac{2}{2L}\int_0^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx = \frac{1}{L}\int_0^{2L} f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx.$$

Now, $f_{ext}(x)$ is antisymmetric about the midpoint of the integration interval, $x=L$, and $\cos(n\pi x/2L)$ is antisymmetric about that point if n is odd and symmetric about that point if n is even. Thus, the integrand $f_{ext}(x)\sin(n\pi x/2L)$ is symmetric about the midpoint $x=L$ if n is odd and is antisymmetric about that point if n is even, so

$$a_n = \begin{cases} \frac{2}{L}\int_0^L f_{ext}(x)\cos\frac{n\pi x}{2L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

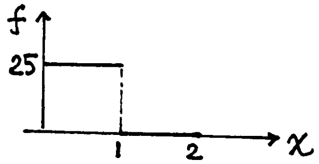
so we have

$$f(x) = \sum_{1,3,\dots}^{\infty} a_n \cos\frac{n\pi x}{2L}, \quad (0<x<L)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx,$$

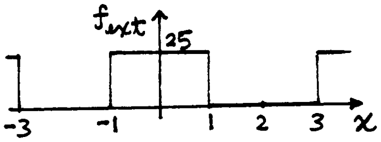
which results agree with (5).

2. (a)



$L=2.$

HRC:

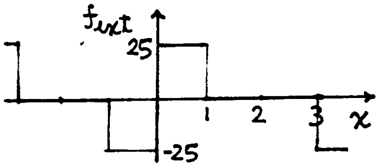


$a_0 = 25/2$

$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 25\cos\frac{n\pi x}{2}\,dx = \frac{50}{n\pi}\sin\frac{n\pi}{2}$

so $f(x) = \frac{25}{2} + \frac{50}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$. $(0<x<2)$

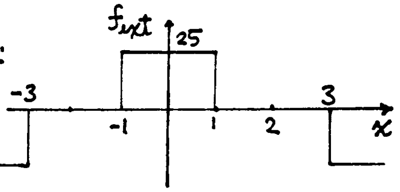
HRS:



$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 25\sin\frac{n\pi x}{2}\,dx = \frac{50}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$

so $f(x) = \frac{50}{\pi}\sum_1^\infty \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$. $(0<x<2)$

QRC:

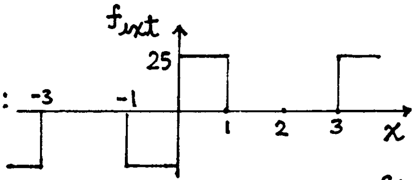


$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{4}\,dx = \int_0^1 25\cos\frac{n\pi x}{4}\,dx = \frac{100}{n\pi}\sin\frac{n\pi}{4}$

so

$f(x) = \frac{100}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\sin\frac{n\pi}{4}\cos\frac{n\pi x}{4}$. $(0<x<2)$

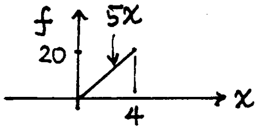
QRS:



$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{4}\,dx = \int_0^1 25\sin\frac{n\pi x}{4}\,dx = \frac{100}{n\pi}\left(1-\cos\frac{n\pi}{4}\right)$

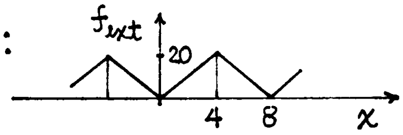
so $f(x) = \frac{100}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\left(1-\cos\frac{n\pi}{4}\right)\sin\frac{n\pi x}{4}$ $(0<x<2)$

(b)



$L = 4.$

HRC:



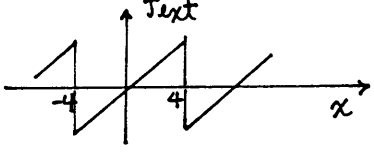
$a_0 = 10$

$a_n = \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx = \frac{1}{2}\int_0^4 5x\cos\frac{n\pi x}{4}\,dx$

$= \frac{40}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} -80/n^2\pi^2, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$

so $f(x) = 10 - \frac{80}{\pi^2}\sum_{1,3,\dots}^\infty \frac{1}{n^2}\cos\frac{n\pi x}{4}$ $(0<x<4)$

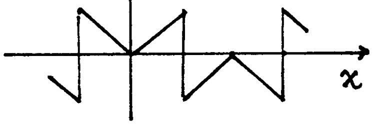
HRS:



$b_n = \frac{2}{4}\int_0^4 5x\sin\frac{n\pi x}{4}\,dx = -\frac{40}{n\pi}(-1)^n$

so $f(x) = -\frac{40}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\sin\frac{n\pi x}{4}$ $(0<x<4)$

QRC:

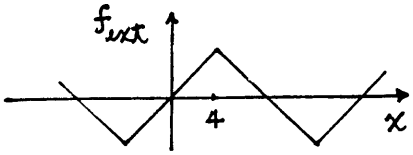


$a_n = \frac{2}{4}\int_0^4 5x\cos\frac{n\pi x}{8}\,dx = \frac{80}{n^2\pi^2}\left[\overset{0 \text{ for } n \text{ odd}}{2\cos\frac{n\pi}{2}} + n\pi\sin\frac{n\pi}{2} - 2\right]$

so

$f(x) = \frac{80}{\pi^2}\sum_{1,3,\dots}^\infty \frac{1}{n^2}\left(n\pi\sin\frac{n\pi}{2} - 2\right)\cos\frac{n\pi x}{8}$. $(0<x<4)$

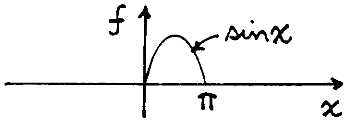
QRS:



$$b_n = \frac{2}{4}\int_0^4 5x\sin\frac{n\pi x}{8}\,dx = \frac{80}{n^2\pi^2}\Big(2\sin\frac{n\pi}{2} - n\pi\underbrace{\cancel{\cos\frac{n\pi}{2}}}_{0 \text{ for } n \text{ odd}}\Big)$$

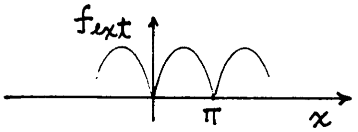
so $f(x) = \frac{160}{\pi^2}\sum_{1,3,\dots}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8} \quad (0<x<4)$

(c)



$L = \pi$.

HRC:

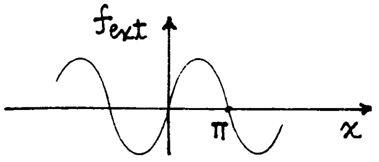


$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x\,dx = 2/\pi$$
$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos\frac{n\pi x}{\pi}\,dx = -\frac{2}{\pi}\,\frac{1+\cos n\pi}{n^2-1}$$

The latter is 0/0 for $n=1$; for that case, l'Hôpital gives $a_1 = 0$. In fact, $a_n = 0$ for all odd n's (due to the $1+\cos n\pi$), so $a_n = \begin{cases} 0, & n \text{ odd} \\ -\frac{4}{\pi}\frac{1}{n^2-1}, & n \text{ even} \end{cases}$

so $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{2,4,\dots}^{\infty} \frac{1}{n^2-1}\cos nx \quad (0<x<\pi)$

HRS:

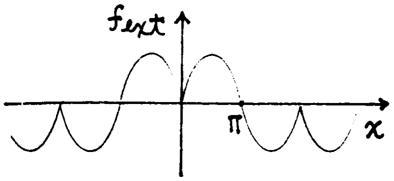


$f_{ext}(x)$ is simply $\sin x$!

so $f(x) = \sin x \quad (0<x<\pi)$

(If we work out the b_n's, using (4), we'd obtain $b_n = 0$ for all $n \neq 1$ and $b_1 = 1$.)

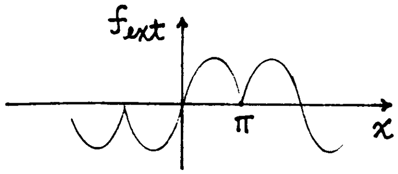
QRC:



$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos\frac{nx}{2}\,dx = -\frac{8}{\pi}\,\frac{\overset{\to 0 \text{ for } n \text{ odd}}{\cancel{\cos\frac{n\pi}{2}}}+1}{n^2-4}$$

so $f(x) = \frac{8}{\pi}\sum_{1,3,\dots}^{\infty} \frac{1}{4-n^2}\cos\frac{nx}{2} \quad (0<x<\pi)$

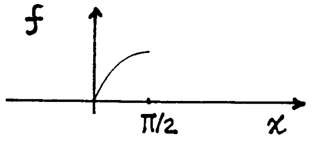
QRS:



$$b_n = \frac{2}{\pi}\int_0^\pi \sin x\sin\frac{nx}{2}\,dx = \frac{8}{\pi}\,\frac{\sin\frac{n\pi}{2}}{4-n^2}$$

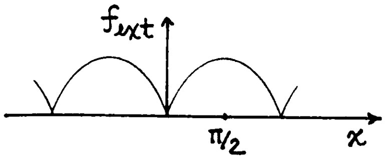
so $f(x) = \frac{8}{\pi}\sum_{1,3,\dots}^{\infty} \frac{\sin\frac{n\pi}{2}}{4-n^2}\sin\frac{nx}{2} \quad (0<x<\pi)$

(d)



$L = \pi/2$.

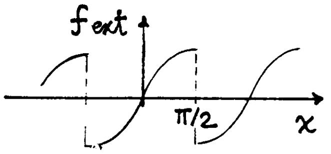
HRC:



$$a_0 = \frac{1}{\pi/2}\int_0^{\pi/2} \sin x\,dx = 2/\pi.$$
$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x\cos 2nx\,dx = -\frac{4}{\pi}\,\frac{1}{4n^2-1}$$

so $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{4n^2-1}\cos 2nx \quad (0<x<\pi/2)$

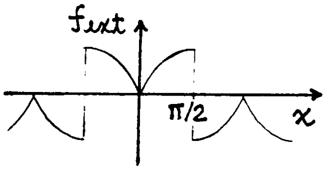
HRS:



$$b_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x \sin 2nx\,dx = -\frac{8}{\pi}\frac{(-1)^n n}{4n^2-1}$$

so $f(x) = -\frac{8}{\pi}\sum_1^\infty \frac{(-1)^n n}{4n^2-1}\sin 2nx \quad (0<x<\pi/2)$

QRC:

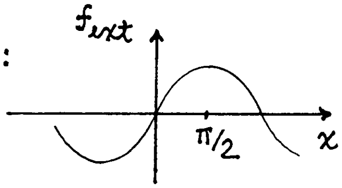


$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x \cos nx\,dx = \frac{4}{\pi}\,\frac{n\sin\frac{n\pi}{2}-1}{n^2-1}$$

The latter gives 0/0 for n=1, and l'Hôpital's rule gives $a_1 = 2/\pi$. Thus, let us write

$$f(x) = \frac{2}{\pi}\cos x + \frac{4}{\pi}\sum_{3,5,\dots}^{\infty} \frac{n\sin\frac{n\pi}{2}-1}{n^2-1}\cos nx \qquad (0<x<\pi/2)$$

QRS:

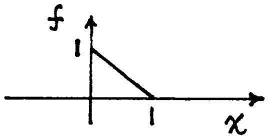


$f_{ext}(x)$ is simply $\sin x$!

so $f(x) = \sin x \quad (0<x<\pi/2)$

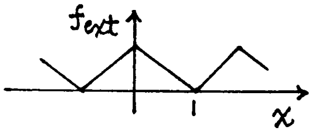
is the QRS expansion of f. That is, it turns out that $b_1=1$, $b_3=b_5=\dots=0$.

(e)



$L=1$

HRC:

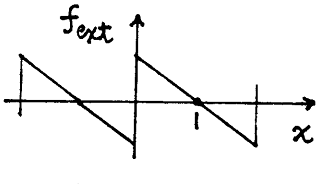


$a_0 = 1/2$

$$a_n = \frac{2}{1}\int_0^1 (1-x)\cos n\pi x\,dx = \frac{2}{n^2\pi^2}(1-\cos n\pi) = \begin{cases} 4/n^2\pi^2, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

so $f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2}\cos n\pi x \quad (0<x<1)$

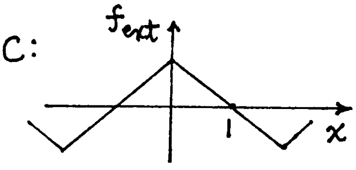
HRS:



$$b_n = \frac{2}{1}\int_0^1 (1-x)\sin n\pi x\,dx = 2/n\pi$$

so $f(x) = \frac{2}{\pi}\sum_1^\infty \frac{1}{n}\sin n\pi x \quad (0<x<1)$

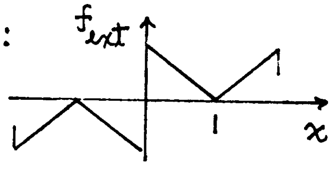
QRC:



$$a_n = \frac{2}{1}\int_0^1 (1-x)\cos\frac{n\pi x}{2}\,dx = \frac{8}{n^2\pi^2}\Big(1-\overset{0 \text{ for } n \text{ odd}}{\cancel{\cos}\frac{n\pi}{2}}\Big)$$

so $f(x) = \frac{8}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$

QRS:



$$b_n = \frac{2}{1}\int_0^1 (1-x)\sin\frac{n\pi x}{2}\,dx = \frac{4}{n^2\pi^2}\Big(n\pi - 2\sin\frac{n\pi}{2}\Big)$$

so $f(x) = \frac{4}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{n\pi-2\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{2} \quad (0<x<1)$

Section 17.5

1. Putting (13) and (14) into (12) gives

$$\sum_1^\infty -(2n-1)^2 b_n \sin(2n-1)t + 0.5\sum_1^\infty b_n \sin(2n-1)t = \frac{8F}{\pi^2}\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)t$$

so, by equating coefficients of $\sin(2n-1)t$ terms, we have

$$b_n = \frac{8F}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2[0.5-(2n-1)^2]} = \frac{8F}{\pi^2}\frac{(-1)^n}{(2n-1)^2[(2n-1)^2-0.5]},$$

in agreement with (15).

2. We use the Weierstrass M-test in each case.

(a) $|a_n(x)| = |e^{-nx}\sin nx| \le e^{-nx} \le e^{-2n}$ on $2<x<5$. Since $\sum_1^\infty e^{-2n} = \sum (e^{-2})^n$ is a convergent geometric series (since $e^{-2}<1$), the original series is uniformly convergent on $2<x<5$ (indeed, on $x_0 \le x < \infty$ for any $x_0 > 0$).

(b) $|a_n(x)| = \left|\frac{\sin 2nx}{n^2-2n+2}\right| \le \frac{1}{n^2-2n+2} \sim \frac{1}{n^2}$ as $n\to\infty$. Now, $\sum_1^\infty \frac{1}{n^2}$ is a convergent p-series (RECALL: The p-series is given in the footnote on pg 875.) since $p=2>1$, so the original series converges uniformly on $-\infty<x<\infty$.

(c) $|a_n(x)| = |e^{-nx}/(n^2+5)| \le 1/(n^2+5)$ on $0<x<\infty$ (indeed, on $0\le x<\infty$) as $1/(n^2+5) \sim 1/n^2$. Now, $\sum_1^\infty 1/n^2$ is a convergent p-series since $p=2>1$, so the original series converges uniformly on the interval.

(d) $|a_n(x)| = |1/(n^2+x^2)| \le 1/n^2$ on $-\infty<x<\infty$. Since $\sum_3^\infty 1/n^2$ is a convergent p-series ($p=2>1$), the given series is uniformly convergent on the interval.

(e) As $n\to\infty$, $\ln(1+x^3/n^3) \sim x^3/n^3$. Thus, on any interval $a<x<b$ we have $|\ln(1+x^3/n^3)| \le C/n^3$ for all sufficiently large n's, where $C = 2\max(|a|^3, |b|^3)$, say. Finally, $\sum 1/n^3$ is a convergent p-series ($p=3>1$) so the given series converges uniformly on $a<x<b$ for any interval $a<x<b$.

2. $|n^{-x}| \le n^{-x_0}$ for $x_0 \le x < \infty$. Now, $\sum_1^\infty n^{-x_0}$ is a convergent p-series for $p = x_0 > 1$, so, by the Weierstrass M-test, the given series converges uniformly on $x_0 \le x < \infty$.

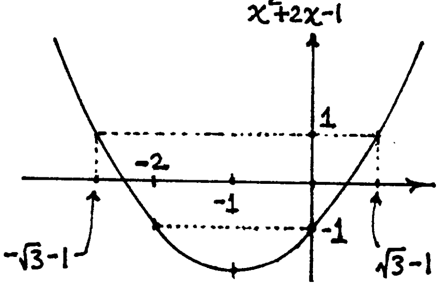
3. (a) $x_0 \le x < \infty$ for any $x_0 > 1$. Alternatively, $-\infty < x < x_0$ for any $x_0 < -1$. Or, on any interval contained in either of these.

(b) Let $0<a<b<\pi$, say. Then, on $a\leq x\leq b$ we have $|a_n(x)|=|n\cos^n x|<$ nK^n where $K=\max\{|\cos a|,|\cos b|\}$. Now use the ratio test for the series ΣnK^n. $\lim\limits_{n\to\infty}\frac{(n+1)K^{n+1}}{nK^n}=\lim\limits_{n\to\infty}K=K<1$, so the given series $\Sigma_1^\infty n(\cos x)^n$ is "dominated" by the positive series of constants $\Sigma_1^\infty nK^n$. Thus, by the Weierstrass M-test the original series converges uniformly on $a\leq x\leq b$.

(c) Let $\pi/2<a<b<3\pi/2$, say. Then, on $a\leq x\leq b$ we have $|a_n(x)|=|n^5(\sin x)^n|$ $\leq n^5K^n$ where $K=\max\{|\sin a|,|\sin b|\}$. Now use the ratio test for the series Σn^5K^n. $\lim\limits_{n\to\infty}\frac{(n+1)^5K^{n+1}}{n^5K^n}=\lim\limits_{n\to\infty}K=K<1$, so the given series $\Sigma_1^\infty n^5(\sin x)^n$ is "dominated" by the series of positive constants $\Sigma_1^\infty n^5K^n$. Thus, by the Weierstrass M-test the original series converges uniformly on $a\leq x\leq b$. Similarly for $3\pi/2<a<b<5\pi/2$, and so on.

(e) On $-1\leq x\leq 1$ we have $|a_n(x)|=|x^n/n^3|\leq 1/n^3$. But $\Sigma\, 1/n^3$ is a convergent p-series, with $p=3>1$. Thus, by the Weierstrass M-test the series $\Sigma_1^\infty x^n/n^3$ converges uniformly on $-1\leq x\leq 1$.

(f) $\frac{d}{dx}(x^2+2x-1)=2x+2=0$ at $x=-1$ and the graph of x^2+2x-1 is as shown at the right. Let $-\sqrt{3}-1<a<b<-2$. Then, on $a\leq x\leq b$ we have
$|a_n(x)|=|x^2+2x-1|^n\leq K^n$ where $K=\max\{|a^2+2a-1|,|b^2+2b-1|\}$. Since $K<1$ the series ΣK^n is a convergent (geometric) series. Hence, by the Weierstrass M-test, the series $\Sigma_1^\infty(x^2+2x-1)^n$ is uniformly convergent on $a\leq x\leq b$. Similarly if $0<a<b<\sqrt{3}-1$.



(g) Let $1-\sqrt{2}<a<b<1$. Then the series is uniformly convergent on $a\leq x\leq b$. Similarly if $1<a<b<1+\sqrt{2}$.

4. (b) $\frac{d}{dx}\left(4+\Sigma_1^\infty\frac{\cos n\pi x}{n^3+1}\right)=0+\Sigma_1^\infty\frac{d}{dx}\frac{\cos n\pi x}{n^3+1}=-\Sigma_1^\infty\frac{n\pi}{n^3+1}\sin n\pi x$ ♀

Now, $|(n\pi/n^3+1)\sin n\pi x|\leq n\pi/(n^3+1)\sim\pi/n^2$ and since $\Sigma\pi/n^2=\pi\Sigma\, 1/n^2$ is a convergent p-series (with $p=2>1$) it follows from the Weierstrass M-test that the last series in ♀ is uniformly convergent on $-\infty<x<\infty$ and then from Theorem 17.5.2 that the termwise differentiation in ♀ is justified.

(c) $\frac{d}{dx}\sum_1^\infty (4x)^n = \sum_1^\infty \frac{d}{dx}(4x)^n = \sum_1^\infty 4^n n x^{n-1}$ ♀

Now, $|4^n n x^{n-1}| \le 4^n n (0.2)^{n-1} = 5n(0.8)^n$ on $-0.2 \le x \le 0.1$.
$\sum 5n(0.8)^n$ converges by the ratio test because $\lim_{n\to\infty} \frac{5(n+1)(0.8)^{n+1}}{5n(0.8)^n} = 0.8 < 1$.

Thus, by the Weierstrass M-test, the last series in ♀ is uniformly convergent on the interval. Thus, by Theorem 17.5.2 the termwise differentiation in ♀ is justified.

(d) $\frac{d}{dx}\sum_1^\infty \frac{x^n}{n^3} = \sum_1^\infty \frac{d}{dx}\left(\frac{x^n}{n^3}\right) = \sum_1^\infty \frac{x^{n-1}}{n^2}$ ♀

Now, $|x^{n-1}/n^2| \le 1/n^2$ on $-1 \le x \le 1$.
$\sum 1/n^2$ is a convergent p-series (with $p = 2 > 1$). Thus, by the Weierstrass M-test the last series in ♀ is uniformly convergent on the interval. Thus, by Theorem 17.5.2 the termwise differentiation in ♀ is justified.

5. (b) First, expand $f(t) = |\cos t| = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\cos 2nt$
so $x'' + x' + x = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\cos 2nt$. ♀

A particular solution of ♀ due to the $2/\pi$ term is $x_p(t) = 2/\pi$ and a particular solution due to the $\cos 2nt$ term is

$$x_p(t) = \frac{(1-4n^2)\cos 2nt + 2n\sin 2nt}{(1-4n^2)^2 + 4n^2},$$

so a particular solution to ♀ is

$$x_p(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\frac{(1-4n^2)\cos 2nt + 2n\sin 2nt}{(1-4n^2)^2+4n^2} \qquad *$$

To verify that $*$ satisfies (5.1) use termwise differentiation to obtain

$$x_p'(t) = -\frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\frac{-2n(1-4n^2)\sin 2nt + 4n^2\cos 2nt}{(1-4n^2)^2+4n^2} \qquad **$$

$$x_p''(t) = -\frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\frac{-4n^2(1-4n^2)\cos 2nt - 8n^3\sin 2nt}{(1-4n^2)^2+4n^2}. \qquad ***$$

Then, summing $*$, $**$, $***$ gives

$$-\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n}{4n^2-1}\left\{\frac{[-8\cancel{n^3} - \cancel{2n} + 8\cancel{n^3} + \cancel{2n}]\sin 2nt + [-4n^2 + 16n^4 + \cancel{4n^2} + 1 - \cancel{4n^2}]\cos 2nt}{16n^4 - 4n^2 + 1}\right\}$$

which does agree with the right side of ♀. Next, let us justify the termwise differentiation in $**$:

$$|a_n(t)| = \left| -\frac{4}{\pi}\frac{(-1)^n}{4n^2-1}\,\frac{-2n(1-4n^2)\sin 2nt + 4n^2\cos 2nt}{(1-4n^2)^2+4n^2} \right|$$

$$\leq \frac{4}{\pi}\,\frac{2n(1+4n^2)+4n^2}{(4n^2-1)[(1-4n^2)^2+4n^2]} \qquad \text{for all } t$$

$$\sim \frac{4}{\pi}\,\frac{8n^3}{4n^2(16n^4)} = \frac{1}{2\pi}\,\frac{1}{n^3} \quad \text{as } n\to\infty .$$

Since $\frac{1}{2\pi}\sum^{\infty} 1/n^3$ is a convergent series of constants (a p-series with $p=3>1$) the series on the right side of ** does indeed converge uniformly for $0 \leq t < \infty$ by the Weierstrass M-test, and so the termwise differentiation in ** is justified by Theorem 17.5.2. Similarly for ***, with $p=2$ instead of 3.

6. (a) To apply Theorem 17.5.4 we need to first show that the given series is a Fourier series. As noted in the middle of pg 877, if a trigonometric series with period 2ℓ converges uniformly on $[-\ell, \ell]$ then it is the Fourier series of its sum function. Indeed, the given series does converge uniformly because it is "dominated" by the convergent series of constants

$$\frac{4}{\pi}\sum_1^{\infty}\frac{1}{(2n-1)^2}.$$

Thus, $\int_0^1 \frac{4}{\pi}\sum_1^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}\,dx = \frac{4}{\pi}\sum_1^{\infty}\int_0^1\frac{\cos(2n-1)x}{(2n-1)^2}\,dx = \frac{4}{\pi}\sum_1^{\infty}\frac{\sin(2n-1)}{(2n-1)^3}$

$$= 1.2732\big(0.84147 + 0.00523 - 0.00767 - 0.00192 + 0.00056 - 0.00075 - 0.00019 - 0.00019 - 0.00020 + 0.00002 + \cdots\big)$$

$$\approx 1.2732(0.8364) \approx 1.06$$

Though not asked for, let us evaluate the integral using these Maple commands:

```
f(x):= sum(cos((2*i-1)*x)/(2*i-1)^2, i=1..5):
int((4/Pi)*f(x), x=0..1);
evalf(");
```

Doing this with i=1..5, then 1..10, then 1..20, then 1..40, then 1..80 gives the results

5: 1.071440
10: 1.070751
20: 1.070804
40: 1.070797
80: 1.070797

Thus, whereas it appeared that the 1.06 was correct to the three significant figures we see that the answer is actually 1.07079.

Section 17.6

1.(a) $$\|\underset{\sim}{E}\|^2 = \langle \underset{\sim}{E},\underset{\sim}{E}\rangle = \int_{-\ell}^{\ell}\left[f(x)-a_0-\sum_{n=1}^{k}\left(a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right)\right]$$
$$\times\left[f(x)-a_0-\sum_{m=1}^{k}\left(a_m\cos\frac{m\pi x}{\ell}+b_m\sin\frac{m\pi x}{\ell}\right)\right]dx$$

Writing out the product and using the following,

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} f\,dx = a_0,\quad \frac{1}{\ell}\int_{-\ell}^{\ell} f\cos\frac{n\pi x}{\ell}\,dx = a_n,\quad \frac{1}{\ell}\int_{-\ell}^{\ell} f\sin\frac{n\pi x}{\ell}\,dx = b_n,$$

$$\int_{-\ell}^{\ell}\cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx = \begin{cases}2\ell, & m=n=0\\ 0, & m\neq n\\ \ell, & m=n\neq 0\end{cases}$$

$$\int_{-\ell}^{\ell}\sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \begin{cases}0, & m\neq n\\ \ell, & m=n\end{cases}$$

gives $$\|\underset{\sim}{E}\|^2 = \int_{-\ell}^{\ell} f^2dx - a_0\int_{-\ell}^{\ell} f\,dx - \sum_{m=1}^{k}\left(a_m\int_{-\ell}^{\ell} f\cos\frac{m\pi x}{\ell}\,dx + b_m\int_{-\ell}^{\ell} f\sin\frac{m\pi x}{\ell}\,dx\right)$$
$$-a_0\int_{-\ell}^{\ell} f\,dx + a_0^2\int_{-\ell}^{\ell} dx + a_0\sum_{m=1}^{k}\int_{-\ell}^{\ell}\left(a_m\overset{0}{\cancel{\cos}}\frac{m\pi x}{\ell}+b_m\overset{0}{\cancel{\sin}}\frac{m\pi x}{\ell}\right)dx$$
$$-\sum_{n=1}^{k}\left(a_n\int_{-\ell}^{\ell} f\cos\frac{n\pi x}{\ell}\,dx + b_n\int_{-\ell}^{\ell} f\sin\frac{n\pi x}{\ell}\,dx\right) + a_0\sum_{n=1}^{k}\int_{-\ell}^{\ell}\left(a_n\overset{0}{\cancel{\cos}}\frac{n\pi x}{\ell}+b_n\overset{0}{\cancel{\sin}}\frac{n\pi x}{\ell}\right)dx$$
$$+\sum_{n=1}^{k}\left(a_n^2\int_{-\ell}^{\ell}\cos^2\frac{n\pi x}{\ell}\,dx + b_n^2\int_{-\ell}^{\ell}\sin^2\frac{n\pi x}{\ell}\,dx\right)$$

$$= \int_{-\ell}^{\ell} f^2dx - 2\ell a_0^2 - \ell\sum_{1}^{k}(a_n^2+b_n^2) - 2\ell a_0^2 + 2\ell a_0^2$$
$$\cancel{-\ell\sum_1^k(a_n^2+b_n^2)} + \cancel{\ell\sum_1^k(a_n^2+b_n^2)} = \int_{-\ell}^{\ell} f^2(x)dx - \ell\left[2a_0^2+\sum_1^k(a_n^2+b_n^2)\right]$$

(b) Since $\|\underset{\sim}{E}\|^2 \geqslant 0$, the right-hand side of (1.2) is $\geqslant 0$. Thus,

$$\int_{-\ell}^{\ell} f^2(x)dx - \ell\left[2a_0^2+\sum_1^k(a_n^2+b_n^2)\right] \geqslant 0$$

so $$2a_0^2+\sum_1^k(a_n^2+b_n^2) \leqslant \frac{1}{\ell}\int_{-\ell}^{\ell} f^2(x)dx \quad ✓$$

(for each $k=1,2,\ldots$).

2. In the results to follow we should find that $\|\underset{\sim}{E}\|$ decreases (or stays the same) as k is increased (with $\|\underset{\sim}{E}\|$ tending to 0 as $k\to\infty$).

(a) $f(x)=|x|$, $\ell=\pi$, $a_0=\pi/2$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos\frac{n\pi x}{\pi}dx = \frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx = \begin{cases}0, & n \text{ even}\\ -\frac{4}{n^2\pi}, & n \text{ odd}\end{cases}$, $b_n=0$

and $\int_{-\pi}^{\pi}|x|^2dx = 2\pi^3/3$

so (1.2) gives

$$\|\underset{\sim}{E}\|^2 = \frac{2\pi^3}{3} - \pi\left[2\frac{\pi^2}{4} + \sum_{n=1,3,\ldots}^{k}\frac{16}{n^4\pi^2}\right]$$

$$k=1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cdots$$
$$= \left(\tfrac{2}{3}-\tfrac{1}{2}\right)\pi^3 - \tfrac{16}{\pi}\left(\tfrac{1}{1^4}+0+\tfrac{1}{3^4}+0+\tfrac{1}{5^4}+0+\tfrac{1}{7^4}+0+\cdots\right)$$

$= 0.0747,\ 0.0747,\ 0.0118,\ 0.0118,\ 0.0037,\ 0.0037,\ 0.0015,\ 0.0015, \ldots$

so $\|\underset{\sim}{E}\| = 0.27,\ 0.27,\ 0.11,\ 0.11,\ 0.06,\ 0.06,\ 0.04,\ 0.04, \ldots$

(b) $f(x)=x$, $\ell=\pi$, $a_0=a_n=0$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$, $\int_{-\pi}^{\pi} x^2 dx = 2\pi^3/3$

so (1.2) gives $\|\underset{\sim}{E}\|^2 = \frac{2\pi^3}{3} - \pi\left[0+\sum_1^k \frac{4}{n^2}\right]$,

$$k=1 \quad 2 \quad 3 \quad \cdots$$
$$\|\underset{\sim}{E}\|^2 = 20.67085 - 12.56637\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\cdots\ \right)$$

$= 8.1045,\ 4.9629,\ 3.5666,\ 2.7812,\ 2.2786,\ 1.9295,\ 1.6730,\ 1.4767, \ldots$

$\|\underset{\sim}{E}\| = 2.847,\ 2.228,\ 1.889,\ 1.668,\ 1.510,\ 1.389,\ 1.293,\ 1.215, \ldots$

Though not asked for, let us use Maple to carry this much further, so we can "see" that $\|\underset{\sim}{E}\| \to 0$ as $k \to \infty$. The commands

```
S: = sum(4/i^2, i=1..8):
e: = sqrt(2*Pi^3/3 - Pi*S);
evalf(");
```

give 1.215, as above. Now increase k and rerun:

k :	8	10^2	10^4	10^6	10^8	10^{10}	
$\|\underset{\sim}{E}\|$:	1.2152	0.3536	0.0354	0.0035	0.0003	0.0001	etc

(c) $f(x)=\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$ so $a_0 = 1/2$, $a_2 = 1/2$, all other a_n's and b_n's $=0$.
$\int_{-\pi}^{\pi} f^2 dx = \int_{-\pi}^{\pi}\cos^4 x\,dx = 3\pi/4$, so (1.2) gives

$$\|\underset{\sim}{E}\|^2 = \frac{3\pi}{4} - \pi\left[2\left(\tfrac{1}{4}\right)+0+\tfrac{1}{4}+0+0+\cdots\right]$$

so for $k=1$, $\|\underset{\sim}{E}\|^2 = \pi/4$, $\|\underset{\sim}{E}\| = 0.8862$

$k \geq 2$, $\|\underset{\sim}{E}\|^2 = 0$, $\|\underset{\sim}{E}\| = 0$.

(d) $f(x)=x^2$, $\ell=\pi$, $a_0=\pi^2/3$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{4}{n^2}(-1)^n$, b_n's $=0$,
$\int_{-\pi}^{\pi} f^2 dx = \int_{-\pi}^{\pi} x^4 dx = 2\pi^5/5$

so (1.2) gives $\|\underset{\sim}{E}\|^2 = \frac{2\pi^5}{5} - \pi\left[2\frac{\pi^4}{9}+\sum_1^k \frac{16}{n^4}\right]$.
Let us use Maple.

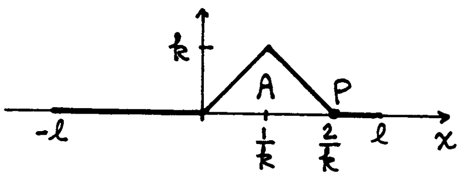
```
S:= sum(16/i^4, i=1..k):
e:= sqrt(8*Pi^5/45 - Pi*S);
evalf(");
```

Running this for various k's gives

k :	2	4	6	8	50	200
$\|\underset{\sim}{E}\|$:	0.998	0.424	0.246	0.165	0.011	0.001

3. $\|f-g\| = \sqrt{\int_{-\ell}^{\ell}[f(x)-g(x)]^2\,dx} = 0$ since $f(x)=g(x)$ everywhere on $[-\ell,\ell]$ except perhaps at a finite number of points, where $f(x)-g(x)$, though nonzero, is, at least, finite.

4. (a) Clearly, $\lim_{k\to\infty} F_k(x) = 0$ for $-\ell \le x \le 0$. For each fixed x in $0 \le x \le \ell$ it is also true that $\lim_{k\to\infty} F_k(x) = 0$ since, no matter how small x is, k eventually reaches some value, say K, such that $F_k(x) = 0$ for all $k \ge K$. That is, for all $k \ge K$ the "knee" P is to the left of x. Next, observe that $\int_{-\ell}^{\ell} F_k(x)\,dx =$ the area $A = (2)(\tfrac12)(k)(\tfrac1k) = 1$ for all $k = 1,2,3,\dots$, so $\lim_{k\to\infty}\int_{-\ell}^{\ell} F_k(x)\,dx = \lim_{k\to\infty} 1 = 1$.



(b) $\lim_{k\to\infty}\int_{-\ell}^{\ell} F_k(x)\,dx = \lim_{k\to\infty}\frac{1}{k} = 0$, whereas $\lim_{k\to\infty} F_k(x) = \begin{cases} 0, & -\ell \le x < 0 \text{ and } 0 < x \le \ell \\ 1, & x = 0 \end{cases}$

(c) $\lim_{k\to\infty}\int_{-\ell}^{\ell} F_k(x)\,dx = \lim_{k\to\infty}(\tfrac1k)(\tfrac1k) = 0$ and $\lim_{k\to\infty} F_k(x) = 0$ on $-\ell \le x \le \ell$ too.

5. $$\|\underset{\sim}{E}\|^2 = \int_{-\ell}^{\ell}\Big[f(x) - a_0 - \sum_{n=1}^{k}\Big(a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\Big)\Big]^2 dx$$

so $$\partial\|\underset{\sim}{E}\|^2/\partial a_0 = (2)(-1)\int_{-\ell}^{\ell}\Big[f(x) - a_0 - \sum_{1}^{k}\Big(a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\Big)\Big]dx$$

$$= -2\Big[\int_{-\ell}^{\ell} f(x)\,dx - 2a_0\ell\Big] = 0 \text{ gives } a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx,$$

$$\partial\|\underset{\sim}{E}\|^2/\partial a_n = (2)(-1)\int_{-\ell}^{\ell}\Big[f(x) - a_0 - \underbrace{\sum_{m=1}^{k}\Big(a_m\cos\frac{m\pi x}{\ell} + b_m\sin\frac{m\pi x}{\ell}\Big)}_{\substack{\text{Important to let the dummy index in here \underline{not} be } n,\\ \text{which is regarded as \underline{fixed}.}}}\Big]\cos\frac{n\pi x}{\ell}\,dx$$

$$= -2\Big[\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx - a_n\ell\Big] = 0 \text{ gives } a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx,$$

$$\partial\|\underset{\sim}{E}\|^2/\partial b_n = (2)(-1)\int_{-\ell}^{\ell}\Big[f(x) - a_0 - \sum_{m=1}^{k}\Big(a_m\cos\frac{m\pi x}{\ell} + b_m\sin\frac{m\pi x}{\ell}\Big)\Big]\sin\frac{n\pi x}{\ell}\,dx$$

$$= -2\Big[\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx - b_n\ell\Big] = 0 \text{ gives } b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx,$$

as the optimal values of a_0, a_n, b_n.

Section 17.7

1. (b) $y(x) = \begin{cases} A+Bx, & \lambda=0 \\ C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x, & \lambda\neq 0 \end{cases}$

$\lambda=0$: $y'(0)=0=B$

$y(L)=0=A+BL$ so $A=B=0$, $y(x)=0$, $\lambda=0$ is not an eigenvalue.

$\lambda\neq0$: $y'(0)=0=0C+\sqrt{\lambda}D \rightarrow D=0$

$y(L)=0=C\cos\sqrt{\lambda}L \rightarrow \cos\sqrt{\lambda}L=0$, $\lambda=(n\pi/2L)^2$, $n=1,3,\dots$

so $\lambda_n=(n\pi/2L)^2$, $\phi_n(x)=\cos\frac{n\pi x}{2L}$, $n=1,3,\dots$.

Then

$$f(x)=\sum\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}\phi_n(x)$$

gives

$$f(x)=1=\sum_{1,3,\dots}^{\infty}a_n\cos\frac{n\pi x}{2L} \quad\text{where } a_n=\frac{\int_0^L(1)\cos\frac{n\pi x}{2L}dx}{\int_0^L\cos^2\frac{n\pi x}{2L}dx}=\frac{\frac{L}{2n\pi}\sin\frac{n\pi}{2}}{\frac{L}{2}}=\frac{\sin\frac{n\pi}{2}}{n\pi}$$

(d) $y(x) = \begin{cases} A+Bx, & \lambda=0 \\ C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x, & \lambda\neq 0 \end{cases}$

$\lambda=0$: $y'(0)=0=B$ so $y(x)=A$

$y(L)+y'(L)=0=A+0$ so $y(x)=0$; $\lambda=0$ is not an eigenvalue.

$\lambda\neq0$: $y'(0)=0=0+\sqrt{\lambda}D \rightarrow D=0$ so $y(x)=C\cos\sqrt{\lambda}x$

$y(L)+y'(L)=0=C(\cos\sqrt{\lambda}L-\sqrt{\lambda}\sin\sqrt{\lambda}L)$ gives $\sqrt{\lambda}=\cot\sqrt{\lambda}L$. Denote the positive roots as $\lambda_1,\lambda_2,\dots$. $\phi_n(x)=\cos\sqrt{\lambda_n}x$, $n=1,2,\dots$.

Then

$$f(x)=50=\sum_1^{\infty}a_n\cos\sqrt{\lambda_n}x \quad\text{where } a_n=\frac{\langle 50,\cos\sqrt{\lambda_n}x\rangle}{\langle\cos\sqrt{\lambda_n}x,\cos\sqrt{\lambda_n}x\rangle}=\frac{\int_0^L 50\cos\sqrt{\lambda_n}x\,dx}{\int_0^L\cos^2\sqrt{\lambda_n}x\,dx}$$

$$=\frac{100\sin\sqrt{\lambda_n}L}{\cos\sqrt{\lambda_n}L\sin\sqrt{\lambda_n}L+\sqrt{\lambda_n}L}$$

(f) $y(x) = \begin{cases} A+Bx, & \lambda=0 \\ C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x, & \lambda\neq 0 \end{cases}$

$\lambda=0$: $y'(-1)=0=B$

$y'(1)=0=B$ so $y(x)=A$. Thus, $\lambda_0=0$, $\phi_0(x)=1$.

$\lambda\neq0$: $y'(-1)=-\sqrt{\lambda}C\sin(-\sqrt{\lambda})+\sqrt{\lambda}D\cos(-\sqrt{\lambda})=0$

$y'(1)=-\sqrt{\lambda}C\sin(\sqrt{\lambda})+\sqrt{\lambda}D\cos(\sqrt{\lambda})=0$

or, since $\lambda\neq0$, $\quad C\sin\sqrt{\lambda}+D\cos\sqrt{\lambda}=0$

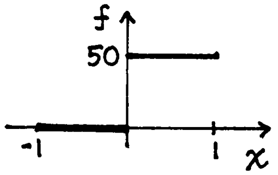
$\quad -C\sin\sqrt{\lambda}+D\cos\sqrt{\lambda}=0$

or, $\begin{pmatrix}\sin\sqrt{\lambda} & \cos\sqrt{\lambda}\\ -\sin\sqrt{\lambda} & \cos\sqrt{\lambda}\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}=\underset{\sim}{0}$ $\quad *$

Set det $=2\sin\sqrt{\lambda}\cos\sqrt{\lambda}=\sin2\sqrt{\lambda}=0\rightarrow 2\sqrt{\lambda}=\overset{\lambda\neq0\text{ here}}{\cancel{0}},\pi,2\pi,\dots$, $\sqrt{\lambda_n}=\frac{n\pi}{2}$.

Now put these λ's into $*$ and solve for the nontrivial C,D's.

Odd n's give C=0, D=arbitrary so $\phi_n(x) = \sin\frac{n\pi x}{2}$ for n=1,3,...
Even n's give C=arb., D=0 so $\phi_n(x) = \cos\frac{n\pi x}{2}$ for n=2,4,...
Summarizing, $\lambda_0 = 0,\ \phi_0(x) = 1$
$\lambda_n = (n\pi/2)^2,\ \phi_n(x) = \sin\frac{n\pi x}{2}$ for n odd
$+\cos\frac{n\pi x}{2}$ for n even



$$f(x) = a_0 \cdot 1 + \sum_{1,3,\dots}^{\infty} b_n \sin\frac{n\pi x}{2} + \sum_{2,4,\dots}^{\infty} c_n \cos\frac{n\pi x}{2}$$

where

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1,1\rangle} = \frac{\int_{-1}^{1} f(x)\,dx}{\int_{-1}^{1} 1^2\,dx} = \frac{1}{2}\int_0^1 50\,dx = 25$$

$$b_n = \frac{\langle f, \sin n\pi x/2\rangle}{\langle \sin\frac{n\pi x}{2}, \sin\frac{n\pi x}{2}\rangle} = \frac{\int_0^1 50 \sin n\pi x/2\,dx}{\int_{-1}^{1}\sin^2\frac{n\pi x}{2}\,dx} = \frac{100}{n\pi}$$

$$c_n = \frac{\langle f, \cos n\pi x/2\rangle}{\langle \cos\frac{n\pi x}{2}, \cos\frac{n\pi x}{2}\rangle} = \frac{\int_0^1 50\cos\frac{n\pi x}{2}\,dx}{\int_{-1}^{1}\cos^2\frac{n\pi x}{2}\,dx} = 0$$

so $f(x) = 25 + \frac{100}{\pi}\sum_{1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$

2. (a) $y'(0)=0,\ y'(L)=0$
(b) $y(0)=0,\ y'(L)=0$
(c) $y'(0)=0,\ y(L)=0$

3. Multiply (3.1) by $\sigma(x)$ where $\sigma(x) \neq 0$ on the interval:
$\sigma A y'' + \sigma B y' + \sigma C y + \lambda \sigma D y = 0$
We need $\sigma B = (\sigma A)'$
$= \sigma' A + \sigma A'$, $\sigma' = \frac{B-A'}{A}\sigma$, $\frac{d\sigma}{\sigma} = \frac{B-A'}{A}dx$, $\sigma(x) = e^{\int \frac{B-A'}{A}dx}$

4. (a) $5 \neq d(x)/dx = 1$ so use $\sigma(x) = \exp\int\frac{5-1}{x}dx = e^{4\ln x} = e^{\ln x^4} = x^4$.
$x^5 y'' + 5x^4 y' + \lambda x^5 y = (x^5 y')' + 0y + \lambda x^5 y = 0$
so $p(x) = x^5,\ q(x) = 0,\ w(x) = x^5$.
(b) $2 \neq d(1)/dx = 0$ so use $\sigma(x) = \exp\int\frac{2-0}{1}dx = e^{2x}$
$e^{2x}y'' + 2e^{2x}y' + xe^{2x}y + \lambda x^2 e^{2x} y = 0,$
$(\underbrace{e^{2x}}_{p(x)} y')' + \underbrace{xe^{2x}}_{q(x)} y + \lambda \underbrace{x^2 e^{2x}}_{w(x)} y = 0$
(c) $1 \neq d(1)/dx = 0$ so use $\sigma(x) = \exp\int\frac{1-0}{1}dx = e^x$
$e^x y'' + e^x y' + \lambda e^x y = (\underbrace{e^x}_{p(x)} y')' + \underbrace{0}_{q(x)} y + \lambda \underbrace{e^x}_{w(x)} y = 0$
(d) $\sigma(x) = e^{-x}$, $e^{-x}y'' - e^{-x}y' + \lambda x e^{-x} y = (\underbrace{e^{-x}}_{p(x)} y')' + \underbrace{0}_{q(x)} y + \lambda \underbrace{xe^{-x}}_{w(x)} y = 0$

5. From Fig. 3 we see that the roots are just to the right of $\sqrt{\lambda}=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, i.e., just to the right of $\lambda \approx 2.47, 22.2, 61.6, 120, 199, 298, 416, 555, \ldots$. This information helps us bracket the search intervals. For instance,

`fsolve (tan(sqrt(x)) + 2 * sqrt(x) = 0, x, 2.47..5 );`

gives $\lambda_1 = 3.373089$. Similarly, we obtain

$\lambda_2 = 23.19234,$

$\lambda_3 = 62.67972, \quad \lambda_5 = 200.8578, \quad \lambda_7 = 417.9900$

$\lambda_4 = 121.8999, \quad \lambda_6 = 299.5544, \quad \lambda_8 = 555.1646$, etc.

6. There was an error in the 1st printing of this second edition. Problem 6 should read as follows:

Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad (0<x<1)$$

$$2y(0) - y(1) + 4y'(1) = 0, \quad y(0) + 2y'(1) = 0.$$

Explain why the latter is not of Sturm-Liouville type. With the help of computer software, determine any two eigenvalues. HINT: You should obtain the characteristic equation $\sin\sqrt{\lambda} = 2\sqrt{\lambda}$. Although the latter has no roots on a real $\sqrt{\lambda}$ axis, we need to search in the complex plane. With $z = x+iy$ write $\sin z = 2z$, use the identity $\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ and obtain the equations

$$\sin x \cosh y = 2x, \quad \cos x \sinh y = 2y.$$

Then, use computer software to find any two solutions for x and y, and hence for λ.

Solution:

$$y(x) = \begin{matrix} A + Bx, & \lambda = 0 \\ C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x, & \lambda \neq 0 \end{matrix}$$

$\lambda=0$: $\left.\begin{matrix} 2y(0) - y(1) + 4y'(1) = 0 = 2A - (A+B) + 4B = A + 3B \\ y(0) + 2y'(1) = 0 = A + 2B \end{matrix}\right]$ These give $A=B=0$, so $\lambda=0$ is not an eigenvalue.

$\lambda \neq 0$: $2y(0) - y(1) + 4y'(1) = 0 = 2C - (C\cos\sqrt{\lambda} + D\sin\sqrt{\lambda}) + 4(-\sqrt{\lambda}C\sin\sqrt{\lambda} + \sqrt{\lambda}D\cos\sqrt{\lambda})$

$y(0) + 2y'(1) = 0 = C + 2(-\sqrt{\lambda}C\sin\sqrt{\lambda} + \sqrt{\lambda}D\cos\sqrt{\lambda})$

or, $\begin{pmatrix} 2 - c - 4\sqrt{\lambda}s & -s + 4\sqrt{\lambda}c \\ 1 - 2\sqrt{\lambda}s & 2\sqrt{\lambda}c \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0}$ where "c" $\equiv \cos\sqrt{\lambda}$ and "s" $\equiv \sin\sqrt{\lambda}$.

Determinant $= 4\cancel{\sqrt{\lambda}c} - 2\sqrt{\lambda}c^2 - 8\cancel{\lambda c}s + s - 4\cancel{\sqrt{\lambda}c} - 2\sqrt{\lambda}s^2 + 8\cancel{\lambda c}s = 0$

or $\sin\sqrt{\lambda} = 2\sqrt{\lambda}$, the graph of which is seen to give no intersections (roots) except $\lambda=0$, which case was handled separately and ruled out. However, we really need to search for roots in the entire complex plane (unlike the Sturm-Liouville case, where we know that λ is real so $\sqrt{\lambda}$ is either purely real or imaginary; the present problem is not of Sturm-Liouville form because the b.c.'s are not separated). Thus, consider $\sin z = 2z$ and let $z = x+iy$.

$$\sin(x+iy) = 2(x+iy)$$

$$\sin x \cosh y + i \sinh y \cos x = 2x + i2y$$

so equating real and imaginary parts gives

$$\sin x \cosh y = 2x, \quad ①$$

$$\cos x \sinh y = 2y. \quad ②$$

Observing that $x=0$ satisfies ①, ② becomes $\sinh y = 2y$ and the Maple command

```
fsolve( sinh(y)= 2*y, y, 0..4);
```

gives $y = 2.177318985$, so $\sqrt{\lambda} = z = x+iy \approx 0 + 2.17732i$

$$\lambda \approx -4.7407.$$

Let us seek at least one more root by applying fsolve to ① and ② in various rectangular regions.

```
fsolve({sin(x)*cosh(y)=2*x, cos(x)*sinh(y)=2*y}, {x,y},
       {x=2..10, y=0..10});
```

say, gives $x = 7.413378$, $y = 3.489028$, so

$$\sqrt{\lambda} = z = x+iy = 7.413378 + 3.489028i,$$

$$\lambda \approx 42.7849 + 51.7310i$$

as another eigenvalue.

7. Not a Sturm-Liouville system because the b.c.'s are not separated.

$$y(x) = \begin{cases} A+Bx, & \lambda=0 \\ C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x, & \lambda\neq 0 \end{cases}$$

$$\lambda=0: \left.\begin{array}{l} y(0)-y(1) = 0 = A-A-B = -B \\ y'(0)+y'(1) = 0 = 2B \end{array}\right\} \text{ so } B=0,\ A=\text{arbitrary. Thus, } \lambda=0 \text{ gives the eigenfunction } \Phi(x)=1.$$

$$\lambda\neq 0: \begin{array}{l} y(0)-y(1) = 0 = C - C\cos\sqrt{\lambda} - D\sin\sqrt{\lambda} \\ y'(0)+y'(1) = 0 = \sqrt{\lambda}D - \sqrt{\lambda}C\sin\sqrt{\lambda} + \sqrt{\lambda}D\cos\sqrt{\lambda} \end{array}$$

or,
$$\begin{pmatrix} 1-c & -s \\ -\sqrt{\lambda}s & \sqrt{\lambda}(1+c) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0}. \qquad (\text{where } c,s \text{ denote } \cos\sqrt{\lambda}, \sin\sqrt{\lambda})$$

Determinant $= \sqrt{\lambda}(1-c^2-s^2) = 0$ for all λ's. The two scalar equations on C and D are redundant so we can drop one, say the second. Then $(1-\cos\sqrt{\lambda})C - \sin\sqrt{\lambda}\,D = 0$ gives D in terms of C and hence the eigenfunction

$$\Phi(x) = (\sin\sqrt{\lambda})\cos\sqrt{\lambda}x + (1-\cos\sqrt{\lambda})\sin\sqrt{\lambda}x.$$

Thus, we have seen that every number λ (real or complex) is an eigenvalue.

8. (a) The ODE is of Cauchy-Euler type so seek $y(x) = x^{\alpha}$:

$$\alpha^2 - \alpha + \alpha + \lambda = 0 \quad \text{gives } \alpha = \pm i\sqrt{\lambda}$$

$$y(x) = Ax^{i\sqrt{\lambda}} + Bx^{-i\sqrt{\lambda}} = Ae^{i\sqrt{\lambda}\ln x} + Be^{-i\sqrt{\lambda}\ln x}$$

$$= C\cos(\sqrt{\lambda}\ln x) + D\sin(\sqrt{\lambda}\ln x)$$

is the general solution — unless $\lambda=0$. In that case $y(x) = E + F\ln x$.

$\lambda=0$: $y(1)=0=E$ $\}$ Thus $E=F=0$ so $\lambda=0$ is
$y(a)=0=E+F\ln a$ $\}$ not an eigenvalue.

$\lambda\neq0$: $y(1)=0=C$
$y(a)=0=C\cos(\sqrt{\lambda}\ln a)+D\sin(\sqrt{\lambda}\ln a)$
gives $C=0$, $D=$ arbitrary, and $\sin(\sqrt{\lambda}\ln a)=0$ 🯅
so $\sqrt{\lambda}\ln a=n\pi$

(since application of Theorem 17.7.2 shows that not only are the λ's real, they are also nonnegative so $\sqrt{\lambda}$ is real; hence it sufficis to look for roots of 🯅 only on the real axis). Thus,

$$\lambda_n=(n\pi/\ln a)^2,\quad \phi_n(x)=\sin\left(n\pi\frac{\ln x}{\ln a}\right)\quad \text{for } n=1,2,\ldots$$

(b) $\sigma(x)=e^{\int\frac{x-2x}{x^2}dx}=1/x$ (from Exercise 3 or by inspection) so

$$xy''+y'+\lambda\frac{1}{x}y=(xy')'+\lambda\frac{1}{x}y=0.$$

Hence the weight function is $w(x)=1/x$ so

$$c_n=\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}=\frac{\int_1^a f(x)\sin\left(n\pi\frac{\ln x}{\ln a}\right)\frac{1}{x}dx}{\int_1^a\sin^2\left(n\pi\frac{\ln x}{\ln a}\right)\frac{1}{x}dx}.$$

9. (a) We find that $\lambda=0$ gives $\lambda_0=0$, $\phi_0(x)=1$
and that $\lambda\neq0$ gives $\lambda_n=n^2$, $\phi_n(x)=\cos nx$
so $f(x)=a_0\cdot1+\sum_1^\infty a_n\cos nx$,
where

$$a_0=\langle f,1\rangle/\langle1,1\rangle=\int_0^2x^4dx\Big/\int_0^\pi1dx=32/5\pi$$

$$\begin{aligned}a_n&=\langle f,\cos nx\rangle/\langle\cos nx,\cos nx\rangle\\&=\int_0^2x^4\cos nx\,dx\Big/\int_0^\pi\cos^2nx\,dx\\&=\frac{16}{\pi n^5}\left[(2n^4+3-6n^2)\sin2n+(4n^3-6n)\cos2n\right]\end{aligned}$$

so $$f(x)=\frac{32}{5\pi}+\frac{16}{\pi}\sum_1^\infty\frac{1}{n^5}\left[(2n^4-6n^2+3)\sin2n+(4n^3-6n)\cos2n\right]\cos nx$$

(b) We find that $\lambda_n=n^2/4$, $\phi_n(x)=\cos\frac{nx}{2}$ for $n=1,3,\ldots$
so $f(x)=\sum_{1,3,\ldots}^\infty a_n\cos\frac{nx}{2}$,
where

$$\begin{aligned}a_n&=\langle f,\phi_n\rangle/\langle\phi_n,\phi_n\rangle=\frac{\int_0^2x^4\cos\frac{nx}{2}dx}{\int_0^\pi\cos^2\frac{nx}{2}dx}\\&=\frac{64}{\pi n^5}\left[(n^4+24-12n^2)\sin n+(4n^3-24n)\cos n\right]\end{aligned}$$

10. Theorem 3.2.3.

11. $(p\phi')'+q\phi+\lambda r\phi=0$; $\alpha\phi(a)+\beta\phi'(a)=0$,
$\gamma\phi(b)+\delta\phi'(b)=0$.

Taking complex conjugates, and remembering that $p(x), q(x), r(x), \alpha, \beta, \gamma, \delta, \lambda$ are all real, we have

$$(p\bar{\Phi}')' + q\bar{\Phi} + \lambda r\bar{\Phi} = 0; \quad \alpha\bar{\Phi}(a)+\beta\bar{\Phi}'(a)=0, \quad \gamma\bar{\Phi}(b)+\delta\bar{\Phi}(b)=0$$

so that if λ is an eigenvalue with eigenfunction $\phi(x)$ then λ also has $\bar{\Phi}(x)$ as an eigenfunction. But according to Theorem 17.7.1, part (b), ϕ and $\bar{\Phi}$ must be LD; i.e., there are constants c_1, c_2 such that $c_1\phi_1(x)+c_2\bar{\Phi}_2(x)=0$ over $[a,b]$, where c_1, c_2 are not both zero. If <u>one</u> of the c_j's is 0 then either $\phi(x)$ or $\bar{\Phi}(x)$ is 0, which is inadmissible since $\phi, \bar{\Phi}$ are eigenfunctions. Thus, neither c_1 nor c_2 is 0. Hence, we can divide by c_1 and express

$$\phi(x) = c\bar{\Phi}(x)$$

or,

$$A(x)e^{iB(x)} = Ce^{iD}A(x)e^{-iB(x)} = CA(x)e^{i[D-B(x)]}.$$

Thus, $B(x) = D - B(x)$ so $B(x) = D/2$ is a constant and hence $\phi(x) = A(x)e^{iB}$ is a perhaps-complex constant times a real function $A(x)$.

12. $(py')' + qy + \lambda wy = 0,$

$$\int_a^b \underbrace{\bar{y}}_{\overset{\uparrow}{u}}\underbrace{(py')'dx}_{dV} + \int_a^b q\bar{y}y\,dx + \lambda\int_a^b w\bar{y}y\,dx = 0,$$

$$\bar{y}py'\Big|_a^b - \int_a^b py'\bar{y}'dx + \int_a^b q\bar{y}y\,dx + \lambda\int_a^b w\bar{y}y\,dx = 0,$$

$$(py'\bar{y})\Big|_a^b - \int_a^b p|y'|^2dx + \int_a^b q|y|^2dx + \lambda\|y\|^2 = 0,$$

$$\lambda\|y\|^2 = -(py'y)\Big|_a^b + \int_a^b p|y'|^2dx - \int_a^b q|y|^2dx,$$

so if $q(x) \le 0$ on $[a,b]$ and $[p(x)y'(x)y(x)]\big|_a^b \le 0$ then it follows from the signs of the terms in ♀ that $\lambda \ge 0$.

13. (a) $$y(x) = \begin{cases} A+Bx+C\sin\sqrt{\lambda}\,x + D\cos\sqrt{\lambda}\,x, & \lambda \ne 0 \\ E+Fx+Gx^2+Hx^3, & \lambda = 0 \end{cases}$$

$\lambda=0$:

$$\left.\begin{aligned} &y(0)=0=E \\ &y'(0)=0=F \\ &y''(L)=0=2G+6HL \\ &y'''(L)-Ky(L)=6H-K(GL^2+HL^3)=0 \end{aligned}\right\} \to \underbrace{\begin{pmatrix} 1 & 3L \\ KL^2 & KL^3-6 \end{pmatrix}}\begin{pmatrix} G \\ H \end{pmatrix} = \underset{\sim}{0}$$

$$\text{Determinant} = KL^3-6-3KL^3 = -2KL^3-6 \ne 0,$$

so $G=H=0$.

Since $E=F=G=H=0$, $\lambda=0$ is not an eigenvalue.

$\lambda \ne 0$:

$$\begin{aligned} &y(0)=0=A+D \\ &y'(0)=0=B+\sqrt{\lambda}\,C \\ &y''(L)=0=-\lambda C\sin\sqrt{\lambda}L - \lambda D\cos\sqrt{\lambda}L \end{aligned}$$

$$\begin{aligned} y'''(L)+\lambda y'(L)-Ky(L)=0= & -\lambda^{3/2}C\cos\sqrt{\lambda}L+\lambda^{3/2}D\sin\sqrt{\lambda}L \\ & +\lambda(B+\sqrt{\lambda}C\cos\sqrt{\lambda}L-\sqrt{\lambda}D\sin\sqrt{\lambda}L) \\ & -K(A+BL+C\sin\sqrt{\lambda}L+D\cos\sqrt{\lambda}L) \end{aligned}$$

or, $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \sqrt{\lambda} & 0 \\ 0 & 0 & s & c \\ -K & \lambda-KL & -Ks & -Kc \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=\underset{\sim}{0}$ where $c, s \equiv \cos\sqrt{\lambda}L, \sin\sqrt{\lambda}L$.

The determinant $= c\sqrt{\lambda}(\lambda-KL)+Ks=0$. $\Lambda\equiv\sqrt{\lambda}$ then gives

$$(\Lambda^2-KL)\Lambda\cos\Lambda L+K\sin\Lambda L=0.$$

To find the eigenfunctions, solve the matrix equation, above, by Gauss elimination, using the characteristic equation:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \sqrt{\lambda} & 0 \\ 0 & \lambda-KL & -Ks & K-Kc \\ 0 & 0 & s & c \end{bmatrix}\rightarrow\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \sqrt{\lambda} & 0 \\ 0 & 0 & \lambda^{3/2}-KL\sqrt{\lambda}+Ks & Kc-K \\ 0 & 0 & s & c \end{bmatrix}\rightarrow$$ (now use the characteristic equation)

$$\rightarrow\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \sqrt{\lambda} & 0 \\ 0 & 0 & s & c \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ so $D=$ arbitrary, say $=1$

$C=-c/s$

$B=\sqrt{\lambda}\,c/s$

$A=-1$

or, scaling them all by $-s/c$, gives $A=\tan\sqrt{\lambda}L, B=-\sqrt{\lambda}, C=1, D=-\tan\sqrt{\lambda}L$, so

$$\phi(x)=\tan\sqrt{\lambda}L-\sqrt{\lambda}x+\sin\sqrt{\lambda}x-\tan\sqrt{\lambda}L\cos\sqrt{\lambda}x.$$

(b) For $K=0$ (i.e, with the lateral spring removed), the characteristic equation simplifies to $\cos\sqrt{\lambda}L=0$, so $\sqrt{\lambda}L=\pi/2, 3\pi/2, \dots$

$$PL^2/EI=\pi^2/4,\ 9\pi^2/4,\ \dots$$

↳ gives $P_{cr}=(\pi/2L)^2EI$

(c) No, neither the ODE nor the boundary conditions are of Sturm-Liouville form. Notice, in particular, the presence of λ in the boundary condition $y'''(L)+\lambda y'(L)-Ky(L)=0$

14. (a) Let $\sin(\sqrt{\lambda}/x)\equiv s$, $\cos(\sqrt{\lambda}/x)\equiv c$. Then

$$y=Axs+Bxc$$

$$y'=As+Bc+Ax(-\sqrt{\lambda}/x^2)c-Bx(-\sqrt{\lambda}/x^2)s=As+Bc-\sqrt{\lambda}Ac/x+\sqrt{\lambda}Bs/x$$

$$y''=A\cancel{(-\tfrac{\sqrt{\lambda}}{x^2})}c-B\cancel{(-\tfrac{\sqrt{\lambda}}{x^2})}s+\sqrt{\lambda}\cancel{A\tfrac{1}{x^2}}c-\sqrt{\lambda}\cancel{B\tfrac{1}{x^2}}s+\sqrt{\lambda}A\tfrac{1}{x}(-\tfrac{\sqrt{\lambda}}{x^2})s+\sqrt{\lambda}B\tfrac{1}{x}(-\tfrac{\sqrt{\lambda}}{x^2})c$$

so

$$x^4y''+\lambda y=-\lambda\cancel{Ax}s-\lambda\cancel{Bx}c+\lambda\cancel{Ax}s+\lambda\cancel{B}\cancel{xc}=0.\ \checkmark$$

NOTE: We can derive (14.2) using (46)-(50) in Sec. 4.6.6:

$$a=0,\ b=\lambda,\ c=-4 \quad \text{so} \quad \alpha = 2/(-4-0+2)=-1,\ \nu=(1-0)/(-4-0+2)=-1/2$$

so $y(x) = x^{1/2} Z_{|-1/2|}(-1\sqrt{\lambda}\, x^{-1}) = \sqrt{x}\, Z_{1/2}(-\sqrt{\lambda}/x)$,

so

$$y(x) = C_1 \sqrt{x}\, J_{1/2}(-\sqrt{\lambda}/x) + C_2 \sqrt{x}\, J_{-1/2}(-\sqrt{\lambda}/x).$$

Then, from Exercise 5 of Section 4.6, we have

$$y(x) = C_1 \sqrt{x} \sqrt{\frac{2}{\pi(-\sqrt{\lambda}/x)}} \sin(-\sqrt{\lambda}/x) + C_2 \sqrt{x} \sqrt{\frac{2}{\pi(-\sqrt{\lambda}/x)}} \cos(-\sqrt{\lambda}/x)$$

$$= x\left[A \sin\left(\frac{\sqrt{\lambda}}{x}\right) + B \cos\left(\frac{\sqrt{\lambda}}{x}\right)\right]$$

(b) $y(a) = 0 = a[A \sin(\sqrt{\lambda}/a) + B\cos(\sqrt{\lambda}/a)]$ ①

$y(b) = 0 = b[A \sin(\sqrt{\lambda}/b) + B\cos(\sqrt{\lambda}/b)]$. ②

Setting the determinant $=0$ gives the characteristic equation

$$\sin\frac{\sqrt{\lambda}}{a} \cos\frac{\sqrt{\lambda}}{b} - \sin\frac{\sqrt{\lambda}}{b}\cos\frac{\sqrt{\lambda}}{a} = 0$$

or,

$$\sin\left(\frac{\sqrt{\lambda}}{a} - \frac{\sqrt{\lambda}}{b}\right) = 0.$$

Thus,

$$\frac{b-a}{ab}\sqrt{\lambda} = n\pi, \quad \lambda_n = (n\pi ab/L)^2.$$

To find the corresponding eigenfunction, note that with $\lambda = \lambda_n$ the equations ① and ② are redundant, so we can discard ② and learn from ① that

$$B = -\frac{\sin(\sqrt{\lambda}/a)}{\cos(\sqrt{\lambda}/a)} A,$$

so $y(x) = xA\left[\sin\frac{\sqrt{\lambda}}{x} - \frac{\sin(\sqrt{\lambda}/a)}{\cos(\sqrt{\lambda}/a)}\cos\frac{\sqrt{\lambda}}{x}\right]$.

Let $A = -\cos(\sqrt{\lambda}/a)$, say. Then

$$y(x) = x\left[-\sin\left(\frac{\sqrt{\lambda}}{x}\right)\cos\frac{\sqrt{\lambda}}{a} + \sin\frac{\sqrt{\lambda}}{a}\cos\left(\frac{\sqrt{\lambda}}{x}\right)\right]$$

$$= x\sin\left(-\frac{\sqrt{\lambda}}{x} + \frac{\sqrt{\lambda}}{a}\right) = x\sin\left[n\pi\frac{b}{L}\left(1-\frac{a}{x}\right)\right].$$

Also,

$$P = \frac{EI_0}{b^4}\left(\frac{n\pi ab}{L}\right)^2,$$ so the smallest one is for $n=1$:

$$P_{cr} = \frac{\pi^2 EI_0 a^2}{b^2 L^2}.$$

15. $x^2 y'' + \lambda y = 0$; $y(a) = 0$, $y(b) = 0$.

Since the ODE is of Cauchy-Euler type, seek $y = x^\alpha$. Then $\alpha^2 - \alpha + \lambda = 0$, $\alpha = (1 \pm \sqrt{1-4\lambda})/2$, so

$$y(x) = \sqrt{x}\left(Ax^{\sqrt{1-4\lambda}/2} + Bx^{-\sqrt{1-4\lambda}/2}\right)$$

is the general solution — provided that $\lambda \neq 1/4$, which case we will treat separately.

$\lambda \neq 1/4$: $y(x) = \sqrt{x}\,(Ax^{\sqrt{1-4\lambda}/2} + Bx^{-\sqrt{1-4\lambda}/2})$

$$y(a) = 0 = \sqrt{a}\,(Aa^{\sqrt{1-4\lambda}/2} + Ba^{-\sqrt{1-4\lambda}/2})$$
$$y(b) = 0 = \sqrt{b}\,(Ab^{\sqrt{1-4\lambda}/2} + Bb^{-\sqrt{1-4\lambda}/2})$$

Set

$$\text{Determinant} = a^{\sqrt{\ }/2}b^{-\sqrt{\ }/2} - a^{-\sqrt{\ }/2}b^{\sqrt{\ }/2} = 0$$

or, $$\left(\frac{a}{b}\right)^{\sqrt{1-4\lambda}/2} = \left(\frac{a}{b}\right)^{-\sqrt{1-4\lambda}/2}$$

or, $$\left(\frac{a}{b}\right)^{\sqrt{1-4\lambda}} = 1,$$

or, $$e^{\sqrt{1-4\lambda}\,\ln(a/b)} = e^{2n\pi i}$$
$$\sqrt{1-4\lambda}\,\ln\frac{a}{b} = 2n\pi i$$

so

$$\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\ln\frac{a}{b})^2} \quad \text{for } n = \cancel{0}, 1, 2, \ldots$$

$\uparrow$ omit since it gives $\lambda = 1/4$, which case will be considered separately. Also, negative n's contribute nothing additional.

$\lambda = 1/4$: Then $\alpha = (1 \pm 0)/2 = 1/2, 1/2$ so we obtain the one solution

$$y(x) = Ax^{1/2}.$$

Seek the missing solution by variation of parameters:

$$y(x) = A(x)\,x^{1/2}.$$

Plugging, $x^2y'' + \frac{1}{4}y = x^{5/2}A'' + x^{3/2}A' - \cancel{\frac{1}{4}Ax^{1/2}} + \cancel{\frac{1}{4}Ax^{1/2}} = 0$

so $xA'' + A' = 0$, $A(x) = C_1 + C_2\ln x$

and $y(x) = \sqrt{x}\,(C_1 + C_2\ln x)$.

Then

$$y(a) = 0 = \sqrt{a}\,(C_1 + C_2\ln a)$$
$$y(b) = 0 = \sqrt{b}\,(C_1 + C_2\ln b)$$

give $C_1 = C_2 = 0$ so $\lambda = 1/4$ is not an eigenvalue.

Since $\lambda = b^2P/EI_0$, the smallest λ_n gives the smallest P, i.e., P_{cr}:

$$\lambda_{min} = \lambda_1 = \frac{1}{4} + \frac{\pi^2}{(\ln\frac{a}{b})^2}$$

so $$P_{cr} = \frac{EI_0}{b^2}\left[\frac{1}{4} + \frac{\pi^2}{(\ln a/b)^2}\right]$$

16. Want to show that $[p(u\bar{v}' - u'\bar{v})]\big|_a^b = 0$ where $u, \bar{u}, v, \bar{v}$ satisfy the conditions

$$\alpha y(a) + \beta y'(a) = 0$$
$$\gamma y(b) + \delta y'(b) = 0,$$

and where α, β are not both zero and γ, δ are not both zero. We distinguish the four cases $\alpha \neq 0$ and $\gamma \neq 0$, $\alpha \neq 0$ and $\delta \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$, $\beta \neq 0$ and $\delta \neq 0$, and treat them separately. Let us do just the first two cases.

$\alpha \neq 0$ and $\gamma \neq 0$: Then

$$[p(u\bar{v}'-u'\bar{v})]\Big|_a^b = p(b)[u(b)\bar{v}'(b)-u'(b)\bar{v}(b)]-p(a)[u(a)\bar{v}'(a)-u'(a)\bar{v}(a)]$$

$$= p(b)\Big[-\frac{\delta}{\gamma}\cancel{u'(b)}\bar{v}'(b)-u'\cancel{(b)}\Big(\cancel{-\frac{\delta}{\gamma}}\bar{v}'(b)\Big)\Big]$$

$$-p(a)\Big[-\frac{\delta}{\gamma}\cancel{u'(a)}\bar{v}'(a)-u'(a)\Big(\cancel{-\frac{\delta}{\gamma}}\bar{v}'(a)\Big)\Big]=0 \checkmark$$

$\alpha \neq 0$ and $\delta \neq 0$: Then

$$[p(u\bar{v}'-u'\bar{v})]\Big|_a^b = p(b)[u(b)\bar{v}'(b)-u'(b)\bar{v}(b)]-p(a)[u(a)\bar{v}'(a)-u'(a)\bar{v}(a)]$$

$$= p(b)\Big[\cancel{u(b)}\Big(\cancel{-\frac{\gamma}{\delta}}\bar{v}(b)\Big)-\Big(\cancel{-\frac{\gamma}{\delta}u(b)}\Big)\bar{v}(b)\Big]$$

$$-p(a)\Big[-\frac{\beta}{\alpha}\cancel{u'(a)}\bar{v}'(a)-u'\cancel{(a)}\Big(\cancel{-\frac{\beta}{\alpha}}\bar{v}'(a)\Big)\Big]=0 \checkmark$$

and similarly for the other two cases.

17. (b) $\langle L[u],v\rangle = \int_0^1 u'v\,dx = uv\big|_0^1 + \int_0^1 -v'u\,dx$

$$= \underbrace{u(1)}_{0}v(1)-\underbrace{u(0)}_{?}\underbrace{v(0)}_{0} + \langle u, L^*[v]\rangle$$

so $L^*[v] = -\frac{dv}{dx}$; $v(0)=0$. Not self adjoint because $L=d/x$ whereas $L^* = -d/dx$ and also because the boundary condition associated with L is $u(1)=0$ whereas that associated with L^* is $v(\underline{\underline{0}})=0$.

(d) $\langle L[u],v\rangle = \int_0^1 u''v\,dx = u'v\big|_0^1 - \int_0^1 u'v'\,dx$

$$= (u'v-uv')\big|_0^1 + \int_0^1 uv''\,dx$$

$$= \underbrace{u'(1)}_{0}v(1)-\underbrace{u(1)}_{?}\underbrace{v'(1)}_{0}-\underbrace{u'(0)}_{0}v(0)+\underbrace{u(0)}_{?}\underbrace{v'(0)}_{0}+\langle u, L^*[v]\rangle$$

so $L^*[v] = \frac{d^2}{dx^2}v$; $v'(0)=0$, $v'(1)=0$. Self-adjoint (i.e., Hermitean).

(f) $\langle L[u],v\rangle = \int_0^1 (u''+u')v\,dx = (u'v+uv)\big|_0^1 - \int_0^1 (u'+u)v'\,dx$

$$= (u'v+uv-uv')\big|_0^1 + \int_0^1 (uv''-uv')\,dx$$

$$= \underbrace{u'(1)}_{?}\underbrace{v(1)}_{0}+\underbrace{u(1)}_{0}v(1)-\underbrace{u(1)}_{0}v'(1)-\underbrace{u'(0)}_{?}\underbrace{v(0)}_{0}-\underbrace{u(0)}_{0}v(0)+\underbrace{u(0)}_{0}v'(0)+\int_0^1 (v''-v')u\,dx$$

so $L^*[v] = v''-v'$; $v(0)=0$, $v(1)=0$. The boundary conditions associated with L^* are the same as those associated with L, but

$$L = \frac{d^2}{dx^2}+\frac{d}{dx} \quad \text{whereas} \quad L^* = \frac{d^2}{dx^2}-\frac{d}{dx} \neq L,$$

so L is <u>not</u> self-adjoint; i.e., it is not Hermitean.

(g) $\langle L[u],v\rangle = \int_0^1 (u'''-u''+2u')v\,dx = (u''v-u'v+2uv)\big|_0^1 + \int_0^1 (-u''v'+u'v' -2uv')\,dx$

$$= (u''v - u'v + 2uv - u'v' + uv')\Big|_0^1 + \int_0^1 (u'v'' - uv'' - 2uv')\,dx$$

$$= (u''v - u'v + 2uv - u'v' + uv' + uv'')\Big|_0^1 + \int_0^1 (-v''' - v'' - 2v')u\,dx$$

$$= u''(1)v(1) - \overbrace{u'(1)}^{0}v(1) + 2u(1)v(1) - \overbrace{u'(1)}^{0}v'(1) + u(1)v'(1) + u(1)v''(1)$$

$$-u''(0)v(0) + \underbrace{u'(0)}_{0}v(0) - 2\underbrace{u(0)}_{0}v(0) + \underbrace{u'(0)}_{0}v'(0) - \underbrace{u(0)}_{0}v'(0) - \underbrace{u(0)}_{0}v''(0) + \int_0^1 uL^*v\,dx$$

$$= \underbrace{u''(1)}_{?}\,\underbrace{v(1)}_{0} + \underbrace{u(1)}_{?}\,[\underbrace{2v(1) + v'(1) + v''(1)}_{0}] - \underbrace{u''(0)}_{?}\,\underbrace{v(0)}_{0} + \int_0^1 uL^*v\,dx$$

so $L^* = -\frac{d^3}{dx^3} - \frac{d^2}{dx^2} - 2\frac{d}{dx}$; $v(0)=0$, $v(1)=0$, $v'(1) + v''(1) = 0$

Not self-adjoint since $L^* \neq L$ and the boundary conditions are different as well.

(h) $\langle L[u], v\rangle = \int_0^1 (u'' - u)v\,dx = u'v\big|_0^1 + \int_0^1 (-u'v' - uv)\,dx$

$$= (u'v - uv')\Big|_0^1 + \int_0^1 (v'' - v)u\,dx$$

$$= \underbrace{u'(1)}_{0}v(1) - u(1)v'(1) - \underbrace{u'(0)}_{= -u(0)}v(0) + u(0)v'(0) + \int_0^1 uL^*[v]\,dx$$

$$= -\underbrace{u(1)}_{?}\,\underbrace{v'(1)}_{0} + \underbrace{u(0)}_{?}\,[\underbrace{v(0) + v'(0)}_{0}] + \int_0^1 uL^*[v]\,dx$$

so $L^* = \frac{d^2}{dx^2} - 1$; $v(0) + v'(0) = 0$, $v'(1) = 0$, so L *is* self-adjoint.

(i) $\langle L[u], v\rangle = \int_0^1 u''v\,dx = u'v\big|_0^1 - \int^1 u'v'\,dx$

$$= (u'v - u\overset{0}{v'})\Big|_0^1 + \int_0^1 uv''\,dx$$

$$= u'(1)v(1) - \underbrace{u(1)}_{-5u'(1)}v'(1) - \underbrace{u'(0)}_{0}v(0) + u(0)v'(0) + \int_0^1 uL^*[v]\,dx$$

$$= \underbrace{u'(1)}_{?}[\underbrace{v(1) + 5v'(1)}_{0}] + \underbrace{u(0)}_{?}\,\underbrace{v'(0)}_{0} + \int_0^1 uL^*[v]\,dx$$

so $L^* = \frac{d^2}{dx^2}$; $v'(0) = 0$, $v(1) + 5v'(1) = 0$, so L *is* self-adjoint.

(j) $\langle L[u], v\rangle = \int_0^1 u''v\,dx = (u'v - uv')\big|_0^1 + \int_0^1 uv''\,dx$

$$= u'(1)v(1) - u(1)v'(1) - \underbrace{u'(0)}_{u'(1)}v(0) + \underbrace{u(0)}_{u(1)}v'(0) + \int_0^1 uL^*[v]\,dx$$

$$= \underbrace{u'(1)}_{?}[\underbrace{v(1) - v(0)}_{0}] + \underbrace{u(1)}_{?}[\underbrace{v'(0) - v'(1)}_{0}] + \int_0^1 uL^*[v]\,dx$$

so $L^* = \frac{d^2}{dx^2}$; $v(1) - v(0) = 0$, $v'(0) - v'(1) = 0$, so L *is* self-adjoint.

Section 17.8

1. $(xy')' + \lambda xy = 0$. $a = c = 1$, $b = \lambda$, $\alpha = 2/2 = 1$, $\nu = 0$, so $y(x) = x^0 Z_0(\sqrt{\lambda}\, x)$
so $y(x) = AJ_0(\sqrt{\lambda}\, x) + BY_0(\sqrt{\lambda}\, x)$

2.(a) $y = \begin{cases} A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x, & \lambda \neq 0 \\ C + Dx, & \lambda = 0. \end{cases}$

$\lambda = 0$: $\left.\begin{array}{l} y(0) - y(4) = 0 = C - C - 4D \\ y'(0) - y'(4) = 0 = D - D \end{array}\right\}$ give $D = 0$, $C =$ arbitrary, so "λ_0" $= 0$, "$\phi_0(x)$" $= 1$

$\lambda \neq 0$: $y(0) - y(4) = 0 = A - A\cos\sqrt{\lambda}4 - B\sin\sqrt{\lambda}4$
$y'(0) - y'(4) = 0 = \sqrt{\lambda}B + \sqrt{\lambda}A\sin\sqrt{\lambda}4 - \sqrt{\lambda}B\cos\sqrt{\lambda}4$, $\begin{pmatrix} 1-\overset{\cos 4\sqrt{\lambda}}{c} & -\overset{\sin 4\sqrt{\lambda}}{s} \\ \sqrt{\lambda}s & \sqrt{\lambda}(1-c) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \underset{\sim}{0}$

Det. $= 0 = \sqrt{\lambda}\,[(1-c)^2 + s^2] \Rightarrow \cos 4\sqrt{\lambda} = 1$ and $\sin 4\sqrt{\lambda} = 0$, so $4\sqrt{\lambda} = 2\pi, 4\pi, 6\pi, \ldots$

$\lambda_n = (n\pi/2)^2 \quad (n = 1, 2, \ldots)$

Then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \underset{\sim}{0}$ gives A, B arbitrary so $\phi_n(x) = \cos\frac{n\pi x}{2}$ and $\sin\frac{n\pi x}{2}$

(i.e., each eigenspace, for $n = 1, 2, \ldots$, has two LI eigenfunctions).

$$f(x) = H(x-2) = a_0 \cdot 1 + \sum_1^\infty \left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right),$$

$$a_0 = \langle H(x-2), 1\rangle / \langle 1, 1\rangle = \int_0^4 H(x-2)\,dx \Big/ \int_0^4 dx = 1/2$$
$$a_n = \langle H(x-2), \cos\tfrac{n\pi x}{2}\rangle / \langle \cos\tfrac{n\pi x}{2}, \cos\tfrac{n\pi x}{2}\rangle = \int_0^4 H(x-2)\cos\tfrac{n\pi x}{2}\,dx \Big/ \int_0^4 \cos^2\tfrac{n\pi x}{2}\,dx = 0$$
$$b_n = \langle H(x-2), \sin\tfrac{n\pi x}{2}\rangle / \langle \sin\tfrac{n\pi x}{2}, \sin\tfrac{n\pi x}{2}\rangle = \int_0^4 H(x-2)\sin\tfrac{n\pi x}{2}\,dx \Big/ \int_0^4 \sin^2\tfrac{n\pi x}{2}\,dx$$
$$= \begin{cases} 0, & n \text{ even} \\ -2/n\pi, & n \text{ odd} \end{cases}$$

so $$H(x-2) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{2}$$

(b) $y(x) = \begin{cases} A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x, & \lambda \neq 0 \\ C + Dx, & \lambda = 0 \end{cases}$

$\lambda = 0$: $\left.\begin{array}{l} y(-1) - y(5) = 0 = (C - D) - (C + 5D) \\ y'(-1) - y'(5) = 0 = D - D \end{array}\right\}$ give $D = 0$, C arbitrary, so "λ_0" $= 0$, "$\phi_0(x)$" $= 1$

$\lambda \neq 0$: $y(-1) - y(5) = 0 = (A\cos\sqrt{\lambda} - B\sin\sqrt{\lambda}) - (A\cos 5\sqrt{\lambda} + B\sin 5\sqrt{\lambda})$
$y'(-1) - y'(5) = 0 = (-\sqrt{\lambda}A\sin\sqrt{\lambda} + \sqrt{\lambda}B\cos\sqrt{\lambda}) - (-\sqrt{\lambda}A\sin 5\sqrt{\lambda} + \sqrt{\lambda}B\cos 5\sqrt{\lambda})$

or, $$\begin{pmatrix} \cos\sqrt{\lambda} - \cos 5\sqrt{\lambda} & -\sin\sqrt{\lambda} - \sin 5\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda} + \sqrt{\lambda}\sin 5\sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda} - \sqrt{\lambda}\cos 5\sqrt{\lambda} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \underset{\sim}{0}$$

Setting the determinant $= 0$ will give a messy characteristic equation and to solve for λ will require skillful use of trigonometric identities. Instead, let us return to the general solution and re-express it equivalently and more conveniently as

$$y(x) = A\cos\sqrt{\lambda}(x+1) + B\sin\sqrt{\lambda}(x+1). \qquad (\text{for } \lambda \neq 0)$$

Then $y(-1)-y(5)=0=(A)-(A\cos 6\sqrt{\lambda}+B\sin 6\sqrt{\lambda})$

$\qquad y'(-1)-y'(5)=0=(\sqrt{\lambda}B)-(-\sqrt{\lambda}A\sin 6\sqrt{\lambda}+\sqrt{\lambda}B\cos 6\sqrt{\lambda})$

or, $\begin{pmatrix} 1-\cos 6\sqrt{\lambda} & -\sin 6\sqrt{\lambda} \\ \sqrt{\lambda}\sin 6\sqrt{\lambda} & \sqrt{\lambda}(1-\cos 6\sqrt{\lambda}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\underset{\sim}{0}$

now, det. $=\sqrt{\lambda}\left[(1-\cos 6\sqrt{\lambda})^2+\sin^2 6\sqrt{\lambda}\right]=0$ gives $1-\cos 6\sqrt{\lambda}=0$ and $\sin 6\sqrt{\lambda}=0$ so that $6\sqrt{\lambda}=2n\pi$ $(n=1,2,\dots)$

$$\lambda_n=(n\pi/3)^2$$

Then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\underset{\sim}{0}$

gives A,B arbitrary, so $\phi_n(x)=\cos\frac{n\pi}{3}(x+1),\ \sin\frac{n\pi}{3}(x+1)$.

Can expand

$$f(x)=x+2=a_0\cdot 1+\sum_1^\infty\left[a_n\cos\frac{n\pi}{3}(x+1)+b_n\sin\frac{n\pi}{3}(x+1)\right]$$

$$a_0=\langle f,1\rangle/\langle 1,1\rangle=\int_{-1}^5(x+2)dx\Big/\int_{-1}^5 dx=4$$

$$a_n=\langle f,\cos\tfrac{n\pi}{3}(x+1)\rangle/\langle\cos\tfrac{n\pi}{3}(x+1),\cos\tfrac{n\pi}{3}(x+1)\rangle$$

$$=\int_{-1}^5(x+2)\cos\frac{n\pi}{3}(x+1)dx\Big/\int_{-1}^5\cos^2\frac{n\pi}{3}(x+1)dx=0$$

$$b_n=\int_{-1}^5(x+2)\sin\frac{n\pi}{3}(x+1)dx\Big/\int_{-1}^5\sin^2\frac{n\pi}{3}(x+1)dx=-6/(n\pi)$$

so $x+2=4-\frac{6}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{n\pi}{3}(x+1)$

(d) $(1-x^2)y''-2xy'+\lambda y=0$ has, as its only bounded solutions, the Legendre polynomials $P_n(x)$, for $\lambda_n=n(n+1)$, for $n=0,1,2,\dots$ Of these, only the odd ones satisfy $y(0)=0$. Thus,

$\qquad\lambda_n=n(n+1),\ \phi_n(x)=P_n(x)$ for $n=1,3,5,\dots$

Then $f(x)=4=\sum_{n=1,3,\dots}^\infty a_n\phi_n(x)=\sum_{1,3,\dots}^\infty a_nP_n(x)$.

$$a_n=\langle f,P_n\rangle/\langle P_n,P_n\rangle=\int_0^1 4P_n(x)dx\Big/\int_0^1 P_n^2(x)dx$$

$a_1=6,\quad a_2=-7/2,\ a_3=11/4,\ \dots\qquad$ so $\ 4=6P_1(x)-\frac{7}{2}P_3(x)+\frac{11}{4}P_5(x)+\cdots$

NOTE: To evaluate $\int_0^1 P_3^2(x)\,dx$, for example, using Maple, use the commands

```
with(orthopoly):
int( P(3,x)^2, x=0..1);
```

and obtain 1/7.

3. $\lambda_n=n(n+1),\ \phi_n(x)=P_n(x),\ n=0,1,2,\dots$

$\qquad f(x)=H(x)=\sum_0^\infty a_nP_n(x),$

$$a_n=\frac{\int_{-1}^1 H(x)P_n(x)dx}{\int_{-1}^1 P_n^2(x)dx}=\frac{2n+1}{2}\int_0^1 P_n(x)dx$$

gives $H(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots$

5. The change of variables $x = \alpha t$ changes the ODE to

$$\alpha^2 t^2 \frac{d}{\alpha dt}\frac{d}{\alpha dt} y + \alpha t \frac{d}{\alpha dt} y + (\lambda\alpha^2 t^2 - 9)y = 0$$

or, $t^2 \ddot{y} + t\dot{y} + (\lambda\alpha^2 t^2 - 9)y = 0.$

Choose $\alpha = 1/\sqrt{\lambda}$, so $t^2\ddot{y} + t\dot{y} + (t^2 - 9)y = 0$

which is now a Bessel equation of order 3, with general solution

$$y = AJ_3(t) + BY_3(t) = AJ_3(\sqrt{\lambda}x) + BY_3(\sqrt{\lambda}x). \longrightarrow ♀$$

$y(0)$ bounded $\Rightarrow B = 0$, so $y(x) = AJ_3(\sqrt{\lambda}x)$.

Then, $y(L) = 0 = AJ_3(\sqrt{\lambda}L)$ gives $\sqrt{\lambda}L = z_1, z_2, z_3, \ldots$ (the positive zeros of $J_3(z)$). Thus,

$$\lambda_n = z_n^2/L^2, \ \phi_n(x) = J_3\left(z_n \frac{x}{L}\right), \quad n = 1, 2, \ldots$$

Actually, ♀ is not the general solution if $\lambda = 0$ since $Y_3(0)$ is undefined, so we need to consider that case separately. For $\lambda = 0$, the ODE is

$$x^2y'' + xy' - 9y = 0 \quad \text{(Cauchy-Euler type)}$$

so $\quad y(x) = Cx^3 + Dx^{-3}.$

Boundedness at $x = 0 \Rightarrow D = 0$ and $y(L) = 0 = CL^3$ gives $C = 0$, so $y(x) = 0$ and $\lambda = 0$ is not an eigenvalue. Finally, what is the weight function? Multiply through by $1/x$ to get into the standard Sturm-Liouville form:

$$xy'' + y' + \left(\lambda x - \frac{9}{x}\right)y = (xy')' - \frac{9}{x}y + \lambda xy = 0$$

↑ weight fn. $w(x) = x$.

6. (a) $(1-x^2)y'' - xy' + \lambda y = 0$. With $x = \cos\theta$, $\frac{d}{dx} = \frac{d}{d\theta}\frac{d\theta}{dx} = \frac{d}{d\theta}\frac{1}{dx/d\theta} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$

so ODE becomes $\sin^2\theta\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\right)\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\right)y - \cos\theta\left(-\frac{1}{\sin\theta}\right)\frac{dy}{d\theta} + \lambda y = 0$

or, with $y(x) = y(\cos\theta) \equiv \Theta(\theta)$,

$$\sin\theta\left(\frac{1}{\sin\theta}\Theta'\right)' + \cot\theta\,\Theta' + \lambda\Theta = 0$$

or,

$$\sin\theta\left(\frac{1}{\sin\theta}\Theta'' + \cancel{\frac{(-1)\cos\theta}{\sin^2\theta}\Theta'}\right) + \cancel{\cot\theta\,\Theta'} + \lambda\Theta = 0,$$

$$\Theta'' + \lambda\Theta = 0.$$

(c) $T_n(x) = \cos(n\cos^{-1}x)$ for $n = 0, 1, 2, \ldots$

$T_0(x) = \cos(0) = 1,$

$T_1(x) = \cos(\cos^{-1}x) = x,$

$T_2(x) = \cos(2\cos^{-1}x) = 2\cos^2(\cos^{-1}x) - 1 = 2x^2 - 1,$

etc.

(d) $\sigma(1-x^2)y'' - \sigma x y' + \lambda\sigma y = 0$, where

$$-\sigma x = [\sigma(1-x^2)]'$$

$$-\sigma x = \sigma'(1-x^2) - 2x\sigma, \quad \frac{d\sigma}{dx}(1-x^2) = (2x-x)\sigma, \quad \frac{d\sigma}{\sigma} = \frac{x}{1-x^2}dx$$, and integration gives $\sigma = (1-x^2)^{-1/2}$. Thus,

$$(\sqrt{1-x^2}\, y')' + \lambda \underset{\text{weight function } w(x).}{\boxed{\frac{1}{\sqrt{1-x^2}}}} y = 0$$

(e) Let us work backwards, starting with Euler's formulas in Sec. 17.3.2. With "l" $=\pi$ we have, for $m \neq n$,

$$0 = \int_{-\pi}^{\pi} \cos mt \cos nt\, dt = 2\int_0^{\pi} \cos mt \cos nt\, dt \overset{x=\cos t}{=} 2\int_1^{-1} \cos(m\cos^{-1}x)\cos(n\cos^{-1}x)\left(-\frac{dx}{\sqrt{1-x^2}}\right)$$

$$= 2\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx.$$

Similarly, with "l" $=\pi$ we have, for $m = n \neq 0$,

$$\pi = \int_{-\pi}^{\pi} \cos nt \cos nt\, dt = 2\int_0^{\pi} \cos^2 nt\, dt = 2\int_1^{-1} \cos^2(n\cos^{-1}x)\left(-\frac{dx}{\sqrt{1-x^2}}\right)$$

so $$\int_{-1}^{1} T_n^2(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

For the case $m = n = 0$ we simply have

$$\int_{-1}^{1} T_0^2(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \int_{\pi}^{0}\frac{1}{\sin t}(-\sin t\, dt) = \pi.$$

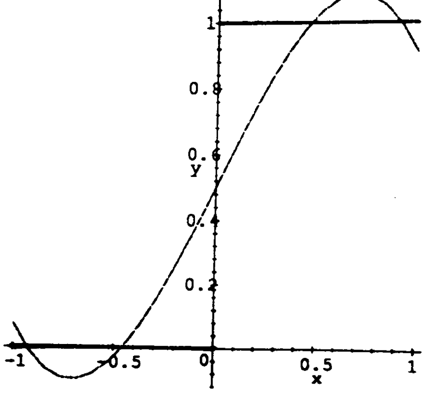
(f) $$a_0 = \frac{1}{\pi}\int_{-1}^{1}\frac{H(x)}{\sqrt{1-x^2}}dx = \frac{1}{\pi}\int_0^1\frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\int_{\pi/2}^{0}\frac{-\cancel{\sin\theta}\, d\theta}{\cancel{\sin\theta}} = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{H(x)T_n(x)}{\sqrt{1-x^2}}dx = \frac{2}{\pi}\int_0^1\frac{T_n(x)}{\sqrt{1-x^2}}dx = \frac{2}{\pi}\int_{\pi/2}^{0}\frac{\cos n\theta}{\cancel{\sin\theta}}(-\cancel{\sin\theta}\, d\theta)$$

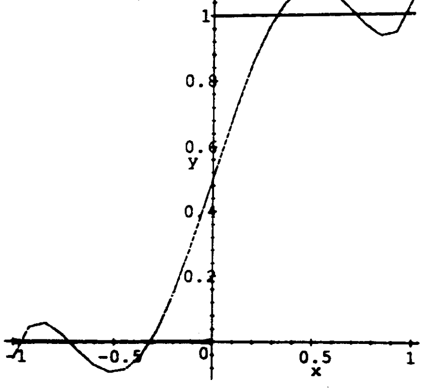
$$= \frac{2}{\pi}\int_0^{\pi/2}\cos n\theta\, d\theta = \frac{2\sin n\theta}{n\pi}\Big|_0^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

so $$H(x) = \frac{1}{2} + \frac{2}{\pi}\sum_1^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\,T_n(x)$$

(g) Summing through $n=4$:



Summing through $n=5$:



The slow convergence is not surprising since $H(x)$ is discontinuous.

(h) $T_n(x)=\cos(n\cos^{-1}x)$ so $T_n(1)=\cos 0=1$, $T_n(-1)=\cos n\pi=(-1)^n$.

$$\begin{aligned}T_{n+1}(x)=\cos[(n+1)\cos^{-1}x] &= \cos(n\cos^{-1}x)\cos(\cos^{-1}x)-\sin(n\cos^{-1}x)\sin(\cos^{-1}x)\\ &= T_n(x)\cdot x-\sqrt{1-\cos^2(n\cos^{-1}x)}\sqrt{1-x^2}\\ &= xT_n(x)-\sqrt{1-x^2}\sqrt{1-T_n^2(x)}\end{aligned}$$

Fine, but not the identity we were after. Instead, use the identity

$$\cos(n+1)\theta+\cos(n-1)\theta=2\cos\theta\cos n\theta$$

or, since $T_n(x)=\cos n\theta$,

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x).$$

(i) $$\begin{aligned}T_6(x)&=2xT_5(x)-T_4(x)\\ &=2x(16x^5-20x^3+5x)-(8x^4-8x^2+1)\\ &=32x^6-48x^4+18x^2-1\end{aligned}$$

Similarly for $T_7, T_8, \dots$

(j) $f(t)=(1-xt)(1-2xt+t^2)^{-1}$, $f(0)=1$

$f'(t)=[-x(1-2xt+t^2)+2(1-xt)(x-t)](1-2xt+t^2)^{-2}$, $f'(0)=x$

and so on, but this gets messy quickly, so let's also use Maple:

```
> taylor((1-x*t)*(1-2*x*t+t^2)^(-1),t=0,4);
```

$$\underset{\uparrow\; T_0(x)=1}{1}+\underset{\llcorner T_1(x)=x}{x}\,t+\underset{\uparrow\; T_2(x)=2x^2-1}{(-1+2x^2)}\,t^2+\underset{\uparrow\; T_3(x)=4x^3-3x}{(-x-2(1-2x^2)x)}\,t^3+O(t^4)$$

```
>
```

Section 17.9

1. $$f_\Omega(x)=\frac{2}{\pi}\int_0^\Omega\frac{\sin\omega}{\omega}\cos\omega x\,d\omega=\frac{1}{\pi}\int_0^\Omega\frac{\sin(1+x)\omega}{\underset{\uparrow\;\text{Let }(1+x)\omega=t}{\omega}}d\omega+\frac{1}{\pi}\int_0^\Omega\frac{\sin(1-x)\omega}{\underset{\uparrow\;\text{Let }(1-x)\omega=t}{\omega}}d\omega$$

$$=\frac{1}{\pi}\int_0^{(1+x)\Omega}\frac{\sin t}{t}dt+\frac{1}{\pi}\int_0^{(1-x)\Omega}\frac{\sin t}{t}dt=\frac{1}{\pi}\mathrm{Si}[(1+x)\Omega]+\frac{1}{\pi}\mathrm{Si}[(1-x)\Omega]$$

2. (b) $a(\omega)=\frac{1}{\pi}\int_0^L x\cos\omega x\,dx=(\omega L\sin\omega L+\cos\omega L-1)/\omega^2$

$b(\omega)=\frac{1}{\pi}\int_0^L x\sin\omega x\,dx=(\sin\omega L-\omega L\cos\omega L)/\omega^2$

so $$f(x)=\int_0^\infty\left[\frac{\omega L\sin\omega L+\cos\omega L-1}{\omega^2}\cos\omega x+\frac{\sin\omega L-\omega L\cos\omega L}{\omega^2}\sin\omega x\right]d\omega$$

(g) $a(\omega)=\frac{1}{\pi}\int_0^\infty e^{-x}\cos\omega x\,dx=\frac{1}{\pi}\frac{1}{1+\omega^2}$

$b(\omega)=\frac{1}{\pi}\int_0^\infty e^{-x}\sin\omega x\,dx=\frac{1}{\pi}\frac{\omega}{1+\omega^2}$

so $$f(x)=\frac{1}{\pi}\int_0^\infty(\cos\omega x+\omega\sin\omega x)\frac{d\omega}{1+\omega^2}$$

(i) $a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\cos\omega x\,dx = \frac{2}{\pi}\int_0^{\infty} e^{-x}\cos\omega x\,dx = \frac{2}{\pi}\,\frac{1}{1+\omega^2}$

$b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|x|}\sin\omega x\,dx = 0$ since $e^{-|x|}\sin\omega x$ = even × odd = odd.

Thus, $e^{-|x|}=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x\,d\omega$

3. (a) $Si(x)=\int_0^x \frac{\sin t}{t}\,dt$, $Si(-x)=\int_0^{-x}\frac{\sin t}{t}\,dt \overset{t=-\tau}{\overset{\downarrow}{=}} \int_0^x \frac{\sin(-\tau)}{-\tau}(-d\tau) = -\int_0^x \frac{\sin\tau}{\tau}\,d\tau$
$= -Si(x).$

(b) $\lim\limits_{\Omega\to\infty} f_\Omega(x) = \lim\limits_{\Omega\to\infty}\frac{1}{\pi}\{Si[\Omega(x+1)] - Si[\Omega(x-1)]\}$

For $x<-1$ this $= \frac{1}{\pi}\{Si(-\infty) - Si(-\infty)\} = 0$ ✓
For $x=-1$ this $= \frac{1}{\pi}\{Si(0) - Si(-\infty)\} = \frac{1}{\pi}\{0-(-\frac{\pi}{2})\} = \frac{1}{2}$ ✓
For $-1<x<1$ this $= \frac{1}{\pi}\{Si(\infty) - Si(-\infty)\} = \frac{1}{\pi}\{\frac{\pi}{2}-(-\frac{\pi}{2})\} = 1$ ✓
For $x=1$ this $= \frac{1}{\pi}\{Si(\infty) - Si(0)\} = \frac{1}{\pi}\{\frac{\pi}{2}-0\} = \frac{1}{2}$ ✓
For $x>1$ this $= \frac{1}{\pi}\{Si(\infty) - Si(\infty)\} = 0$ ✓

4. Not true. $a(0)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos 0\,dx = \lim\limits_{A\to\infty}\frac{1}{\pi}\int_{-A}^{A} f(x)\,dx \neq \lim\limits_{A\to\infty}\frac{1}{2A}\int_{-A}^{A} f(x)\,dx.$

For instance, in Example 1 $a(0)=2/\pi$ whereas the average value of f is 0.

Section 17.10

1. $F\{H(x)e^{-ax}\} = \int_{-\infty}^{\infty} H(x)e^{-ax}e^{-i\omega x}\,dx$ $(a=\alpha+i\beta$ with $\alpha>0)$

$$= \int_0^{\infty} e^{-[\alpha+i(\beta+\omega)]x}\,dx$$
$$= \frac{e^{-[\alpha+i(\beta+\omega)]x}}{-[\alpha+i(\beta+\omega)]}\Bigg|_0^{\infty} = \frac{\overset{\text{because }\alpha>0}{0}-1}{-(a+i\omega)} = \frac{1}{a+i\omega}$$

3. $F\{e^{-a|x|}\} = \int_{-\infty}^{\infty} e^{-a|x|}e^{-i\omega x}\,dx = \int_{-\infty}^{0} e^{ax}e^{-i\omega x}\,dx + \int_0^{\infty} e^{-ax}e^{-i\omega x}\,dx$

$$= \frac{e^{(a-i\omega)x}}{a-i\omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(a+i\omega)x}}{-a-i\omega}\Bigg|_0^{\infty} = \frac{1-0}{a-i\omega} - \frac{0-1}{a+i\omega} = \frac{2a}{\omega^2+a^2}$$

4. (c) $F^{-1}\{[H(\omega)-H(\omega-1)]\omega\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[H(\omega)-H(\omega-1)]\omega e^{i\omega x}\,d\omega$

$$= \frac{1}{2\pi}\int_0^1 \omega e^{i\omega x}\,d\omega = \frac{1}{2\pi}\,\frac{(1-ix)e^{ix}-1}{x^2}$$

(d) $F^{-1}\{H(\omega)e^{-a\omega}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-a\omega}e^{i\omega x}\,d\omega = \frac{1}{2\pi}\int_0^{\infty} e^{-a\omega}e^{i\omega x}\,d\omega$

$$= \frac{1}{2\pi}\,\frac{e^{-(a-ix)\omega}}{-(a-ix)}\Bigg|_0^{\infty} = -\frac{1}{2\pi}\,\frac{0-1}{a-ix} = \frac{1}{2\pi}\,\frac{1}{a-ix}$$

(e) $F^{-1}\{[H(\omega+1)-H(\omega-1)]|\omega|\} = \frac{1}{2\pi}\int_{-1}^{1}|\omega|e^{i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1}|\omega|(\cos\omega x+\overset{0 \text{ since odd}}{\cancel{i\sin\omega x}})\,d\omega$

$$= \frac{2}{2\pi}\int_{0}^{1}\omega\cos\omega x\,d\omega = \frac{1}{\pi x^2}(\cos x + x\sin x - 1)$$

(f) $F^{-1}\{\delta(\omega-a)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-a)e^{i\omega x}\,d\omega = \frac{1}{2\pi}e^{iax}$

5. (g) $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

$$(d/d\omega)\hat{f}(\omega) = \int_{-\infty}^{\infty}(-ix)f(x)e^{-i\omega x}\,dx$$

$$(d^2/d\omega^2)\hat{f}(\omega) = \int_{-\infty}^{\infty}(-ix)^2 f(x)e^{-i\omega x}\,dx$$

$$\vdots$$

$$(d^n/d\omega^n)\hat{f}(\omega) = \int_{-\infty}^{\infty}(-ix)^n f(x)e^{-i\omega x}\,dx = (-i)^n\, F\{x^n f(x)\}$$

Then,

$$i^n\frac{d^n}{d\omega^n}\hat{f}(\omega) = i^n(-i)^n F\{x^n f(x)\} = (-i^2)^n F\{x^n f(x)\}$$

$$= F\{x^n f(x)\}$$

(h) $F\{f^{(n)}(x)\} = \int_{-\infty}^{\infty} f^{(n)}(x)\underbrace{e^{-i\omega x}}_{v}\,dx = \cancel{e^{-i\omega x}f^{(n-1)}(x)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f^{(n-1)}(x)(-i\omega)e^{-i\omega x}\,dx$

$$= i\omega\int_{-\infty}^{\infty} f^{(n-1)}(x)e^{-i\omega x}\,dx = \text{in same manner} = (i\omega)^2\int_{-\infty}^{\infty} f^{(n-2)}(x)e^{-i\omega x}\,dx$$

$$= \text{etc.} = (i\omega)^n\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = (i\omega)^n\hat{f}(\omega).$$

6. (a) By 4, $F\{e^{-3|x|}\} = \frac{6}{\omega^2+9}$. By 17, $F\{x^2e^{-3|x|}\} = i^2\frac{d^2}{d\omega^2}\frac{6}{\omega^2+9}$

$$= -6\frac{d}{d\omega}[-1(2\omega)(\omega^2+9)^{-2}] = 12\,\frac{9-3\omega^2}{(\omega^2+9)^3},$$ so $F\{4x^2e^{-3|x|}\} = 144\,\frac{3-\omega^2}{(\omega^2+9)^3}$

Check by Maple: `readlib(fourier):`

`fourier(4*x^2 * exp(-3* abs(x)), x, w);`

does give same result. Actually, using xmapleVr4 the commands are different; namely, use

`with(inttrans):`

in place of `readlib(fourier):`

(b) $F\{e^{-x^2}\} = \sqrt{\pi}\,e^{-\omega^2/4}$ per 5. Then by 17, $F\{xe^{-x^2}\} = i\frac{d}{d\omega}(\sqrt{\pi}\,e^{-\omega^2/4})$

$$= -i\frac{\sqrt{\pi}}{2}\omega e^{-\omega^2/4}$$

(c) By 1, $F\{\frac{1}{x^2+2}\} = \frac{\pi}{\sqrt{2}}e^{-\sqrt{2}|\omega|}$. By 13 $(a=1, c=3)$,

$$F\{\frac{\cos 3x}{x^2+2}\} = \frac{1}{2}\left(\frac{\pi}{\sqrt{2}}e^{-\sqrt{2}|\omega-3|} + \frac{\pi}{\sqrt{2}}e^{-\sqrt{2}|\omega+3|}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\left(e^{-\sqrt{2}|\omega-3|} + e^{-\sqrt{2}|\omega+3|}\right)$$

NOTE: The result obtained using Maple is less compact since in place of $e^{-\sqrt{2}|\omega-3|}$ maple gives the equivalent expression $H(3-\omega)e^{\sqrt{2}(\omega-3)} + H(\omega-3)e^{-\sqrt{2}(\omega-3)}$, and similarly for $e^{-\sqrt{2}|\omega+3|}$.

(d) By 1, $F\{1/(4x^2+3)\} = \frac{1}{4}F\{\frac{1}{x^2+3/4}\} = \frac{1}{4}\frac{\pi}{\sqrt{3/4}}e^{-\sqrt{3/4}|\omega|} = \frac{\pi}{2\sqrt{3}}e^{-(\sqrt{3}/2)|\omega|}$.

By 14, $F\{\frac{\sin 2x}{4x^2+3}\} = \frac{1}{2i}\left[\frac{\pi}{2\sqrt{3}}e^{-(\sqrt{3}/2)|\omega-2|} - \frac{\pi}{2\sqrt{3}}e^{-(\sqrt{3}/2)|\omega+2|}\right]$

$$= \frac{\pi i}{4\sqrt{3}}\left(e^{-(\sqrt{3}/2)|\omega+2|} - e^{-(\sqrt{3}/2)|\omega-2|}\right)$$

Actually, in the first step we not only used 1, we also used linearity, 18.

(e) By 18, $F\{\frac{3}{2x^2+1} - 5e^{-|x|}\} = \frac{3}{2}F\{\frac{1}{x^2+\frac{1}{2}}\} - 5F\{e^{-|x|}\}$

and by 1 and 4 it $= \frac{3}{2}\frac{\pi}{\sqrt{1/2}}e^{-\sqrt{1/2}|\omega|} - 5\frac{2}{\omega^2+1} = \frac{3\pi}{\sqrt{2}}e^{-|\omega|/\sqrt{2}} - \frac{10}{\omega^2+1}$

(f) By 18, $F\{e^{-|x|} + e^{-3|x+2|}\} = F\{e^{-|x|}\} + F\{e^{-3|x+2|}\}$

By 4, $F\{e^{-|x|}\} = 2/(\omega^2+1)$

By 4, $F\{e^{-3|x|}\} = 6/(\omega^2+9)$

By 11, $F\{e^{-3|x+2|}\} = e^{i2\omega}\,6/(\omega^2+9)$, so the answer is:

$F\{e^{-|x|} + e^{-3|x+2|}\} = \frac{2}{\omega^2+1} + \frac{6e^{i2\omega}}{\omega^2+9}$

(g) By 18, $F^{-1}\{\frac{4\sin\omega}{\omega} - \frac{1}{\sqrt{|\omega|}}\} = 4F^{-1}\{\frac{\sin\omega}{\omega}\} - F^{-1}\{\frac{1}{\sqrt{\omega}}\}$

$= (4)(\frac{1}{2})[H(x+1) - H(x-1)] - \frac{1}{\sqrt{2\pi|x|}}$ by 9, 7, 18

$= 2[H(x+1) - H(x-1)] - \frac{1}{\sqrt{2\pi|x|}}$

(h) By 1 and 18, $F^{-1}\{e^{-2|\omega|}\} = \frac{2}{\pi}\frac{1}{x^2+4}$.

Then, by 12 with $a=1$ and $b=-3$, $F^{-1}\{e^{-2|\omega-3|}\} = e^{i3x}\frac{2}{\pi}\frac{1}{x^2+4}$

Checking by maple, using xmapleVr4,

```
with(inttrans):
invfourier(exp(-2*abs(w-3)), w, x);
```

does give the same result.

(i) We want to obtain the form in entry 2 or 3:

$$F^{-1}\left\{\frac{9}{2\omega+i}\right\} = F^{-1}\left\{\frac{9i}{2i\omega-1}\right\} = -\frac{9i}{2}F^{-1}\left\{\frac{1}{\frac{1}{2}-i\omega}\right\} \text{ by } 18$$

$$= -\frac{9i}{2}H(-x)\,e^{x/2} \text{ by } 3.$$

(j) $$F^{-1}\{e^{-\omega^2+4\omega}\} = F^{-1}\{e^4 e^{-\omega^2+4\omega-4}\} = e^4 F^{-1}\{e^{-(\omega-2)^2}\} \text{ by } 18$$

Now, $e^{-\omega^2} \to \frac{1}{2\sqrt{\pi}}e^{-x^2/4}$ by 6 so, by 12, with $a=1$ and $b=-2$,

$$e^{-(\omega-2)^2} \to e^{i2x}\frac{1}{2\sqrt{\pi}}e^{-x^2/4}, \text{ so } F^{-1}\{e^{-\omega^2+4\omega}\} = \frac{e^4}{2\sqrt{\pi}}e^{i2x-x^2/4}$$

(k) By 1 and 18, $e^{-|\omega|} \to \frac{1}{\pi}\frac{1}{x^2+1}$. Then, by 15 (with $c=1$) and 18,

$$e^{-|\omega|}\cos\omega \to \frac{1}{2}\left(\frac{1}{\pi}\frac{1}{(x+1)^2+1} + \frac{1}{\pi}\frac{1}{(x-1)^2+1}\right)$$

so $$F^{-1}\{e^{-|\omega|}\cos\omega\} = \frac{1}{2\pi}\left(\frac{1}{x^2+2x+2} + \frac{1}{x^2-2x+2}\right)$$

(l) By 6, $e^{-3\omega^2} \to \frac{1}{2\sqrt{3\pi}}e^{-x^2/12}$ and by 19 (and 18)

$$F^{-1}\{\omega e^{-3\omega^2}\} = \frac{1}{i}F^{-1}\{i\omega e^{-3\omega^2}\} = \frac{1}{i}\frac{d}{dx}F^{-1}\{e^{-3\omega^2}\}$$

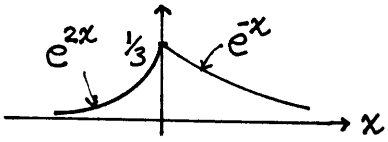
$$= -i\frac{d}{dx}\left(\frac{1}{2\sqrt{3\pi}}e^{-x^2/12}\right) = \frac{ix}{12\sqrt{3\pi}}e^{-x^2/12}$$

(m) $\omega^2+i\omega+2=0$ gives $\omega = (-i\pm\sqrt{-1-8})/2 = -2i, i$

so $$F^{-1}\left\{\frac{1}{\omega^2+i\omega+2}\right\} = F^{-1}\left\{\frac{1}{(\omega-i)(\omega+2i)}\right\} = F^{-1}\left\{\frac{1/3i}{\omega-i} - \frac{1/3i}{\omega+2i}\right\}$$

$$= \frac{1}{3}F^{-1}\left\{\frac{1}{1+i\omega}\right\} + \frac{1}{3}F^{-1}\left\{\frac{1}{2-i\omega}\right\} \text{ by } 18$$

$$= \frac{1}{3}H(x)e^{-x} + \frac{1}{3}H(-x)e^{2x} \text{ by 2 and 3.}$$



Alternatively, we could have used the convolution theorem, as we did in Example 7:

$$F^{-1}\left\{\frac{1}{\omega^2+i\omega+2}\right\} = F^{-1}\left\{\frac{1}{(\omega-i)(\omega+2i)}\right\} = F^{-1}\left\{\frac{1}{\omega-i}\right\} * F^{-1}\left\{\frac{1}{\omega+2i}\right\},$$

but this approach is a bit harder.

(n) By 18, $$F^{-1}\left\{\frac{\omega}{\omega^2+1}\right\} = \frac{1}{i}F^{-1}\left\{\frac{i\omega}{\omega^2+1}\right\} = \frac{1}{2i}\frac{d}{dx}F^{-1}\left\{\frac{2}{\omega^2+1}\right\} \text{ by 19 and 18}$$

$$= \frac{1}{2i}\frac{d}{dx}e^{-|x|} \text{ (by 4) } = \frac{-1}{2i}e^{-|x|}\boxed{\frac{d}{dx}|x|} \leftarrow +1 \text{ for } x>0, -1 \text{ for } x<0$$

$$= \frac{i}{2}[2H(x)-1]\,e^{-|x|}$$

8. $$u(x) = \frac{\alpha W}{\sqrt{2}k}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}\sin(\alpha|x-\xi|+\tfrac{\pi}{4})\,d\xi$$

$$= \frac{\alpha W}{\sqrt{2}k}\Big\{\underset{\text{Here, set } x-\xi=t/\alpha}{\int_{-\infty}^{x} e^{-\alpha(x-\xi)}\sin[\alpha(x-\xi)+\tfrac{\pi}{4}]\,d\xi} + \underset{\text{Here, set } \xi-x=t/\alpha}{\int_{x}^{\infty} e^{-\alpha(\xi-x)}\sin[\alpha(\xi-x)+\tfrac{\pi}{4}]\,d\xi}\Big\}$$

$$= \frac{\alpha W}{\sqrt{2}k}\Big\{\int_{\infty}^{0} e^{-t}\sin(t+\tfrac{\pi}{4})(-\tfrac{dt}{\alpha}) + \int_{0}^{\infty} e^{-t}\sin(t+\tfrac{\pi}{4})(\tfrac{dt}{\alpha})\Big\}$$

$$= \frac{2W}{\sqrt{2}k}\int_{0}^{\infty} e^{-t}\sin(t+\tfrac{\pi}{4})\,dt = \frac{2W}{\sqrt{2}k}\frac{1}{\sqrt{2}} = \frac{W}{k}$$

9. (a) $$f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-i\omega x}\,d\omega = \frac{1}{2\pi}\int_{\infty}^{-\infty}\hat{f}(-\mu)e^{i\mu x}(-d\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overset{\substack{\text{because } \hat{f} \text{ is even}\\ \downarrow}}{\hat{f}(\mu)}e^{i\mu x}\,d\mu = f(x)$$

so $\hat{f}$ even $\Rightarrow$ f even. Next,

$$\hat{f}(-\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \int_{\infty}^{-\infty} f(-\xi)e^{-i\omega\xi}(-d\xi) = \int_{-\infty}^{\infty}\overset{\substack{\text{because } f \text{ is even}\\ \downarrow}}{f(\xi)}e^{-i\omega\xi}\,d\xi = \hat{f}(\omega)$$

so f even $\Rightarrow$ $\hat{f}$ even. Thus, $f(\omega)$ even $\Leftrightarrow$ $f(x)$ even. Similarly for part (b).

10. $$H(x)e^{-ax} = \frac{H(x)e^{-ax}+H(-x)e^{ax}}{2} + \frac{H(x)e^{-ax}-H(-x)e^{ax}}{2} \quad \text{(i.e., even part + odd part)}$$

$$\frac{1}{a+i\omega}\frac{a-i\omega}{a-i\omega} = \frac{a}{\omega^2+a^2} - \frac{i\omega}{\omega^2+a^2} \quad \text{(i.e., even part + odd part)}$$

Thus, $$F\Big\{\frac{H(x)e^{-ax}+H(-x)e^{ax}}{2}\Big\} = \frac{a}{\omega^2+a^2}$$

and $$F\Big\{\frac{H(x)e^{-ax}-H(-x)e^{ax}}{2}\Big\} = -\frac{i\omega}{\omega^2+a^2}$$

or, equivalently, $F\{e^{-a|x|}\} = 2a/(\omega^2+a^2)$

and $F\{(\operatorname{sgn} x)e^{-a|x|}\} = -2i\omega/(\omega^2+a^2)$

11(a) In the formula $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega$, set $x \to -\omega$ and $\omega \to x$, obtaining

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(x)e^{-i\omega x}\,dx = \frac{1}{2\pi}F\{\hat{f}(x)\}, \text{ or,}$$

$$F\{\hat{f}(x)\} = 2\pi f(-\omega).$$

Equivalently, $$F^{-1}\{f(-\omega)\} = \frac{1}{2\pi}\hat{f}(x).$$

(b) Entry 4 says $$F\{\underbrace{e^{-a|x|}}_{f(x)}\} = \underbrace{2a/(\omega^2+a^2)}_{\hat{f}(\omega)}$$

so $\hat{f}(x)$ is $2a/(x^2+a^2)$ and $f(-\omega)$ is $e^{-a|-\omega|} = e^{-a|\omega|}$. Thus, (11.1) gives

$$F\Big\{\frac{2a}{x^2+a^2}\Big\} = 2\pi e^{-a|\omega|}.$$

(c) Entry 9 says $F\{\underbrace{H(x+a)-H(x-a)}_{f(x)}\} = \underbrace{\frac{2\sin\omega a}{\omega}}_{\hat{f}(\omega)}$

so $\hat{f}(x)$ is $2\sin ax/x$ and $f(-\omega)$ is $H(-\omega+a)-H(-\omega-a)$. Thus, (11.1) gives

$$F\{2\frac{\sin ax}{x}\} = 2\pi[H(-\omega+a)-H(-\omega-a)]$$

or, since $H(-x) = 1-H(x)$ and by linearity (to cancel the 2's),

$$F\{\frac{\sin ax}{x}\} = \pi[1-H(\omega-a)-1+H(\omega+a)]$$
$$= \pi[H(\omega+a)-H(\omega-a)]$$

(d) Entry 3 says $F\{\underbrace{H(-x)e^{ax}}_{f(x)}\} = \underbrace{\frac{1}{a-i\omega}}_{\hat{f}(\omega)}$

so $\hat{f}(x)$ is $1/(a-ix)$ and $f(-\omega)$ is $H(\omega)e^{-a\omega}$. Thus, (11.1) gives

$$F\{\frac{1}{a-ix}\} = 2\pi H(\omega)e^{-a\omega}$$

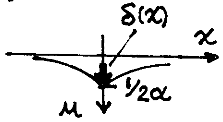
12.(a) Fourier transforming gives $-\omega^2\hat{u}-\alpha^2\hat{u} = -\hat{f}$,

$$\hat{u} = +\hat{f}(\omega)\frac{1}{\omega^2+\alpha^2}$$

and the convolution theorem gives $u(x) = +f(x) * F^{-1}\{\frac{1}{\omega^2+\alpha^2}\}$

$$= +f(x) * \frac{1}{2\alpha}\exp(-\alpha|x|)$$
$$= +\frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}f(\xi)\,d\xi$$

(b) If $f(x)=\delta(x)$, $u(x) = +\frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}\delta(\xi)\,d\xi = +\frac{1}{2\alpha}e^{-\alpha|x|}$



(c) Note first that $\frac{d}{dx}|x| = \begin{cases} +1, & x>0 \\ -1, & x<0 \end{cases} = 2H(x)-1$ and recall that $H'(x)=\delta(x)$ (see Exercise 3 of Sec. 5.6), and that $\delta(-x)=\delta(x)$.

$$u(x) = \frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}f(\xi)\,d\xi$$
$$u'(x) = \frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}\frac{\partial}{\partial x}(-\alpha|x-\xi|)f(\xi)\,d\xi$$
$$= -\frac{1}{2}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}[2H(x-\xi)-1]\,d\xi$$

$$u''(x) = -\frac{1}{2}\int_{-\infty}^{\infty}\{e^{-\alpha|x-\xi|}(-\alpha)\underbrace{[2H(x-\xi)-1]^2}_{=1} + e^{-\alpha|x-\xi|}2\delta(x-\xi)\}f(\xi)\,d\xi$$
$$= \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}f(\xi)\,d\xi - \frac{1}{2}2f(x)$$
$$= +\alpha^2u(x)-f(x),$$

so $u''-\alpha^2u = -f$. ✓

(a) $-k\omega^2\hat{c} - iU\omega\hat{c} - \beta\hat{c} = -Q/A$,

$$\hat{c} = \frac{Q}{A}\,\frac{1}{k\omega^2 + iU\omega + \beta}$$

(b) $\omega = \dfrac{-iU \pm \sqrt{-U^2-4k\beta}}{2k}$ so $\omega_\pm = -i\left[\dfrac{U\pm\sqrt{U^2+4k\beta}}{2k}\right] \equiv i\Omega_\mp$, where

$\Omega_- = (-U-\sqrt{U^2+4k\beta})/2k < 0$ and $\Omega_+ = (-U+\sqrt{U^2+4k\beta})/2k > 0$.

Thus,

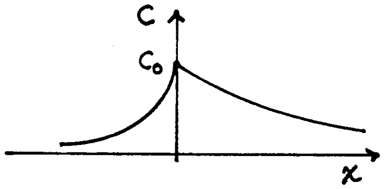
$$\hat c = \frac{Q}{kA}\,\frac{1}{(\omega-\omega_+)(\omega-\omega_-)} = \frac{Q}{kA(\omega_+-\omega_-)}\left(\frac{1}{\omega-\omega_+}-\frac{1}{\omega-\omega_-}\right)$$

$$= \frac{iQ}{A\sqrt{U^2+4k\beta}}\left(\frac{1}{\omega-i\Omega_-}-\frac{1}{\omega-i\Omega_+}\right) = \frac{Q}{A\sqrt{U^2+4k\beta}}\left(\frac{1}{-\Omega_- - i\omega}-\frac{1}{\Omega_+ + i\omega}\right)$$

and (since $-\Omega_->0$ and $\Omega_+>0$) entries 2 and 3 give

$$c(x) = \frac{Q}{A\sqrt{U^2+4k\beta}}\begin{cases} e^{-\Omega_- x}+0, & x<0\\ 0+e^{-\Omega_+ x}, & x>0\end{cases} = \begin{cases} c_0 e^{-\Omega_- x}, & x<0 \\ c_0 e^{-\Omega_+ x}, & x>0\end{cases}$$

(c)



As β increases, the peak $c_0 = Q/A\sqrt{U^2+4k\beta}$ diminishes and the upstream exponent Ω_- and the downstream exponent Ω_+ both increase. These results make sense since β manifests a chemical *decay*. Note also that $|\Omega_-| \geq |\Omega_+|$, which also makes sense since the stream U sweeps material downstream (except as $U\to 0$, in which case $|\Omega_-|=|\Omega_+|$ and the distribution becomes symmetric about $x=0$.

Section 17.11

1. Consider the odd extension shown in Fig. 1c. The Fourier transform of f_{ext} is

$$\hat f_{ext}(\omega) = \int_{-\infty}^{\infty} f_{ext}(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} \overset{\text{odd}}{f_{ext}(x)}(\overset{\text{even}}{\cancel{\cos\omega x}} - i\overset{\text{odd}}{\sin\omega x})\,dx = -2i\int_0^\infty f_{ext}(x)\sin\omega x\,dx$$

and the inversion formula gives

$$f_{ext}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat f_{ext}(\omega)e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{\text{odd}}{\hat f_{ext}(\omega)}(\overset{\text{even}}{\cancel{\cos\omega x}}+i\overset{\text{odd}}{\sin\omega x})\,d\omega = \frac{2i}{2\pi}\int_0^\infty \hat f_{ext}(\omega)\sin\omega x\,d\omega$$

so $$f_{ext}(x) = \frac{i}{\pi}\int_0^\infty\left\{-2i\int_0^\infty f_{ext}(\xi)\sin\omega\xi\,d\xi\right\}\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty\left\{\int_0^\infty f_{ext}(\xi)\sin\omega\xi\,d\xi\right\}\sin\omega x\,d\omega. \qquad (-\infty<x<\infty)$$

We can drop the "ext" on the right-hand side because the integral is from 0 to ∞, on which interval $f_{ext}(x) = f(x)$, and since we are only interested in $0<x<\infty$ we can also drop the "ext" on the left-hand side. Thus,

$$f(x) = \frac{2}{\pi}\int_0^\infty \left\{ \int_0^\infty f(\xi)\sin\omega\xi \, d\xi \right\} \sin\omega x \, d\omega \qquad (0<x<\infty)$$

If we define $\quad F_s\{f(x)\} = \hat{f}_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx,$

then the inverse is

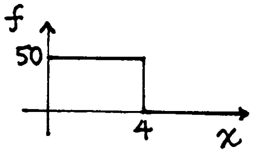
$$F_s^{-1}\{\hat{f}_s(\omega)\} = f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\,d\omega,$$

as in (5a,b).

2. $$F_s\{f'(x)\} = \int_0^\infty f'(x)\sin\omega x\,dx = f(x)\sin\omega x\Big|_0^\infty - \int_0^\infty f(x)\,\omega\cos\omega x\,dx$$
$$= 0 - 0 - \omega\int_0^\infty f(x)\cos\omega x\,dx$$
$$= -\omega\hat{f}_c(\omega).$$

3.(a) $$F_c\{f''''(x)\} = \int_0^\infty f''''(x)\cos\omega x\,dx = f'''(x)\cos\omega x\Big|_0^\infty - \int_0^\infty f'''(x)(-\omega\sin\omega x)\,dx$$
$$= 0 - f'''(0) + \omega\left[f''(x)\sin\omega x\Big|_0^\infty - \int_0^\infty f''(x)(\omega\cos\omega x)\,dx\right]$$
$$= -f'''(0) + \omega\left[0 - 0 - \omega\int_0^\infty f''(x)\cos\omega x\,dx\right]$$
$$= -f'''(0) - \omega^2\left[f'(x)\cos\omega x\Big|_0^\infty - \int_0^\infty f'(x)(-\omega\sin\omega x)\,dx\right]$$
$$= -f'''(0) - \omega^2\left[0 - f'(0) + \omega\int_0^\infty f'(x)\sin\omega x\,dx\right]$$
$$= -f'''(0) + \omega^2 f'(0) - \omega^3\left[f(x)\sin\omega x\Big|_0^\infty - \int_0^\infty f(x)(\omega\cos\omega x)\,dx\right]$$
$$= -f'''(0) + \omega^2 f'(0) + \omega^4\hat{f}_c(\omega)$$

(b) $$F_s\{f''''(x)\} = \int_0^\infty f''''(x)\sin\omega x\,dx = f'''(x)\sin\omega x\Big|_0^\infty - \int_0^\infty f'''(x)(\omega\cos\omega x)\,dx$$
$$= -\omega\int_0^\infty f'''(x)\cos\omega x\,dx = -\omega\left[f''(x)\cos\omega x\Big|_0^\infty - \int_0^\infty f''(x)(-\omega\sin\omega x)\,dx\right]$$
$$= +\omega f''(0) - \omega^2\int_0^\infty f''(x)\sin\omega x\,dx$$
$$= \omega f''(0) - \omega^2\left[f'(x)\sin\omega x\Big|_0^\infty - \int_0^\infty f'(x)(\omega\cos\omega x)\,dx\right]$$
$$= \omega f''(0) + \omega^3\int_0^\infty f'(x)\cos\omega x\,dx$$
$$= \omega f''(0) + \omega^3\left[f(x)\cos\omega x\Big|_0^\infty - \int_0^\infty f(x)(-\omega\sin\omega x)\,dx\right]$$
$$= \omega f''(0) - \omega^3 f(0) + \omega^4\hat{f}_s(\omega)$$

4. 

$$\hat{f}_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx = \int_0^4 50\cos\omega x\,dx = 50\,\frac{\sin 4\omega}{\omega}$$
$$\hat{f}_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx = \int_0^4 50\sin\omega x\,dx = \frac{50}{\omega}(1-\cos 4\omega)$$

5. (a) $F_c\{e^{-ax}\} = \int_0^\infty e^{-ax}\cos\omega x\,dx = ?$

The Maple commands

```
assume(a>0);
int(exp(-a*x)*cos(w*x), x=0..infinity);
simplify(");
```

give the result $a/(a^2+w^2)$, as in entry 1C.

(c) $F_c\{xe^{-ax}\} = \int_0^\infty xe^{-ax}\cos\omega x\,dx = ?$

The Maple commands

```
assume(a>0);
int(x*exp(-a*x)*cos(w*x), x=0..infinity);
simplify(");
```

give the result $\dfrac{a^2-\omega^2}{(\omega^2+a^2)^2}$,

and entry 2C gives $\dfrac{\mathrm{Re}(a+i\omega)^2}{(\omega^2+a^2)}$, which does $= \dfrac{a^2-\omega^2}{(\omega^2+a^2)^2}$

6. $u'' - 9u = 50e^{-2x}$

$s^2\bar{u} - s\underbrace{u(0)}_{u_0} - \underbrace{u'(0)}_{?} - 9\bar{u} = \dfrac{50}{s+2}$ so $(s^2-9)\bar{u} = u_0 s + u'(0) + \dfrac{50}{s+2}$,

$\bar{u} = u_0\dfrac{s}{s^2-9} + \dfrac{u'(0)}{s^2-9} + \dfrac{50}{(s^2-9)(s+2)}$

so $u(x) = u_0\cosh 3x + u'(0)\dfrac{\sinh 3x}{3} + 50\,\dfrac{\sinh 3x}{3} * e^{-2x}$.

Now,

$$\sinh 3x * e^{-2x} = \int_0^x \sinh 3\xi\, e^{-2(x-\xi)}\,d\xi = e^{-2x}\int_0^x \frac{e^{5\xi}-e^{-\xi}}{2}\,d\xi = \frac{e^{3x}}{10} + \frac{e^{-3x}}{2} - \frac{3}{5}e^{-2x}$$

so $u(x) = u_0\left(\dfrac{e^{3x}+e^{-3x}}{2}\right) + u'(0)\left(\dfrac{e^{3x}-e^{-3x}}{6}\right) + \dfrac{50}{3}\left(\dfrac{e^{3x}}{10} + \dfrac{e^{-3x}}{2} - \dfrac{3}{5}e^{-2x}\right)$ ♀

To satisfy the condition $u(\infty)$ bounded we set the coefficient of e^{3x} equal to 0:

$0 = \dfrac{u_0}{2} + \dfrac{u'(0)}{6} + \dfrac{5}{3}$ so $u'(0) = -3u_0 - 10$

and ♀ becomes

$u(x) = \left(\dfrac{u_0}{2} - \dfrac{(-3u_0-10)}{6} + \dfrac{25}{3}\right)e^{-3x} - 10e^{-2x} = (u_0+10)e^{-3x} - 10e^{-2x}$. ✓

7. Actually, it will work – but inconveniently (just as the Laplace transform worked, in Exercise 6, but was inconvenient). That is, if we apply the Fourier cosine transform to (10a) we obtain

$-\omega^2\hat{u}_c - u'(0) - 9\hat{u}_c = 50\dfrac{2}{\omega^2+4}$.

We can solve for $u(x)$ but our solution will contain the unknown $u'(0)$. Finally, we can impose the boundedness condition (as in Exercise 6) to evaluate $u'(0)$.

8. $u'' - 9u = 50e^{-2x}$ $(0<x<\infty)$, $u'(0) = u_0'$, $u(\infty) = \text{bdd}$

This time choose a cosine transform: $-\omega^2\hat{u}_c - \underbrace{u'(0)}_{u_0'} - 9\hat{u}_c = 50\dfrac{2}{\omega^2+4}$

so $\hat{u}_c = -u_0'\dfrac{1}{\omega^2+9} - 100\dfrac{1}{\omega^2+9}\dfrac{1}{\omega^2+4} = -u_0'\dfrac{1}{\omega^2+9} + \dfrac{100}{5}\left(\dfrac{1}{\omega^2+9} - \dfrac{1}{\omega^2+4}\right)$

$= (20-u_0')\dfrac{1}{\omega^2+9} - 20\dfrac{1}{\omega^2+4}$, $u(x) = F_c^{-1}\left\{(20-u_0')\dfrac{1}{\omega^2+9} - 20\dfrac{1}{\omega^2+4}\right\}$,

$$u(x) = (20-u_0')F_c^{-1}\left\{\frac{1}{\omega^2+9}\right\} - 20F_c^{-1}\left\{\frac{1}{\omega^2+4}\right\} = \frac{20-u_0'}{3}e^{-3x} - 10e^{-2x}.$$

9.(a) Sine transform gives $-\omega^2\hat{u}_s + \omega\cancelto{0}{u(0)} - 9\hat{u}_s = 50\omega/(\omega^2+9)$
so $\hat{u}_s = -50\dfrac{\omega}{(\omega^2+9)^2}$.

With $n=1$, entry 23 gives $xe^{-ax} \rightarrow \dfrac{2a\omega}{(\omega^2+a^2)^2}$, so $\dfrac{xe^{-3x}}{6} \rightarrow \dfrac{\omega}{(\omega^2+9)^2}$
and
$$u(x) = -50\,xe^{-3x}/6 = -\frac{25}{3}xe^{-3x}.$$
Of course, in place of the table we could use computer software to carry out the inversion integral,
$$u(x) = \frac{2}{\pi}\int_0^\infty -50\frac{\omega}{(\omega^2+9)^2}\sin\omega x\,d\omega.$$

For example, the Maple commands

```
assume(x>0);
int((2/Pi)*(-50)*w*sin(w*x)/(w^2+9)^2, w=0..infinity);
simplify(");
```

gives the result
$$-\frac{25}{3}x(-\sinh 3x + \cosh 3x),$$
or, $-25xe^{-3x}$, as above.

10.(a) $\displaystyle\int_0^\infty\left\{\frac{1}{2}\int_0^\infty [f(|x-\xi|)+f(x+\xi)]g(\xi)d\xi\right\}\cos\omega x\,dx$

$$= \frac{1}{2}\int_0^\infty g(\xi)d\xi\Big[\underbrace{\int_0^\infty f(|x-\xi|)\cos\omega x\,dx}_{\text{Let } x-\xi=t} + \underbrace{\int_0^\infty f(x+\xi)\cos\omega x\,dx}_{\text{Let } x+\xi=t}\Big]$$

$$= \frac{1}{2}\int_0^\infty g(\xi)d\xi\Big[\int_{-\xi}^\infty f(|t|)\cos\omega(t+\xi)dt + \int_\xi^\infty f(t)\cos\omega(t-\xi)dt\Big]$$

$$= \quad " \quad \Big[\int_{-\xi}^0 f(|t|)\cos\omega(t+\xi)\,dt - \int_0^\xi f(t)\cos\omega(t-\xi)dt + \int_0^\infty f(|t|)\cos\omega(t+\xi)^0dt + \int_0^\infty f(t)\cos\omega(t-\xi)dt\Big]$$

$$= \quad " \quad \Big[\int_\xi^0 f(|-\tau|)\cos\omega(-\tau+\xi)(-d\tau) - \int_0^\xi f(t)\cos\omega(t-\xi)dt + \int_0^\infty f(t)\cos\omega(t+\xi)dt + \int_0^\infty f(t)\cos\omega(t-\xi)\,dt\Big]$$

$$= \quad " \quad \Big[\int_0^\xi f(\tau)\cancel{\cos\omega(-\tau+\xi)}d\tau - \int_0^\xi f(t)\cancel{\cos\omega(t-\xi)}dt + \int_0^\infty f(t)(\cos\omega t\cos\omega\xi - \sin\omega\cancel{t\sin}\omega\xi + \cos\omega t\cos\omega\xi + \sin\omega\cancel{t\sin}\omega\xi)dt\Big]$$

$$= \frac{1}{2}\int_0^\infty g(\xi)d\xi\int_0^\infty f(t)\,2\cos\omega t\cos\omega\xi\,dt = \int_0^\infty g(\xi)\cos\omega\xi\,d\xi\int_0^\infty f(t)\cos\omega t\,dt$$
$$= \hat{f}_c(\omega)\;\hat{g}_c(\omega).\ \checkmark$$

(c) $\frac{1}{2}\int_0^\infty [f(|x-\xi|)+f(x+\xi)]g(\xi)d\xi = \frac{1}{2}\int_0^\infty (e^{-|x-\xi|}+e^{-(x+\xi)})e^{-3\xi}d\xi$

$$=\frac{1}{2}\left(\int_0^x e^{-(x-\xi)}e^{-3\xi}d\xi+\int_x^\infty e^{-(\xi-x)}e^{-3\xi}d\xi+\int_0^\infty e^{-(x+\xi)}e^{-3\xi}d\xi\right)$$

$$=\frac{1}{2}\left(e^{-x}\int_0^x e^{-2\xi}d\xi+e^{x}\int_x^\infty e^{-4\xi}d\xi+e^{-x}\int_0^\infty e^{-4\xi}d\xi\right) = \frac{1}{8}(3e^{-x}-e^{-3x})$$

$$F_c\{\cdot\cdot\} = \int_0^\infty \frac{1}{8}(3e^{-x}-e^{-3x})\cos\omega x\,dx = \frac{3}{(\omega^2+1)(\omega^2+9)}$$

$$\hat{f}_c\hat{g}_c = \int_0^\infty e^{-x}\cos\omega x\,dx\int_0^\infty e^{-3x}\cos\omega x\,dx = \frac{1}{\omega^2+1}\,\frac{3}{\omega^2+9} \quad ✓$$

CHAPTER 18

Section 18.2

2. (b) $L[Au+Bv]-AL[u]-BL[v]$
$= (Au+Bv)_t + \alpha(Au+Bv)(Au_x+Bv_x) + \beta(Au+Bv)_{xxx}$
$\quad - A(u_t+\alpha uu_x+\beta u_{xxx}) - B(v_t+\alpha vv_x+\beta v_{xxx})$
$= \alpha(A^2-A)uu_x + \alpha(B^2-B)vv_x + \alpha AB(uv)_x$
is not identically zero for all constants A,B and functions u,v; e.g., if $A=0, B=2, u=8, v=x$, then it $= 0+\alpha(4-2)x = 2\alpha x \neq 0$. Thus, L is nonlinear. (We used A,B rather than α,β because of the α,β in the PDE.)

(c) linear

(d) $L[\alpha u+\beta v]-\alpha L[u]-\beta L[v]$
$= (\cancel{\alpha u}+\cancel{\beta v})_{xx} + x(\cancel{\alpha u}+\cancel{\beta v})_{yy} - \alpha(\cancel{u_{xx}}+\cancel{xu_{yy}}) - \beta(\cancel{v_{xx}}+\cancel{xv_{yy}}) = 0$; linear

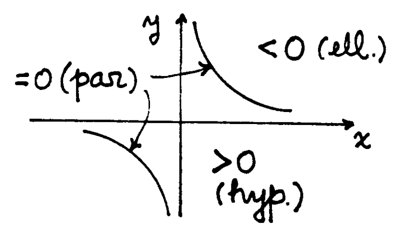
(e) nonlinear (due to the e^u term)

(f) linear (g) linear NOTE: $L = \partial^2/\partial x^2 + 5\partial^2/\partial x\partial y - x$ does not include the e^x.

(h) nonlinear (due to the uu_y term). Let's show it:
$L[\alpha u+\beta v]-\alpha L[u]-\beta L[v]$
$= x(\cancel{\alpha u}+\cancel{\beta v})_x + (\alpha u+\beta v)(\alpha u+\beta v)_y - \alpha(\cancel{xu_x}+uu_y) - \beta(\cancel{xv_x}+vv_y)$
$= (\alpha^2-\alpha)uu_y + (\beta^2-\beta)vv_y + \alpha\beta(uv)_y$
is not identically zero for all constants α,β and functions u,v; e.g. if $\alpha=0, \beta=3, u=\sin x, v=y^2$, it $= 0+6y^2\cdot 2y+0 = 12y^3 \neq 0$. Thus, L is nonlinear.

3. (a) $A=1, B=1/2, C=0$, so $B^2-AC = 1/4 > 0$, so hyperbolic (everywhere in the x,y plane)

(b) $A=x, B=-1/2, C=y$ so $B^2-AC = \frac{1}{4}-xy$ so elliptic in the two disjoint regions shown, hyperbolic in between, and parabolic on the hyperbolas $xy = 1/4$.



(c) $A=C=0, B=1/2$, so $B^2-AC = 1/4 > 0$, so hyperbolic everywhere

(d) $A=x, B=0, C=-(\sin^2 y+1)$, so $B^2-AC = x(\sin^2 y+1)$, so elliptic in the left half-plane ($x<0$), hyperbolic in the right half-plane ($x>0$), and parabolic on the y axis ($x=0$).

(e) $A=1, B=1/2, C=1$, so $B^2-AC = -3/4 < 0$ so elliptic everywhere

(f) $A=1, B=0, C=\cos x$, so $B^2-AC = -\cos x$, so elliptic in the strips $\pi/2<x<3\pi/2$, $-3\pi/2<x<-\pi/2$, $5\pi/2<x<7\pi/2$, $-7\pi/2<x<-5\pi/2, \dots$, hyperbolic in the strips $-\pi/2<x<\pi/2$, $-5\pi/2<x<-3\pi/2$, $3\pi/2<x<5\pi/2, \dots$, and parabolic along the lines $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

(g) $A=1, B=0, C=0,\ B^2-AC=0$, parabolic everywhere

(h) $A=0, B=1/2, C=-1,\ B^2-AC = 1/4 > 0$, hyperbolic everywhere

4. $$kAu_x\big|_{x+\Delta x} - kAu_x\big|_x = \frac{\partial}{\partial t}(A\Delta x\sigma c u),$$

$$k\,\frac{(Au_x)|_{x+\Delta x} - (Au_x)|_x}{\Delta x} = A\sigma c u_t\ ,\quad \Delta x \to 0 \longrightarrow \frac{1}{A(x)}[A(x)u_x]_x = \sigma c u_t .$$

Section 18.3

2.(a) $\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = +K^2$ gives $X''-K^2X=0\ ,\ X = \begin{cases} A\cosh Kx + B\sinh Kx, & K\neq 0 \\ C+Dx &, K=0 \end{cases}$

$T' - K^2\alpha^2 T = 0,\ T = \begin{cases} E\exp(K^2\alpha^2 t), & K\neq 0 \\ F &, K=0 \end{cases}$

$$u = (C+Dx)F + (A\cosh Kx + B\sinh Kx)Ee^{K^2\alpha^2 t} = C'+D'x+(A'\cosh Kx + B'\sinh Kx)e^{K^2\alpha^2 t}$$

$u(0,t) = u_1 = C' + A'\exp(K^2\alpha^2 t) \Rightarrow C' = u_1, A'=0$

so $u(x,t) = u_1 + D'x + B'\sinh Kx \exp(K^2\alpha^2 t)$

$u(L,t) = u_2 = u_1 + D'L + B'\sinh KL \exp(\ \text{"}\ ) \Rightarrow D' = (u_2-u_1)/L$, and $B'\sinh KL = 0$.

Of the choices $B'=0$ and $\sinh KL = 0$ we choose the latter:

$$\sinh KL = \tfrac{1}{i}\sin iKL = -i\sin iKL = 0 \Rightarrow iKL = n\pi \quad (n=1,2,\ldots)$$

so $K = -n\pi i/L$ or, equivalently, $K = n\pi i/L$ since K appears originally as K^2, so we can never distinguish between ± values. Okay, $K = n\pi i/L$ gives

$$u(x,t) = u_1 + (u_2-u_1)\frac{x}{L} + B'\sinh\left(i\frac{n\pi x}{L}\right)e^{-(n\pi\alpha/L)^2 t}$$
$$= u_1 + (u_2-u_1)\frac{x}{L} + iB'\sin\frac{n\pi x}{L}\,e^{-(n\pi\alpha/L)^2 t}$$

or, renaming iB' as G', say, and using superposition,

$$u(x,t) = u_1 + (u_2-u_1)\frac{x}{L} + \sum_1^\infty G_n'\sin\frac{n\pi x}{L}\exp[-(n\pi\alpha/L)^2 t],$$

which is the same as (22).

(b) Using $-K^2$ in (6), as we did, the relevant St.-Liou. problem is

$$X'' + K^2X = 0 \quad (0<x<L)$$
$$X(0)=0,\ X(L)=0,$$

as noted in Example 3. Thus, $p(x)=1, q(x)=0, w(x)=1$, K^2 is λ. Then $q(x)$ is ≤ 0 on $[0,L]$ and $[p(x)\phi_n(x)\phi_n'(x)]\big|_0^L = 0$ (because the ϕ_n's are 0 at 0 and L) is ≤ 0. Hence, by Theorem 17.7.2, $\lambda_n = K_n^2 \geq 0$ so that K_n^2 must be nonnegative and we see that our use of $-K^2$ in (6) is justified.

3. Here are the only conditions under which the graph of $u(x,t)$, plotted versus x, does not change its shape (although its magnitude might vary with time):

(i) If $f(x) = u_1 + (u_2 - u_1)x/L$ then $F(x) = 0$ in (28), so the solution simply remains a constant with time, namely, $u(x,t) = u_1 + (u_2 - u_1)x/L$.

(ii) If $u_1 = u_2$ and $f(x)$ is of the form $C \sin n\pi x/L$ for some constant C and some integer n, then the solution is the single term

$$u(x,t) = C \sin \frac{n\pi x}{L} \exp[-(n\pi\alpha/L)^2 t],$$

which is of product form. Its shape is $C \sin \frac{n\pi x}{L}$, modulated in amplitude by the $\exp[-(n\pi\alpha/L)^2 t]$ factor.

4. (b) $u = XT$ gives $\frac{X'' + 2X'}{X} = \frac{T'}{T} = -k^2$, $X'' + 2X' + k^2 X = 0$, $T' + k^2 T = 0$.
Seeking $X = e^{\lambda x}$ gives $\lambda^2 + 2\lambda + k^2 = 0$, $\lambda = (-2 \pm \sqrt{4 - 4k^2})/2 = -1 \pm \sqrt{1 - k^2}$
so we obtain distinct roots and hence the general solution – provided that $k \neq 1$; if $k = 1$ then $\lambda = -1, -1$ and the solutions are e^{-x} and $\underline{x} e^{-x}$. Thus,

$$X(x) = \begin{cases} A e^{(-1+\sqrt{1-k^2})x} + B e^{(-1-\sqrt{1-k^2})x}, & k \neq 1 \\ (C + Dx) e^{-x}, & k = 1 \end{cases}$$

and

$$T(t) = \begin{cases} E e^{-k^2 t}, & k \neq 1 \\ F e^{-t}, & k = 1 \end{cases}$$

so we can form

$$\begin{aligned} u(x,t) &= (C + Dx) e^{-x} F e^{-t} + e^{-x}(A e^{\sqrt{1-k^2}x} + B e^{-\sqrt{1-k^2}x}) E e^{-k^2 t} \\ &= (C' + D'x) e^{-(x+t)} + e^{-x}(\underbrace{A' e^{\sqrt{1-k^2}x} + B' e^{-\sqrt{1-k^2}x}}_{*}) e^{-k^2 t} \end{aligned}$$

NOTE: Of course, we could use the form $A'' \cosh \sqrt{1-k^2}x + B'' \sinh \sqrt{1-k^2}x$ in place of $*$ if we wish. Further, note that the latter form is fine if k^2 turns out to be smaller than 1, but if it is greater than 1 then we are well-advised to re-express

$$\begin{aligned} A'' \cosh \sqrt{1-k^2}x + B'' \sinh \sqrt{1-k^2}x &= A'' \cosh i\sqrt{k^2-1}x + B'' \sinh i\sqrt{k^2-1}x \\ &= A'' \cos \sqrt{k^2-1}x + iB'' \sin \sqrt{k^2-1}x \\ &= A'' \cos \sqrt{k^2-1}x + B''' \sin \sqrt{k^2-1}x \end{aligned}$$

Anticipating an eventual Fourier (or, more generally, eigenfunction) expansion it is probably best to use the cosine and sine version rather than the cosh, sinh version.

(d) $u = XT$ gives $X''/X + 2X'T'/XT = T''/T$. Can't be separated due to the $X'T'/XT$ term.

NOTE: The following problem might be useful for lecture: $u_{xx} + 2u_x = u_{tt}$. Solution:

$u = XT$ gives $\frac{X'' + 2X'}{X} = \frac{T''}{T} = -k^2$, $X'' + 2X' + k^2 X = 0$, $T'' + k^2 T = 0$.
Proceeding as in (b), above, write

$$X(x) = \begin{cases} e^{-x}(A \cos \sqrt{k^2-1}x + B \sin \sqrt{k^2-1}x), & k \neq 1 \\ (C + Dx) e^{-x}, & k = 1 \end{cases}$$

Then, $T'' + k^2 T = 0$ gives $T(t) = E \cos kt + F \sin kt$. The latter is the general solution for all $k \neq 0$, since the sine term drops out if $k = 0$. The X solution dictated distinguishing the cases $k \neq 1$, $k = 1$ and the T solution dictates also distinguishing the case $k = 0$. Thus, we have

$$X(x) = \begin{cases} e^{-x}(A\cos\sqrt{k^2-1}\,x + B\sin\sqrt{k^2-1}\,x) \\ (C+Dx)e^{-x} \\ E+Fe^{-2x} \end{cases} \qquad T(t) = \begin{cases} G\cos kt + H\sin kt, & k \neq 0,1 \\ I\cos t + J\sin t, & k=1 \\ L+Mt, & k=0 \end{cases}$$

so

$$u(x,t) = (E+Fe^{-2x})(L+Mt) + (C+Dx)e^{-x}(I\cos t + J\sin t) + e^{-x}(A\cos\sqrt{k^2-1}\,x + B\sin\sqrt{k^2-1}\,x)(G\cos kt + H\sin kt)$$

5. No, this is a serious error. We can superimpose various solutions of the same (linear) ODE, but here we would be superimposing solutions of different ODE's. Namely, $A\cos kx + B\sin kx$ is a solution of $X'' + k^2X = 0$ for $k \neq 0$, and $D+Ex$ is a solution of $X''=0$ (i.e., for $k=0$); these are different ODE's! If not convinced, put $(A\cos kx + B\sin kx + D + Ex)(Fe^{-k^2\alpha^2t} + G)$ into $\alpha^2 u_{xx} = u_t$ and you will see that it does not work.

NOTE: This error is a common one, and is similar to the error in saying that if the eigenvalue problem $Ax = \lambda x$ has eigenpairs λ_1, e_1 and λ_2, e_2 then the solution of $Ax = \lambda x$ is $x = C_1e_1 + C_2e_2$.

6. (b) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2t}$

$u(0,t) = 10 = A + C\,e^{-k^2\alpha^2t} \longrightarrow A = 10, C = 0$ so

$u(x,t) = 10 + Bx + D\sin kx\; e^{-k^2\alpha^2t}$

$u_x(2,t) = -5 = B + kD\cos 2k\; e^{-k^2\alpha^2t} \rightarrow B = -5,\ 2k = n\pi/2\ (n=1,3,\dots)$ so

$u(x,t) = 10 - 5x + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}\, \exp[-(n\pi\alpha/4)^2 t]$ ①

$u(x,0) = f(x) = \cancel{10} = \cancel{10} - 5x + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}$

or,

$5x = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0 < x < 2)$ Here, $L=2$.

QRS: $D_n = \frac{2}{2}\int_0^2 5x \sin\frac{n\pi x}{4}\,dx = \frac{40}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right)$ ②

Solution given by ① and ②. $u_s(x) = 10 - 5x$.

(c) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2t}$

$u(0,t) = 0 = A + C\exp(-k^2\alpha^2t) \rightarrow A = C = 0$ so

$u(x,t) = Bx + D\sin kx\,\exp(-k^2\alpha^2t)$

$u_x(2,t) = 0 = B + kD\cos 2k\, \exp(\ \text{"}\ ) \rightarrow B = 0,\ 2k = n\pi/2\ (n=1,3,\dots)$ so

$u(x,t) = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}\, \exp[-(n\pi\alpha/4)^2 t]$ ①

$u(x,0) = f(x) = 50\sin\frac{\pi x}{2} = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0 < x < 2) \quad (L=2)$

QRS: $D_n = \frac{2}{2}\int_0^2 50\sin\frac{\pi x}{2}\sin\frac{n\pi x}{4}\,dx = -\frac{400}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2-4}$ ②

Solution given by ① and ②. $u_s(x) = 0$.

(e) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\, e^{-k^2\alpha^2 t}$

$u(0,t) = 25 = A + C\exp(-k^2\alpha^2 t) \rightarrow A = 25,\ C = 0$ so

$u(x,t) = 25 + Bx + D\sin kx\ \exp(-k^2\alpha^2 t)$

$u_x(4,t) = 0 = B + kD\cos 4k\ \exp(\ {}''\ ) \rightarrow B = 0,\ 4k = n\pi/2\ (n = 1,3,\dots)$ so

$u(x,t) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8}\ \exp[-(n\pi\alpha/8)^2 t]$ ①

$u(x,0) = 25 = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8}$ or, $0 = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{8}$

QRS: D_n's $= 0$ by inspection! Thus,
$u(x,t) = 25$

(With hindsight, this was "obvious.") $u_s(x) = 25$.

(f) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\, e^{-k^2\alpha^2 t}$

$u(0,t) = 25 = A + C\exp(-k^2\alpha^2 t) \rightarrow A = 25,\ C = 0$ so, updating our solution,*

$u(x,t) = 25 + Bx + D\sin kx\ \exp(-k^2\alpha^2 t)$

$u_x(2,t) = 0 = B + kD\cos 2k\ \exp(\ {}''\ ) \rightarrow B = 0,\ 2k = n\pi/2\ (n = 1,3,\dots)$ so

$u(x,t) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}\ \exp[-(n\pi\alpha/4)^2 t]$ ①

$u(x,0) = f(x) = 25 + \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}$ or, $f(x) - 25 = \sum_{1,3,\dots}^{\infty} D_n \sin\frac{n\pi x}{4}\quad (0 < x < 2)$

QRS: $D_n = \frac{2}{2}\int_0^2 [f(x) - 25]\sin\frac{n\pi x}{4}\,dx = \int_0^1 -25\sin\frac{n\pi x}{4}\,dx + \int_1^2 0\,dx = -\frac{100}{n\pi}\left(1 - \cos\frac{n\pi}{4}\right)$ ②

Solution given by ① and ②. $u_s(x) = 25$.

*** NOTE: As a procedural matter, we recommend "updating" the solution before moving on to the next boundary or initial condition. Also, it is helpful to write the arguments: for example, u(0,t) rather than just u, so we do not mistake a boundary conditon u(0,t) = etc. for the solution u(x,t) = etc.**

(h) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\, e^{-k^2\alpha^2 t}$

$u_x(0,t) = 0 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = D = 0$ so

$u(x,t) = A + C\cos kx\ \exp(-k^2\alpha^2 t)$

$u_x(3\pi,t) = 0 = -kC\sin 3\pi k\ \exp(\ {}''\ ) \rightarrow 3\pi k = n\pi\ (n = 1,2,\dots)$ so

$u(x,t) = A + \sum_{1}^{\infty} C_n \cos\frac{nx}{3}\ \exp[-(n\pi\alpha/3)^2 t]$ ①

$u(x,0) = f(x) = A + \sum_{1}^{\infty} C_n \cos\frac{nx}{3}\quad (0 < x < 3\pi)\ (L = 3\pi)$

HRC:

$A = \frac{1}{3\pi}\int_0^{3\pi} f\,dx = \frac{1}{3\pi}\int_{2\pi}^{3\pi} 60\,dx = 20,\ C_n = \frac{2}{3\pi}\int_0^{3\pi} f\cos\frac{nx}{3}\,dx = -\frac{120}{n\pi}\sin\frac{2n\pi}{3}$ ②

Solution given by ① and ②. $u_s(x) = 20$.

(i) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\,e^{-k^2\alpha^2 t}$

$u_x(0,t) = 5 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = 5,\ D = 0$ so

$u(x,t) = A + 5x + C\cos kx \exp(\;''\;)$

$u_x(10,t) = 5 = 5 - kC\sin 10k \exp(\;''\;) \rightarrow 10k = n\pi \quad (n = 1, 2, \dots)$ so

$$u(x,t) = A + 5x + \sum_1^\infty C_n \cos\frac{n\pi x}{10} \exp[-(n\pi\alpha/10)^2 t]$$

$$u(x,0) = f(x) = 45 + \cancel{5x} = A + \cancel{5x} + \sum_1^\infty C_n \cos\frac{n\pi x}{10}$$

$$45 = A + \sum_1^\infty C_n \cos\frac{n\pi x}{10} \quad (0 < x < 10)$$

HRC: By inspection (or by the integral formulas) $A = 45$, $C_n = 0$, so

$$u(x,t) = 45 + 5x$$

(j) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\,e^{-k^2\alpha^2 t}$

$u_x(0,t) = 3 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = 3,\ D = 0$ so

$u(x,t) = A + 3x + C\cos kx \exp(\;''\;)$

$u_x(5,t) = \cancel{3} = \cancel{3} - kC\sin 5k \exp(\;''\;) \rightarrow 5k = n\pi \quad (n = 1, 2, \dots)$ so

$$u(x,t) = A + 3x + \sum_1^\infty C_n \cos\frac{n\pi x}{5} \exp[-(n\pi\alpha/5)^2 t] \quad ①$$

$$u(x,0) = 2x = A + 3x + \sum_1^\infty C_n \cos\frac{n\pi x}{5}$$

or, $-x = A + \sum_1^\infty C_n \cos\frac{n\pi x}{5} \quad (0 < x < 5) \quad (L = 5)$

HRC: $A = \frac{1}{5}\int_0^5 (-x)\,dx = -5/2$, $C_n = \frac{2}{5}\int_0^5 (-5x)\cos\frac{n\pi x}{5}\,dx = \begin{cases} 0, & n = 2, 4, \dots \\ \frac{100}{n^2\pi^2}, & n = 1, 3, \dots \end{cases}$ ②

so ① and ② give $u(x,t) = -\frac{5}{2} + 3x + \frac{100}{\pi^2}\sum_{1,3,\dots}^\infty \frac{1}{n^2}\cos\frac{n\pi x}{5}\exp[-(n\pi\alpha/5)^2 t]$,

$u_s(x) = -\frac{5}{2} + 3x$.

(k) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)\,e^{-k^2\alpha^2 t}$

$u(0,t) = 0 = A + C\exp(-k^2\alpha^2 t) \rightarrow A = C = 0$ so

$u(x,t) = Bx + D\sin kx \exp(\;''\;)$

$u(5,t) = 0 = 5B + D\sin 5k \exp(\;''\;) \rightarrow B = 0,\ 5k = n\pi \quad (n = 1, 2, \dots)$ so

$$u(x,t) = \sum_1^\infty D_n \sin\frac{n\pi x}{5}\exp[-(n\pi\alpha/5)^2 t] \quad ①$$

$$u(x,0) = \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5} = \sum_1^\infty D_n \sin\frac{n\pi x}{5} \quad (0 < x < 5)$$

HRS:

By inspection, $D_1 = -37$, $D_5 = 1$, $D_9 = 6$, all other D_n's $= 0$, so

$$u(x,t) = -37\sin\frac{\pi x}{5}\exp[-(\pi\alpha/5)^2 t] + \sin\pi x \exp[-(\pi\alpha)^2 t] + 6\sin\frac{9\pi x}{5}\exp[-(9\pi\alpha/5)^2 t]$$

and $u_s(x) = 0$.

(ℓ) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u(0,t) = 0 = A + C\exp(-k^2\alpha^2 t) \rightarrow A = C = 0$ so

$u(10,t) = 100 = 10B + D\sin 10k\,\exp(-k^2\alpha^2 t) \rightarrow B = 10,\ 10k = n\pi$ $(n = 1,2,\ldots)$ so

$u(x,t) = 10x + \sum_1^\infty D_n \sin\frac{n\pi x}{10}\exp[-(n\pi\alpha/10)^2 t]$ ①

$u(x,0) = 0 = 10x + \sum_1^\infty D_n \sin\frac{n\pi x}{10}$ or, $-10x = \sum_1^\infty D_n \sin\frac{n\pi x}{10}$ $(0<x<10)$

HRS:

$$D_n = \frac{2}{10}\int_0^{10}(-10x)\sin\frac{n\pi x}{10}\,dx = \frac{200}{n\pi}(-1)^n \quad ②$$

Solution given by (1) and (2). $u_s(x) = 10x$.

(m) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u_x(0,t) = 2 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = 2,\ D = 0$ so

$u(x,t) = A + 2x + C\cos kx\,\exp('')$

$u(6,t) = 12 = A + 12 + C\cos 6k\,\exp('') \rightarrow A = 0,\ 6k = n\pi/2$ $(n = 1,3,\ldots)$ so

$u(x,t) = 2x + \sum_{1,3,\ldots}^\infty C_n \cos\frac{n\pi x}{12}\exp('')$ ①

$u(x,0) = 0 = 2x + \sum_{1,3,\ldots}^\infty C_n \cos\frac{n\pi x}{12}$ $(0<x<6)$

or,

$-2x = \sum_{1,3,\ldots}^\infty C_n \cos\frac{n\pi x}{12}.$ $(0<x<6)$ #

NOTE: As usual, we move any known terms on the right-hand side of #, namely, the 2x term, to the left, and then seek to identify the series as HRC, HRS, QRC, or QRS. It helps to write, to the right of the equation, the interval on which the expansion is to hold (in this case $0 < x < 6$) since then we can see the $\cos(n\pi x/12)$ tearm as being of the form $\cos(n\pi x/2L)$. That fact, together with the absence of a constant term and the fact that the series is over n = 1,3,... tell us that the series is a QRC series.

QRC: $C_n = \frac{2}{6}\int_0^6(-2x)\cos\frac{n\pi x}{12}\,dx = \frac{48}{n^2\pi^2}\left(2 - n\pi\sin\frac{n\pi}{2}\right)$ ②

Solution given by ① and ②. $u_s(x) = 2x$.

(n) $u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$

$u_x(0,t) = 0 = B + kD\exp(-k^2\alpha^2 t) \rightarrow B = D = 0$ so

$u(x,t) = A + C\cos kx\,\exp(\ ''\ )$

$u(6,t) = 0 = A + C\cos 6k\,\exp(\ ''\ ) \rightarrow A = 0,\ 6k = n\pi/2$ $(n = 1,3,\ldots)$ so

$u(x,t) = \sum_{1,3,\ldots}^\infty C_n \cos\frac{n\pi x}{12}\exp(\ ''\ )$ ①

$u(x,0) = \sin x = \sum_{1,3,\ldots}^\infty C_n \cos\frac{n\pi x}{12}$ $(0<x<6)$

QRC: $C_n = \frac{2}{6}\int_0^6 \sin x \cos\frac{n\pi x}{12}\,dx = 4\,\frac{n\pi \sin 6 \sin\frac{n\pi}{2} - 12}{n^2\pi^2 - 144}$ ②

Solution given by ① and ②. $u_S(x) = 0$.

8. $K_n = \frac{2}{L}\int_0^L 40\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = -80\,\frac{\sin n\pi}{n^2-1} = 0$ because $\sin n\pi = 0$. However, for $n=1$ it is 0/0 and hence indeterminate. L'Hôpital's rule gives $K_1 = -80\lim_{n\to 1}\frac{\sin n\pi}{n^2-1} = -80\lim_{n\to 1}\frac{\pi\cos n\pi}{2n} = (-80)(-\frac{1}{2}) = 40$. Alternatively, we could work out K_1 separately: $K_1 = \frac{80}{L}\int_0^L \sin^2\frac{\pi x}{L}\,dx = \frac{80}{L}\frac{L}{2} = 40$.

9. (a) We obtain $u(x,t) = \sum_1^\infty D_n \sin\frac{n\pi x}{2}\, e^{-(n\pi\alpha/2)^2 t}$ ①

$u(x,0) = \sum_1^\infty D_n \sin\frac{n\pi x}{2} \quad (0<x<2)$

HRS:

$$D_n = \frac{2}{2}\int_0^2 u(x,0)\sin\frac{n\pi x}{2}\,dx = \int_0^1 50x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (100-5x)\sin\frac{n\pi x}{2}\,dx$$
$$= -\frac{100}{n^2\pi^2}\left(n\pi\cos\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}\right) - \frac{10}{n^2\pi^2}\left[18(-1)^n n\pi - 19n\pi\cos\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}\right]$$
$$= -\frac{100}{n^2\pi^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]$$

so $u(x,t) = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}\, e^{-(n\pi\alpha/2)^2 t}$

(b) With $\alpha^2 = 2.9\times 10^{-5}$,

$u(1,3600) = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - 0.9n\pi\cos\frac{n\pi}{2} - 2.2\sin\frac{n\pi}{2}\right]\sin\frac{n\pi}{2}\, e^{-(n\pi/2)^2(0.1044)}$

The Maple commands

```
S:= sum((1/i^2)*(1.8*(-1)^i*i*Pi-.9*i*Pi*cos(i*Pi/2)-2.2sin(i*Pi/2))
        *sin(i*Pi/2)*exp(-.1044*(i*Pi/2)^2), i=1..1):
    u:= -100*S/Pi^2;
```

gives $u(1,3600) = 61.51290869$.

Changing i=1..1 to i=1..3 gives 59.87675495
" " " i=1..5 " 59.89647346
" " " i=1..10 " <u>59.89644798</u>

and further increase of the upper limit of summation gives (to this many decimal places) no further change. (Of course, in most applications we don't <u>need</u> this many correct significant figures.)

(c) We wish to solve

$$u(1,t) = 5 = -\frac{100}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left[1.8(-1)^n n\pi - \overset{*}{\boxed{0.9n\pi\cos\frac{n\pi}{2}}} - 2.2\sin\frac{n\pi}{2}\right]\overset{♀}{\boxed{\sin\frac{n\pi}{2}}}$$
$$\times \exp\left[-(n\pi/2)^2(0.000029)t\right]$$

for t. Actually, the * term can be omitted since the ♀ term is nonzero only for n odd, and if n is odd then the $\cos n\pi/2$ is 0.

To solve, use the Maple commands

```
u: = -(100/Pi^2)*sum((1/i^2)*(1.8*(-1)^i*i*Pi-2.2*sin(i*Pi/2))
          *sin(i*Pi/2)* exp(-(i*Pi/2)^2*.000029*t, i=1..1):
  fsolve( u=5, t);
```

and obtain $t = 38675.42518$ seconds ($\simeq 10.74$ hrs.)

To see how many significant figures can be believed, let us change $i=1..1$ to $i=1..3$ (which sums the first $\underline{3}$ terms — the 2nd term being 0 due to the $\sin n\pi/2$ factor). In that case we obtain $t=-2642$, which is obviously incorrect. To provide some help, include a search interval option in fsolve, such as fsolve(u=5, t, t=0..50000); and obtain

$$t = 38675.42517.$$

Evidently we already have 10 significant figure accuracy, the reason being that the $\exp[-(n\pi/2)^2(0.000029)t]$ factor causes faster and faster convergence as t increases.

(d) $t = 61167.86024$

10. (a) $\alpha^2 u_s'' = 0$, $u_s(x) = A + Bx$, $u_s(0) = u_1 = A$, $u_s'(L) = Q_2 = B$, so $u_s(x) = u_1 + Q_2 x$.

(b) $\alpha^2 u_s'' = 0$, $u_s(x) = A + Bx$, $u_s'(0) = Q_1 = B$, $u_s(L) = u_2 = A + BL$ gives $A = u_2 - Q_1 L$ and $B = Q_1$, so $u_s(x) = u_2 - Q_1 L + Q_1 x$

(c) $\alpha^2 u_s'' = 0$, $u_s(x) = A + Bx$, $u_s'(0) = Q_1 = B$, $u_s'(L) = Q_2 = B$, which give no solution if $Q_1 \neq Q_2$. Physically, Fourier's law of heat conduction tells us that

$$\text{Heat in at left end} = -u_x(0,t)kA = -Q_1 kA$$
$$\text{Heat out at right end} = u_x(L,t)kA = Q_2 kA$$

so $\quad$ net heat input $= (Q_2 - Q_1)kA$.

If the latter is nonzero then the temperature will increase indefinitely (if $Q_2 > Q_1$, and decrease indefinitely if $Q_2 < Q_1$) and a steady state will not exist! Only if $(Q_2 - Q_1)kA = 0$, i.e. if $Q_2 = Q_1$, will there exist a steady state. Let $Q_1 = Q_2 \equiv Q$. Then, from above, $B = Q$ and $u_s(x) = A + Qx$. To determine A, integrate the PDE on x:

$$\begin{aligned} \alpha^2 \int_0^L u_{xx}\,dx &= \int_0^L u_t\,dx \\ \alpha^2 u_x\Big|_0^L &= \frac{d}{dt}\int_0^L u(x,t)\,dx \\ \alpha^2(Q-Q) &= \quad '' \\ 0 &= \quad '' \end{aligned}$$

so $\quad \int_0^L u(x,t)\,dx = \text{constant},$

which result gives us a connection between the steady state and the initial condition:

$$\begin{aligned} \int_0^L u(x,\infty)\,dx &= \int_0^L u(x,0)\,dx, \\ \int_0^L (A+Qx)\,dx &= \int_0^L f(x)\,dx, \\ AL + QL^2/2 &= \int_0^L f(x)\,dx, \end{aligned}$$

so $A = \frac{1}{L}\int_0^L f(x)\,dx - QL/2$ and

$$u_s(x) = \left(\frac{1}{L}\int_0^L f(x)\,dx - QL/2\right) + Qx.$$

(d) $u_s'' - \frac{H}{\alpha^2}u_s = 0$, $u_s(x) = A\sinh\frac{\sqrt{H}}{\alpha}x + B\cosh\frac{\sqrt{H}}{\alpha}x$

$$\left.\begin{aligned} u_s(0) &= u_1 = B \\ u_s(L) &= u_2 = A\sinh\frac{\sqrt{H}}{\alpha}L + B\cosh\frac{\sqrt{H}}{\alpha}L \end{aligned}\right\} \begin{aligned} &B = u_1, \\ &A = (u_2 - u_1\cosh\frac{\sqrt{H}}{\alpha}L)/\sinh\frac{\sqrt{H}}{\alpha}L \end{aligned}$$

so $u_s(x) = (u_2 - u_1\cosh\frac{\sqrt{H}}{\alpha}L)\frac{\sinh\sqrt{H}x/\alpha}{\sinh\sqrt{H}L/\alpha} + u_1\cosh\frac{\sqrt{H}}{\alpha}x$

(e) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{aligned} u_s'(0) &= Q_1 = \sqrt{H}A/\alpha \\ u_s(L) &= u_2 = A\sinh\sqrt{H}L/\alpha + B\cosh\sqrt{H}L/\alpha \end{aligned}\right\} \begin{aligned} &A = \alpha Q_1/\sqrt{H} \\ &B = (u_2 - \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}L}{\alpha})/\cosh\frac{\sqrt{H}L}{\alpha} \end{aligned}$$

so $u_s(x) = \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}x}{\alpha} + (u_2 - \frac{\alpha Q_1}{\sqrt{H}}\sinh\frac{\sqrt{H}L}{\alpha})\frac{\cosh\sqrt{H}x/\alpha}{\cosh\sqrt{H}L/\alpha}$

(f) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{aligned} u_s(0) &= u_1 = B \\ u_s'(L) &= Q_2 = \frac{\sqrt{H}}{\alpha}A\cosh\frac{\sqrt{H}L}{\alpha} + \frac{\sqrt{H}}{\alpha}B\sinh\frac{\sqrt{H}L}{2} \end{aligned}\right\} \begin{aligned} &B = u_1 \\ &A = (\frac{\alpha Q_2}{\sqrt{H}} - u_1\sinh\frac{\sqrt{H}L}{\alpha})/\cosh\frac{\sqrt{H}L}{\alpha} \end{aligned}$$

so $u_s(x) = (\frac{\alpha Q_2}{\sqrt{H}} - u_1\sinh\frac{\sqrt{H}L}{\alpha})\frac{\sinh\sqrt{H}x/\alpha}{\cosh\sqrt{H}L/\alpha} + u_1\cosh\frac{\sqrt{H}x}{\alpha}$

(g) $u_s(x) = A\sinh\sqrt{H}x/\alpha + B\cosh\sqrt{H}x/\alpha$

$$\left.\begin{aligned} u_s'(0) &= Q_1 = \sqrt{H}A/\alpha \\ u_s'(L) &= Q_2 = \frac{\sqrt{H}}{\alpha}A\cosh\frac{\sqrt{H}L}{\alpha} + \frac{\sqrt{H}}{\alpha}B\sinh\frac{\sqrt{H}L}{\alpha} \end{aligned}\right\} \begin{aligned} &A = \alpha Q_1/\sqrt{H} \\ &B = (Q_2 - Q_1\cosh\frac{\sqrt{H}L}{\alpha})/\frac{\sqrt{H}}{\alpha}\sinh\frac{\sqrt{H}L}{\alpha} \end{aligned}$$

so $u_s(x) = \frac{\alpha Q_1}{\sqrt{H}}\sinh\sqrt{H}x/\alpha + \left(Q_2 - Q_1\cosh\frac{\sqrt{H}L}{\alpha}\right)\frac{\alpha}{\sqrt{H}}\frac{\cosh\sqrt{H}x/\alpha}{\sinh\sqrt{H}L/\alpha}$

(h) $u_s'' - \frac{V}{\alpha^2}u_s' = 0$, $u_s(x) = A + Be^{Vx/\alpha^2}$

$$\left.\begin{aligned} u_s(0) &= u_1 = A + B \\ u_s(L) &= u_2 = A + Be^{VL/\alpha^2} \end{aligned}\right\} \begin{aligned} &B = (u_1 - u_2)/(1 - e^{VL/\alpha^2}) \\ &A = u_1 - (u_1 - u_2)/(1 - e^{VL/\alpha^2}) \end{aligned}$$

so $u_s(x) = \frac{u_2 - u_1\exp(VL/\alpha^2)}{1 - \exp(VL/\alpha^2)} + \frac{u_1 - u_2}{1 - \exp(VL/\alpha^2)}\exp(Vx/\alpha^2)$

(i) $u_s(x) = A + Be^{Vx/\alpha^2}$,
$$\left.\begin{aligned} u_s'(0) &= Q_1 = BV/\alpha^2 \\ u_s(L) &= u_2 = A + Be^{VL/\alpha^2} \end{aligned}\right\} \begin{aligned} &B = \alpha^2 Q_1/V \\ &A = u_2 - \frac{\alpha^2 Q_1}{V}e^{VL/\alpha^2} \end{aligned}$$

so $u_s(x) = u_2 - \frac{\alpha^2 Q_1}{V}e^{VL/\alpha^2} + \frac{\alpha^2 Q_1}{V}e^{Vx/\alpha^2} = u_2 - \frac{\alpha^2 Q_1}{V}(e^{VL/\alpha^2} - e^{Vx/\alpha^2})$

(j) $u_s(x) = A + Be^{Vx/\alpha^2}$,
$$\left.\begin{aligned} u_s(0) + 5u_s'(0) &= 3 = A + B + 5BV/\alpha^2 \\ u_s(L) &= 10 = A + Be^{VL/\alpha^2} \end{aligned}\right\} \begin{aligned} &B = 7/(e^{VL/\alpha^2} - 1 - 5V/\alpha^2) \\ &A = 10 - 7e^{VL/\alpha^2}/(e^{VL/\alpha^2} - 1 - \frac{5V}{\alpha^2}) \end{aligned}$$

$$u_s(x) = \frac{3\exp(VL/\alpha^2) - 10 - 50V/\alpha^2 + 7\exp(Vx/\alpha^2)}{\exp(VL/\alpha^2) - 1 - 5V/\alpha^2}$$

11. If there is a steady state $u_S(x)$ then it satisfies $u_S''(x) = F(x)$. Integrating,

$$\alpha^2 \int_0^L u_S''(x)\,dx = \int_0^L F(x)\,dx,$$
$$\alpha^2 u_S'(x)\Big|_0^L = \text{"}$$
$$\alpha^2 (Q_2 - Q_1) = \int_0^L F(x)\,dx, \qquad ①$$

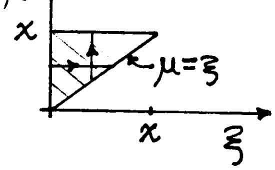
which relation must be satisfied by $Q_1, Q_2, F(x)$. In words, ① says that the net heat flux into the rod through its ends must equal the net absorption of heat by the distributed "sink" $F(x)$ if a steady state is to be maintained. Assuming that ① is satisfied let us solve for $u_S(x)$.

$$\alpha^2 u_S''(x) = F(x)$$
$$\alpha^2 u_S'(x) = \int_0^x F(\xi)\,d\xi + A \qquad ②$$

and setting $x=0$ in ② gives $A = \alpha^2 Q_1$. Integrating again,

$$u_S(x) = \frac{1}{\alpha^2}\int_0^x \int_0^\mu F(\xi)\,d\xi\,d\mu + Q_1 x + B \qquad ③$$

We can reduce the double integral in ③ to a single integral by reversing the order of integration:



$$\int_0^x \int_0^\mu F(\xi)\,d\xi\,d\mu = \int_0^x \int_\xi^x F(\xi)\,d\mu\,d\xi = \int_0^x (x-\xi)F(\xi)\,d\xi$$

so $$u_S(x) = \frac{1}{\alpha^2}\int_0^x (x-\xi)F(\xi)\,d\xi + Q_1 x + B. \qquad ④$$

To evaluate B we establish a conservation principle relating $u_S(x)$ to the initial condition $u(x,0) = f(x)$, as we did in Exercise 10c. Integrating $\alpha^2 u_{xx} = u_t + F(x)$ on x from 0 to L and then using ① gives

$$\alpha^2 \int_0^L u_{xx}\,dx = \int_0^L u_t\,dx + \int_0^L F(x)\,dx$$
$$\cancel{\alpha^2 (Q_2 - Q_1)} = \frac{d}{dt}\int_0^L u(x,t)\,dx + \cancel{\int_0^L F(x)\,dx}$$

so

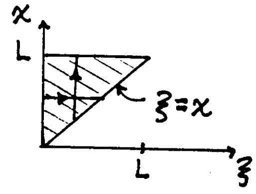
$$\frac{d}{dt}\int_0^L u(x,t)\,dx = 0, \quad \text{or,} \quad \int_0^L u(x,t)\,dx = \text{constant}.$$

Hence, $$\int_0^L u(x,\infty)\,dx = \int_0^L u(x,0)\,dx$$
$$\int_0^L u_S(x)\,dx = \int_0^L f(x)\,dx$$

or, from ④,

$$\frac{1}{\alpha^2}\int_0^L \int_0^x (x-\xi)F(\xi)\,d\xi\,dx + Q_1\frac{L^2}{2} + BL = \int_0^L f(x)\,dx.$$

Solving for B and reducing the double integral to a single integral by reversing the order of integration,



$$B = \frac{1}{L}\int_0^L f(x)\,dx - \frac{Q_1 L}{2} - \frac{1}{\alpha^2 L}\int_0^L \int_\xi^L (x-\xi)F(\xi)\,dx\,d\xi$$
$$= \frac{1}{L}\int_0^L f(x) - \frac{Q_1}{2}L + \frac{1}{\alpha^2 L}\int_0^L \frac{(\xi-L)^2}{2}F(\xi)\,d\xi$$

so

$$u_S(S) = \frac{1}{\alpha^2}\int_0^x (x-\xi)F(\xi)\,d\xi + Q_1\left(x - \frac{L}{2}\right) + \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{2\alpha^2 L}\int_0^L (\xi-L)^2 F(\xi)\,d\xi.$$

12. (20) was $\frac{k}{c\sigma}u_{xx} - \frac{hs}{Ac\sigma}(u-u_\infty) - \frac{v}{c}u_x = u_t$. In steady state $u=u(x)$ so we have

$$u'' - \frac{hs}{Ak}(u-u_\infty) - \frac{v\sigma}{k}u' = 0.$$

With $v\sigma/k \equiv 2a$ and $h_f s/Ak \equiv b_f$, $h_a s/Ak \equiv b_a$, we have

$x<0$: $u''-2au'-b_f u = -b_f u_f$ | $x>0$: $u''-2au'-b_a u = -b_a u_a$

$u(-\infty) = u_f$ | $u(\infty) = u_a$

And at $x=0$ match u and u' from the two solutions.

Solving, $u(x) = e^{ax}(Be^{\sqrt{a^2+b_f}\,x} + \cancel{C}e^{-\sqrt{a^2+b_f}\,x}) + u_f$ in $x<0$

$u(x) = e^{ax}(\cancel{D}e^{\sqrt{a^2+b_a}\,x} + E\,e^{-\sqrt{a^2+b_a}\,x}) + u_a$ in $x>0$

$u(x) \to u_f$ as $x \to -\infty \Rightarrow C=0$

$u(x) \to u_a$ as $x \to +\infty \Rightarrow D=0$.

Then, matching u and u' at $x=0$ gives

$$B+u_f = E+u_a,$$

$$(a+\sqrt{a^2+b_f})B = (a-\sqrt{a^2+b_a})E$$

Solving for E (we don't need B if we desire only the solution for $x>0$) gives

$$E = \frac{a+\sqrt{a^2+b_a}}{2a+\sqrt{a^2+b_a}-\sqrt{a^2+b_f}}(u_f-u_a)$$

so, over $0<x<\infty$,

$$u(x) = u_a + \frac{a+\sqrt{a^2+b_a}}{2a+\sqrt{a^2+b_a}-\sqrt{a^2+b_f}}(u_f-u_a)e^{(a-\sqrt{a^2+b_a})x}$$

$u(L) =$ etc.

13. (a) I will denote $c_A(x,t)$ as $c(x,t)$, and $c_{A_s}(x)$ as $c_s(x)$, for brevity.

$c(x,t) = A+Bx+(C\cos\kappa x + D\sin\kappa x)\,e^{-\kappa^2 Dt}$

$c_x(0,t) = 0 = B + \kappa E\exp(-\kappa^2 Dt) \to B=E=0$ so

$c(x,t) = A + C\cos\kappa x\,\exp(\;''\;)$.

$c_x(L,t) = 0 = -\kappa C\sin\kappa L\,\exp(\;''\;) \to \kappa L = n\pi \;(n=1,2,\dots)$ so

$c(x,t) = A + \sum_1^\infty C_n\cos\frac{n\pi x}{L}\exp[-(n\pi/L)^2 Dt]$ ①

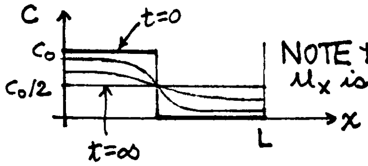
Then,

$c(x,0) = A + \sum_1^\infty C_n\cos\frac{n\pi x}{L}$ $(0<x<L)$

HRC: $A = \frac{1}{L}\int_0^L c(x,0)\,dx = c_0/2$ ②

$C_n = \frac{2}{L}\int_0^L c(x,0)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^{L/2} c_0\cos\frac{n\pi x}{L}dx = \frac{2c_0}{n\pi}\sin\frac{n\pi}{2}$ ③

Solution is given by ①-③. Sketch:



NOTE that the slope u_x is 0 at $x=0$ and $x=L$, for all t.

(b) $D\int_0^L c_{xx}\,dx = \int_0^L c_t\,dx$,

$Dc_x(x,t)\Big|_0^L = \frac{d}{dt}\int_0^L c(x,t)\,dx$,

$0 = \frac{d}{dt}\int_0^L c(x,t)\,dx$,

so $\int_0^L c(x,t)\,dx =$ constant.

(c) $Dc_S''(x)=0$ gives $c_S(x)=C_1+C_2x$. $c_S'(0)=0$ and $c_S'(L)=0$ give $C_2=0$ so
$c_S(x)=C_1$.

Now use (13.5) between $t=0$ and $t=\infty$:

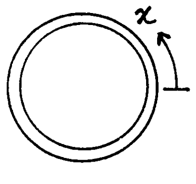
$$\int_0^L c(x,\infty)\,dx = \int_0^L c(x,0)\,dx$$
$$\int_0^L C_1\,dx = \int_0^{L/2} c_0\,dx + \int_{L/2}^{L} 0\,dx$$

gives $C_1L=\frac{c_0L}{2}$ so

$$c_S(x)=\frac{c_0}{2},$$

in agreement with the result obtained in (a).

14.



(a) See Answers to Selected Exercises, in text.

(b) Integrating,

$$\alpha^2\int_0^L u_{xx}\,dx = \int_0^L u_t\,dx$$
$$\alpha^2 u_x\Big|_0^L = \frac{d}{dt}\int_0^L u\,dx$$

But $u_x(L,t)=u_x(0,t)$ so

$$0=\frac{d}{dt}\int_0^L u(x,t)\,dx, \text{ or, } \int_0^L u(x,t)\,dx = \text{constant}.$$

(c) $\alpha^2u_s''=0$ gives $u_s(x)=A+Bx$

$$u_s(0)-u_s(L)=0=A-(A+BL)=-BL \rightarrow B=0$$
$$u_s'(0)-u_s'(L)=0=B-B$$

so $u_s(x)=A$ (i.e., a constant). To evaluate A use (14.2):

$$\int_0^L u(x,\infty)\,dx = \int_0^L u(x,0)\,dx$$
$$\int_0^L u_s(x)\,dx = \int_0^L f(x)\,dx$$
$$AL = \text{"} \quad , \quad A=\frac{1}{L}\int_0^L f(x)\,dx$$

and $c_s(x)=\frac{1}{L}\int_0^L f(x)\,dx$.

15. Seeking $u(x,t)=u_s(x)+X(x)T(t)$, we have, upon substitution,

$$\alpha^2(\cancel{u_s''})+\alpha^2X''T = XT'-\cancel{F}. \qquad ①$$

Ask u_s to be such that $\alpha^2u_s''(x)=F$.

Then

$$u_s'(x)=\frac{F}{\alpha^2}x+A$$
$$u_s(x)=\frac{Fx^2}{2\alpha^2}+Ax+B$$
$$\left.\begin{array}{l} u_s(0)=0=B \\ u_s(L)=50=\frac{FL^2}{2\alpha^2}+AL+B \end{array}\right\} \text{give } B=0,\ A=(100\alpha^2-FL^2)/2\alpha^2L$$

so $u_s(x)=\frac{100\alpha^2-FL^2}{2\alpha^2L}x+\frac{F}{2\alpha^2}x^2 = \frac{50x}{L}+\frac{F}{2\alpha^2}x(x-L).$ ②

Next, ① gives X,T in the usual way, so

$$u(x,t)=u_s(x)+C+Dx+(P\cos kx+Q\sin kx)e^{-k^2\alpha^2t}$$
$$u(0,t)=0=0+C+P\exp(-k^2\alpha^2t) \rightarrow C=P=0 \text{ so}$$
$$u(x,t)=u_s(x)+Dx+Q\sin kx\,\exp(-k^2\alpha^2t)$$
$$u(L,t)=50=50+DL+Q\sin kL\,\exp(\text{"}) \rightarrow D=0,\ kL=n\pi\ (n=1,2,\dots)$$

so $u(x,t)=u_s(x)+\sum_1^\infty Q_n\sin\frac{n\pi x}{L}\exp[-(n\pi\alpha/L)^2t]$ ③

$u(x,0)= f(x) = u_s(x) + \sum_1^{\infty} Q_n \sin\frac{n\pi x}{L}$

or, $f(x)-u_s(x)=\sum_1^{\infty} Q_n \sin\frac{n\pi x}{L} \quad (0<x<L)$

HRS: $Q_n = \frac{2}{L}\int_0^L [f(x)-u_s(x)]\sin\frac{n\pi x}{L}\,dx$ ④

Solution is given by ②-④.

16.(b) $\alpha^2 u_s'' = F$ gives $u_s(x) = \frac{F}{2\alpha^2}x^2 + Ax + B$

$\left.\begin{aligned} u_s'(0)&=0=0+A \\ u_s(L)&=0=\frac{FL^2}{2\alpha^2}+AL+B \end{aligned}\right\}$ give $A=0$, $B=-FL^2/2\alpha^2$

so

$u_s(x) = \frac{F}{2\alpha^2}(x^2-L^2)$ ①

$u(x,t) = u_s(x) + C + Dx + (P\cos kx + Q\sin kx)e^{-k^2\alpha^2 t}$

$u_x(0,t) = 0 = 0 + D + kQ\exp(-k^2\alpha^2 t) \rightarrow D=Q=0$ so

$u(x,t) = u_s(x) + C + P\cos kx\exp(-k^2\alpha^2 t)$

$u(L,t) = 0 = 0 + C + P\cos kL\exp(\;"\;) \rightarrow C=0$, $kL = n\pi/2$ $(n=1,3,\dots)$, so

$u(x,t) = u_s(x) + \sum_{1,3,\dots}^{\infty} P_n\cos\frac{n\pi x}{2L}\exp[-(n\pi\alpha/2L)^2 t]$ ②

$u(x,0) = 0 = u_s(x) + \sum_{1,3,\dots}^{\infty} P_n\cos\frac{n\pi x}{2L}$

so $P_n = -\frac{2}{L}\int_0^L u_s(x)\cos\frac{n\pi x}{2L}\,dx.$ ③

Solution is given by ①-③.

(d) $\alpha^2 u_s'' = F$ gives $u_s(x) = \frac{F}{2\alpha^2}x^2 + Ax + B$

$\left.\begin{aligned} u_s(0)&=0=B \\ u_s'(L)&=-20=\frac{F}{\alpha^2}L+A \end{aligned}\right\}$ so $B=0$, $A=-20-FL/\alpha^2$, so

$u_s(x) = -20x + \frac{Fx}{2\alpha^2}(x-2L)$ ①

$u(x,t) = u_s(x) + C + Dx + (P\cos kx + Q\sin kx)e^{-k^2\alpha^2 t}$

$u(0,t) = 0 = 0 + C + P\exp(-k^2\alpha^2 t) \rightarrow C=0, P=0$ so

$u(x,t) = u_s(x) + Dx + Q\sin kx\exp(-k^2\alpha^2 t)$

$u_x(L,t) = -20 = -20 + D + kQ\cos kL\exp(\;"\;) \rightarrow D=0$, $kL = n\pi/2$, $k = n\pi/2L$ (n odd), so

$u(x,t) = u_s(x) + \sum_{1,3,\dots}^{\infty} Q_n\sin\frac{n\pi x}{2L}\exp[-(n\pi\alpha/2L)^2 t]$ ②

$u(x,0) = 0 = u_s(x) + \sum_{1,3,\dots}^{\infty} Q_n\sin\frac{n\pi x}{2L}$,

$Q_n = -\frac{2}{L}\int_0^L u_s(x)\sin\frac{n\pi x}{2L}\,dx$ ③

Solution given by ①-③.

17. (a) Putting (17.2) and (17.3) into the PDE gives

$$\alpha^2 \sum_1^\infty -\left(\frac{n\pi}{L}\right)^2 g_n \sin\frac{n\pi x}{L} = \sum_1^\infty g_n' \sin\frac{n\pi x}{L} - \sum_1^\infty F_n \sin\frac{n\pi x}{L}$$

so, equating coefficients of sines,

$$g_n'(t) + \left(\frac{n\pi\alpha}{L}\right)^2 g_n(t) = F_n(t). \qquad (1a)$$

Now, putting $t=0$ into (17.2) and using the initial condition $u(x,0)=0$ gives

$$0 = \sum g_n(0) \sin\frac{n\pi x}{L} \quad (0<x<L)$$

so that

$$g_n(0) = 0 \qquad (1b)$$

for each n. Solving (1a) subject to the initial condition (1b) gives [from (24) on page 24, with $b=0$]

$$g_n(t) = e^{-(n\pi\alpha/L)^2 t}\int_0^t e^{(n\pi\alpha/L)^2\tau} F_n(\tau)\,d\tau$$
$$= \int_0^t e^{(n\pi\alpha/L)^2(\tau-t)} F_n(\tau)\,d\tau$$

so

$$u(x,t) = \sum_1^\infty \left[\int_0^t F_n(\tau) e^{(n\pi\alpha/L)^2(\tau-t)}\,d\tau\right] \sin\frac{n\pi x}{L},$$

where the $F_n(\tau)$'s are given by (17.4).

(b) If $F(x,t) = e^{-t}$, then

$$F_n(\tau) = \frac{2}{L}\int_0^L e^{-t} \sin\frac{n\pi x}{L}\,dx = \frac{2e^{-t}}{L}\,\frac{-\cos\frac{n\pi x}{L}}{n\pi/L}\Bigg|_0^L = \begin{cases} 4e^{-t}/n\pi, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

and

$$u(x,t) = \sum_{1,3,\dots}^\infty \left[\int_0^t \frac{4e^{-\tau}}{n\pi} e^{(n\pi\alpha/L)^2(\tau-t)}\,d\tau\right] \sin\frac{n\pi x}{L}$$
$$= \frac{4}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\,\frac{e^{-t} - e^{-(n\pi\alpha/L)^2 t}}{(n\pi\alpha/L)^2 - 1} \sin\frac{n\pi x}{L}$$

(c) In this case the eigenfunctions from the relevant Sturm-Liouville problem

$$X'' + \kappa^2 X = 0; \quad X'(0) = 0,\ X(L) = 0$$

will be $\cos n\pi x/2L$ $(n=1,3,\dots)$, so this time seek

$$u(x,t) = \sum_{1,3,\dots}^\infty g_n(t) \cos\frac{n\pi x}{2L},$$

where

$$F(x,t) = \sum_{1,3,\dots}^\infty F_n(t) \cos\frac{n\pi x}{2L}.$$

Once again we obtain equations (1a) and (1b), as in (a), but with $L \to 2L$, so

$$u(x,t) = \sum_{1,3,\dots}^\infty \left[\int_0^t F_n(\tau) e^{(n\pi\alpha/2L)^2(\tau-t)}\,d\tau\right] \cos\frac{n\pi x}{2L}$$

18.
$$\alpha^2 u_{1xx} = u_{1t} + g(x,t)$$
$$\alpha^2 u_{2xx} = u_{2t}$$
$$\alpha^2 u_{3xx} = u_{3t}$$
$$\underline{\alpha^2 u_{4xx} = u_{4t}}$$

Addition gives $\alpha^2(u_{1xx} + \cdots + u_{4xx}) = (u_{1t} + \cdots + u_{4t}) + g(x,t)$, or,

$$\alpha^2(u_1 + \cdots + u_4)_{xx} = (u_1 + \cdots + u_4)_t + g(x,t) \qquad ①$$

Likewise, add the boundary conditions and initial conditions:

$$\begin{array}{lll}
u_1(0,t)=0 & u_1(L,t)=0 & u_1(x,0)=0 \\
u_2(0,t)=p(t) & u_2(L,t)=0 & u_2(x,0)=0 \\
u_3(0,t)=0 & u_3(L,t)=q(t) & u_3(x,0)=0 \\
u_4(0,t)=0 & u_4(L,t)=0 & u_4(x,0)=f(x) \\
\hline
u_1(0,t)+\cdots+u_4(0,t)=p(t), & u_1(L,t)+\cdots+u_4(L,t)=q(t), & u_1(x,0)+\cdots+u_4(x,0)=f(x)
\end{array}$$

so we see that $u(x,t) \equiv u_1(x,t)+\cdots+u_4(x,t)$ satisfies the PDE, boundary conditions, and initial condition in (18.1).

19. This problem represents the class of problems where $u_x(0,t) \neq u_x(L,t)$ so there is a net heat influx and a steady state does not exist. Let us begin with separation of variables, nonetheless, so we can see how it fails.

$$u(x,t) = A+Bx+(C\cos kx + D\sin kx)e^{-k^2\alpha^2 t}$$

$$\left.\begin{array}{l} u_x(0,t)=-1 = B+kD\exp(-k^2\alpha^2 t) \rightarrow B=-1,\ D=0 \\ u_x(L,t)=0 = B+kD\cos kL\exp(\text{"}) \rightarrow B=0,\ kL=n\pi/2\ (n\text{ odd}) \end{array}\right\rangle \text{contradiction}$$

Following the hint, seek

$$u(x,t) = \frac{(x-L)^2}{2L} + v(x,t) \qquad ①$$

That gives the following problem on v:

$$\alpha^2 v_{xx} = v_t - \frac{\alpha^2}{L}, \quad v_x(0,t)=0, \quad v_x(L,t)=0, \quad v(x,0) = -\frac{(x-L)^2}{2L}. \qquad ②$$

The idea, then, is that the change of variables ① led to <u>homogeneous</u> Neumann b.c.'s. It's true that we now have a nonzero source term and initial condition, but we can solve this problem by the method outlined in Exercise 15. Actually, it is nice to break ② down first by superposition as

$$v(x,t) = v_1(x,t) + v_2(x,t)$$

where

$$\begin{array}{ll}
\alpha^2 v_{1xx} = v_{1t} & \alpha^2 v_{2xx} = v_{2t} - \frac{\alpha^2}{L} \\
v_{1x}(0,t) = v_{1x}(L,t) = 0,\ v_1(x,t) = -\frac{(x-L)^2}{2L}, & v_{2x}(0,t) = v_{2x}(L,t) = v_2(x,0) = 0,
\end{array}$$

because the v_2 problem is solved easily by inspection:

$$v_2(x,t) = \alpha^2 t/L.$$

20. $\alpha^2 u_{xx} = u_t$; $u(0,t) = p(t)$, $u(L,t) = q(t)$, $u(x,0) = f(x)$

Setting $u(x,t) = v(x,t) + (1-\frac{x}{L})p(t) + \frac{x}{L}q(t)$,

$$u_{xx} = v_{xx}$$

$$u_t = v_t + (1-\tfrac{x}{L})p'(t) + \tfrac{x}{L}q'(t)$$

so the v problem is

$$\alpha^2 v_{xx} = v_t + \left[(1-\tfrac{x}{L})p'(t) + \tfrac{x}{L}q(t)\right] \leftarrow \text{call this } -F(x,t)$$

$v(0,t) = 0$ because $u(0,t) = p(\cancel{t}) = v(0,t) + p(\cancel{t})$

$v(L,t) = 0$ because $u(L,t) = q(\cancel{t}) = v(L,t) + q(\cancel{t})$

$v(x,t) = f(x) - (1-\frac{x}{L})p(0) - \frac{x}{L}q(0)$ because $u(x,0) = f(x) = v(x,0) + (1-\frac{x}{L})p(0) + \frac{x}{L}q(0)$.

21. (a) $v_s(x)$ satisfies $\alpha^2 v_s'' = h v_s$

$$v_s'' - \frac{h}{\alpha^2}v_s = 0\,; \quad v_s(0)=50,\ v_s(L)=50$$

Solving,

$$v_s(x) = A\cosh\frac{\sqrt{h}}{\alpha}x + B\sinh\frac{\sqrt{h}}{\alpha}x$$

$$v_s(0) = 50 = A$$

$$v_s(L) = 50 = A\cosh\frac{\sqrt{h}L}{\alpha} + B\sinh\frac{\sqrt{h}}{\alpha}L$$

so $A = 50$,

$$B = 50\left(1-\cosh\frac{\sqrt{h}L}{\alpha}\right)\Big/\sinh\frac{\sqrt{h}L}{\alpha},$$

$$v_s(x) = 50\cosh\frac{\sqrt{h}}{\alpha}x + 50\left(1-\cosh\frac{\sqrt{h}L}{\alpha}\right)\frac{\sinh\frac{\sqrt{h}}{\alpha}x}{\sinh\frac{\sqrt{h}}{\alpha}L}$$

Though not essential, we can simplify the latter a <u>bit</u> using the identity $\sinh(x+y) = \sinh x\cosh y + \sinh y \cosh x$:

$$v_s(x) = 50\,\frac{\sinh(\sqrt{h}(L-x)/\alpha) + \sinh(\sqrt{h}x/\alpha)}{\sinh(\sqrt{h}L/\alpha)}. \qquad ①$$

$$X'' + k^2X = 0 \longrightarrow X = \begin{cases} C\cos kx + D\sin kx, & k \neq 0 & \text{(2a)} \\ E + Fx, & k = 0 & \text{(2b)} \end{cases}$$

$$T' + (k^2\alpha^2 + h)T = 0 \to T = \begin{cases} Ge^{-(k^2\alpha^2+h)t}, & k \neq 0 & \text{(3a)} \\ He^{-ht}, & k = 0 & \text{(3b)} \end{cases}$$

so $v(x,t) = v_s(x) + (E+Fx)He^{-ht} + (C\cos kx + D\sin kx)Ge^{-(k^2\alpha^2+h)t}$

$= v_s(x) + (E'+F'x)e^{-ht} + (C'\cos kx + D'\sin kx)e^{-(k^2\alpha^2+h)t}$

$v(0,t) = 50 = 50 + E'e^{-ht} + C'\exp[-(k^2\alpha^2+h)t] \to E' = C' = 0$ so

$v(x,t) = v_s(x) + F'xe^{-ht} + D'\sin kx\, \exp[-(k^2\alpha^2+h)t]$

$v(L,t) = 50 = 50 + F'Le^{-ht} + D'\sin kL\, \exp[\quad " \quad] \to F' = 0,\ kL = n\pi\ (n=1,2,..)$

so $v(x,t) = v_s(x) + \sum_1^\infty D_n' \sin\frac{n\pi x}{L}\exp[-(n^2\pi^2\alpha^2/L^2 + h)t]$ ④

Finally,

$$v(x,0) = f(x) = v_s(x) + \sum_1^\infty D_n'\sin\frac{n\pi x}{L},$$

$$f(x) - v_s(x) = \sum_1^\infty D_n'\sin\frac{n\pi x}{L} \qquad (0<x<L)$$

HRS:

$$D_n' = \frac{2}{L}\int_0^L [f(x) - v_s(x)]\sin\frac{n\pi x}{L}\,dx \qquad ⑤$$

and the solution is given by ④ and ⑤, where $v_s(x)$ is given by ①.

(b) Looking over the solution to part (a), it is tempting to believe that inclusion of the $v_s(x)$ term in the solution form $v(x,t) = v_s(x) + X(x)T(t)$ is essential. Actually, we can <u>omit</u> the $v_s(x)$ term — provided that we distinguish, in

② and ③, one more case, the case $k=(\sqrt{h}/\alpha)i$, because then $T'+(k^2\alpha^2+h)T=0$ reduces to $T'=0$, i.e., <u>steady state</u>. Thus, seeking $v(x,t)=X(x)T(t)$,

$$X=\begin{cases} A\cos kx+B\sin kx, & k\neq 0, \sqrt{h}i/\alpha \\ C+Dx, & k=0 \\ E\cosh\frac{\sqrt{h}}{\alpha}x+F\sinh\frac{\sqrt{h}}{\alpha}x, & k=\sqrt{h}i/\alpha \end{cases} \qquad T=\begin{cases} G\exp[-(k^2\alpha^2+h)t], & k\neq 0, \sqrt{h}i/\alpha \\ H\exp(-ht), & k=0 \\ I, & k=\sqrt{h}i/\alpha \end{cases}$$

so

$$v(x,t)=\underbrace{E\cosh\frac{\sqrt{h}}{\alpha}x+F\sinh\frac{\sqrt{h}}{\alpha}x}_{\uparrow \text{ this will give the ``} v_s(x)\text{'' part}}+(C+Dx)He^{-ht}+(A\cos kx+B\sin kx)Ge^{-(k^2\alpha^2+h)t}$$

22. $u(x,t)=u_\infty+e^{-ht}w(x,t)$

$u_{xx}=e^{-ht}w_{xx}$, $u_t=-he^{-ht}w+e^{-ht}w_t$

so (21.1) becomes

$$\alpha^2e^{-ht}w_{xx}=-h\cancel{e^{-ht}w}+e^{-ht}w_t+h\cancel{e^{-ht}w}$$

or, $\alpha^2w_{xx}=w_t$.

23. (a) The idea is that the initial condition for the $0<t<\infty$ problem is the steady-state solution for the $-\infty<t<0$ part, namely, the solution to

$$v_s''-rgv_s=0;\quad v_s(0)=12,\ v_s(L)=6 \qquad ①$$

Taking the solution of ① to be the initial condition $v(x,0)$ for the $0<t<\infty$ part, we have

$$v(x,0)=12\cosh\sqrt{rg}\,x+(6-12\cosh\sqrt{rg}\,L)\frac{\sinh\sqrt{rg}\,x}{\sinh\sqrt{rg}\,L}. \qquad ②$$

(b) Next, we will need the steady-state solution for the $0<t<\infty$ problem, namely, the solution of

$$v_s''-rgv_s=0;\quad v_s(0)=0,\ v_s(L)=6,$$

namely, $$v_s(x)=6\frac{\sinh\sqrt{rg}\,x}{\sinh\sqrt{rg}\,L}. \qquad ③$$

(c) $v(x,t)=\overset{\curvearrowright}{v_s(x)}+X(x)T(t)$ gives $\dfrac{X''}{X}=\dfrac{rCT'+rgT}{T}=-k^2$

$$X''+k^2X=0\rightarrow X=\begin{cases} A\cos kx+B\sin kx, & k\neq 0 \\ D+Ex, & k=0 \end{cases}$$

$$T'+\frac{k^2+rg}{rC}T=0\rightarrow T=\begin{cases} Fe^{-\beta t}, & k\neq 0 \\ Ge^{-gt/C}, & k=0 \end{cases} \quad \left(\beta=\frac{k^2+rg}{rC}\right)$$

so $v(x,t)=v_s(x)+(H+Ix)e^{-gt/C}+(J\cos kx+M\sin kx)e^{-\beta t}$

$v(0,t)=0=0+\quad He^{-gt/C}+Je^{-\beta t}\rightarrow H=J=0$

$v(L,t)=6=6+ILe^{-gt/C}+M\sin kL\,e^{-\beta t}\rightarrow I=0,\ k=n\pi/L\ (n=1,2,\dots)$

so $$v(x,t)=v_s(x)+\sum_1^\infty M_n\sin\frac{n\pi x}{L}e^{-\beta_nt}. \quad \left(\beta_n=\frac{(n\pi/L)^2+rg}{rC}\right) \qquad ④$$

Finally,

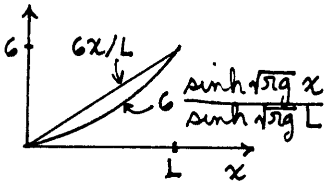
$$v(x,0) = v_s(x) + \sum_1^\infty M_n \sin\frac{n\pi x}{L}$$

$$v(x,0) - v_s(x) = \sum_1^\infty M_n \sin\frac{n\pi x}{L} \qquad (0 < x < L)$$

HRS:

$$M_n = \frac{2}{L}\int_0^L [v(x,0) - v_s(x)] \sin\frac{n\pi x}{L}\,dx \qquad ⑤$$

Solution given by ④ and ⑤, where $v(x,0)$ is given by ② and $v_s(x)$ by ③.
NOTE: It is interesting to examine the effects of the leakage g. The leakage
(i) reduces the steady state from $6x/L$ (for $g=0$) to $6\sinh\sqrt{rg}\,x/\sinh\sqrt{rg}\,L$, as sketched at the right. This makes sense, physically.
(ii) increase the β_n's and therefore speeds the decay of the transients. This makes sense too.



24. (a) $u = v/\rho$, $u_\rho = v_\rho/\rho - v/\rho^2$, $u_{\rho\rho} = v_{\rho\rho}/\rho - v_\rho/\rho^2 - v_\rho/\rho^2 + 2v/\rho^3$
so (24.1) becomes $\alpha^2\left(\frac{v_{\rho\rho}}{\rho} - \cancel{\frac{v_\rho}{\rho^2}} - \cancel{\frac{v_\rho}{\rho^2}} + \cancel{\frac{2v}{\rho^3}} + \frac{2}{\rho}\left(\cancel{\frac{v_\rho}{\rho}} - \cancel{\frac{v}{\rho^2}}\right)\right) = \frac{v_t}{\rho}$

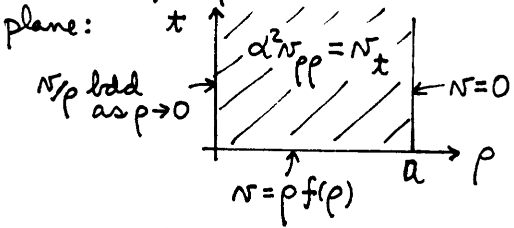
$$\text{or,}\quad \alpha^2 v_{\rho\rho} = v_t.$$

(b) With $u(\rho,t) = v(\rho,t)/\rho$, the problem on v is

$$\alpha^2 v_{\rho\rho} = v_t, \quad (0<\rho<a,\ 0<t<\infty)$$
$$v(a,t) = 0, \quad (0<t<\infty)$$
$$v(\rho,0) = \rho f(\rho), \quad (0<\rho<a)$$

where $v(\rho,t)/\rho$ is bounded as $\rho \to 0$.

ρ,t plane:



$$v(\rho,t) = A + B\rho + (C\cos k\rho + D\sin k\rho)e^{-k^2\alpha^2 t}$$
$$v/\rho = \frac{A}{\rho} + B + \left(C\frac{\cos k\rho}{\rho} + D\frac{\sin k\rho}{\rho}\right)e^{-k^2\alpha^2 t}$$

bounded as $\rho\to 0 \Rightarrow A = 0$ and $C = 0$, so

$$v(\rho,t) = B\rho + D\sin k\rho\,\exp(-k^2\alpha^2 t)$$
$$v(a,t) = 0 = Ba + D\sin ka\,\exp(-k^2\alpha^2 t) \Rightarrow B=0,\ ka = n\pi\ (n=1,2,\dots)$$
$$v(\rho,t) = \sum_1^\infty D_n \sin\frac{n\pi\rho}{a}\exp[-[n\pi\alpha/a]^2 t] \qquad ①$$

Finally,

$$v(\rho,0) = \rho f(\rho) = \sum_1^\infty D_n \sin\frac{n\pi\rho}{a} \qquad (0<\rho<a)$$

HRS: $D_n = \frac{2}{a}\int_0^a \rho f(\rho)\sin\frac{n\pi\rho}{a}\,d\rho$ ②

Solution given by ① and ②, where $u(\rho,t) = v(\rho,t)/\rho$.

25. (a) $\frac{d}{dt}\int_0^L w^2(x,t)\,dx = \int_0^L 2ww_t\,dx$ by the Leibniz rule

$= 2\alpha^2\int_0^L ww_{xx}dx$ since $\alpha^2 w_{xx} = w_t$

$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right)$

$= -2\alpha^2\int_0^L w_x^2 dx$ since $w(0,t) = w(L,t) = 0$

Now integrate on t from 0 to t:

$$d\int_0^L w^2(x,t)\,dx = -2\alpha^2\left(\int_0^L w_x^2\,dx\right)dt$$

$$\int_0^L w^2(x,t)\,dx - \int_0^L \underbrace{w^2(x,0)}_{0}\,dx = -2\alpha^2\int_0^t\int_0^L w_x^2(x,\tau)\,dx\,d\tau$$

so $\int_0^L w^2(x,t)\,dx = -2\alpha^2\int_0^t\int_0^L w_x^2(x,\tau)\,dx\,d\tau$.

The left-hand side is ≥ 0 and the right-hand side is ≤ 0, so they must both $= 0$. Finally, if $w(x,t)$ is a continuous function of x, for each t, and $\int_0^L w^2(x,t)\,dx = 0$, then $w(x,t) = 0$ over $0 \leq x \leq L$ for each $t \geq 0$.

(b) Then $\alpha^2 w_{xx} = w_t$

$w(0,t) = 0,\ w_x(L,t) = 0,\ w(x,0) = 0.$

$\frac{d}{dt}\int_0^L w^2(x,t)\,dx = \int_0^L 2ww_t\,dx$ (Leibniz)

$= 2\alpha^2\int_0^L ww_{xx}\,dx$ since $\alpha^2 w_{xx} = w_t$

$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right)$

$= -2\alpha^2\int_0^L w_x^2\,dx$ since $w(0,t) = 0$ and $w_x(L,t) = 0$

Then proceed as in (a).

(c) This time consider a mixed boundary condition (i.e., of Robin type) at $x = L$.

$w = u_1 - u_2$ gives $\alpha^2 w_{xx} = w_t$; $w(0,t) = 0$,

$w(L,t) + \beta w_x(L,t) = 0$,

$w(x,0) = 0$.

$$\frac{d}{dt}\int_0^L w^2 dx = 2\int_0^L ww_t\,dx = 2\alpha^2\int_0^L \underbrace{w}_{u}\,\underbrace{w_{xx}\,dx}_{dv}$$

$$= 2\alpha^2\left(ww_x\big|_0^L - \int_0^L w_x^2\,dx\right) = 2\alpha^2\left[-\beta w_x^2(L,t) - \int_0^L w_x^2\,dx\right]$$

so

$$\int_0^L w^2(x,t)\,dx - \int_0^L \underset{0}{w^2(x,0)}\,dx = -2\alpha^2\int_0^t \left[\beta w_x^2(x,\tau) + \int_0^L w_x^2(x,\tau)\,dx\right]d\tau \leq 0,$$

so $w(x,t) \equiv 0$.

26. Show that $\sum_1^\infty M_n = Q\sum_1^\infty n e^{-(n\pi\alpha/L)^2 t_0}$ converges, by the ratio test.

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\frac{Q(n+1)\exp[-[(n+1)\pi\alpha/L]^2 t_0]}{Qn\exp[-(n\pi\alpha/L)^2 t_0]} = \lim_{n\to\infty}\frac{e^{-(n^2+2n+1)(\pi\alpha/L)^2 t_0}}{e^{-n^2(\pi\alpha/L)^2 t_0}}$$

$= \lim_{n\to\infty} e^{-(2n+1)(\pi\alpha/L)^2 t_0} = 0$, which is <1. Thus, convergent.

27. The only difference is in the final step — satisfaction of the initial condition. For brevity, we will focus just on that last step.

(b) $u(x,0) = 10 = 10-5x + \sum_{1,3,\ldots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0<x<2)$

or,

$$5x = \sum_{1,3,\ldots}^{\infty} D_n \sin\frac{n\pi x}{4} \quad (0<x<2)$$

The St.-Liou. problem is $X''+k^2X=0$ on $(0<x<2)$, with $X(0)=0, X'(2)=0$.
The weight function (for all cases in this exercise) is 1, so

$$D_n = \frac{\langle 5x, \sin n\pi x/4\rangle}{\langle \sin n\pi x/4, \sin n\pi x/4\rangle} = \frac{\int_0^2 5x \sin n\pi x/4\,dx}{\int_0^2 \sin^2 n\pi x/4\,dx} = \text{as in Exercise 6(b).}$$

(h) $u(x,0) = f(x) = A + \sum_{1}^{\infty} C_n \cos\frac{nx}{3} \quad (0<x<3\pi)$

The Sturm-Liouville problem is $X''+k^2X=0 \quad (0<x<3\pi)$

$X'(0)=0,\ X'(3\pi)=0$

The eigenfunctions are 1 and $\cos\frac{nx}{3}$ $(n=1,2,\ldots)$ so

$$A = \frac{\langle f(x),1\rangle}{\langle 1,1\rangle} = \frac{\int_0^{3\pi} f\,dx}{\int_0^{3\pi} 1\,dx} = \frac{1}{3\pi}\int_0^{3\pi} f\,dx = \text{as in Exercise 6(h)}$$

(k) $u(x,0) = \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5} = \sum_{1}^{\infty} D_n \sin\frac{n\pi x}{5} \quad (0<x<5)$

The St.-Liou. problem is $X''+k^2X=0 \quad (0<x<5)$

$X(0)=0,\ X(5)=0$

The eigenfunctions are $\sin n\pi x/5$ $(n=1,2,\ldots)$ so

$$D_n = \frac{\langle \sin\pi x - 37\sin\frac{\pi x}{5} + 6\sin\frac{9\pi x}{5}, \sin\frac{n\pi x}{5}\rangle}{\langle \sin\frac{\pi x}{5}, \sin\frac{\pi x}{5}\rangle} = \text{as in Exercise 6(k)}$$

(m) $u(x,0) = 0 = 2x + \sum_{1,3,\ldots}^{\infty} C_n \cos\frac{n\pi x}{12}$,

$$-2x = \sum_{1,3,\ldots}^{\infty} C_n \cos\frac{n\pi x}{12} \quad (0<x<6)$$

The St-Liou. problem is $X''+k^2X=0 \quad (0<x<6)$

$X'(0)=0,\ X(6)=0$

The eigenfunctions are $\cos n\pi x/12$ $(n=1,3,\ldots)$ so

$$C_n = \frac{\langle -2x, \cos n\pi x/12\rangle}{\langle \cos n\pi x/12, \cos n\pi x/12\rangle} = \frac{\int_0^6 (-2x)\cos n\pi x/12\,dx}{\int_0^6 \cos^2 n\pi x/12\,dx}$$

28. $\alpha^2(u_{\rho\rho} + \frac{2}{\rho}u_\rho) = u_t \quad (0<\rho<a,\ 0<t<\infty)$

$u(a,t)=0,\ u(\rho,0)=f(\rho),\ u(0,t)=\text{bounded}.$

$u(\rho,t) = R(\rho)T(t)$ gives $\frac{R''+\frac{2}{\rho}R'}{R} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$

$R''+\frac{2}{\rho}R'+k^2R=0$, or, $\rho^2R''+2\rho R'+k^2\rho^2R=0$, or, $(\rho^2R')'+k^2\rho^2R=0$.
Use (46) on page 238: $a=2,\ b=k^2,\ c=2$, so $\alpha=2/2=1$ and $\nu=-1/2$.

Then (50) on pg 239 gives $R(\rho)=\rho^{-1/2}Z_{1/2}(\kappa\rho)=\rho^{-1/2}(AJ_{1/2}(\kappa\rho)+BJ_{-1/2}(\kappa\rho))$

$$=\frac{1}{\sqrt{\rho}}\left(A\sqrt{\frac{2}{\pi\kappa\rho}}\sin\kappa\rho+B\sqrt{\frac{2}{\pi\kappa\rho}}\cos\kappa\rho\right)$$

$$=C\frac{\sin\kappa\rho}{\rho}+D\frac{\cos\kappa\rho}{\rho}.$$

The latter is the general solution if $\kappa\neq 0$, but if $\kappa=0$ we lose the $\sin\kappa\rho/\rho$ solution. For $\kappa=0$ the ODE is $R''+\frac{2}{\rho}R'=0$ with solution $E+F/\rho$. Thus,

$$R=\begin{cases}C\sin\kappa\rho/\rho+D\cos\kappa\rho/\rho, & \kappa\neq 0\\ E+F/\rho, & \kappa=0\end{cases}\qquad T=\begin{cases}Ge^{-\kappa^2\alpha^2t}, & \kappa\neq 0\\ H, & \kappa=0\end{cases}$$

so

$$u(\rho,t)=(E+\frac{F}{\rho})H+(C\frac{\sin\kappa\rho}{\rho}+D\frac{\cos\kappa\rho}{\rho})(Ge^{-\kappa^2\alpha^2t})$$
$$=E'+\frac{F'}{\rho}+(C'\frac{\sin\kappa\rho}{\rho}+D'\frac{\cos\kappa\rho}{\rho})\,e^{-\kappa^2\alpha^2t}$$

$u(0,t)$ bounded $\rightarrow F'=D'=0$ (but the $\sin\kappa\rho/\rho$ term <u>is</u> bounded as $\rho\rightarrow 0$), so

$$u(\rho,t)=E'+C'\frac{\sin\kappa\rho}{\rho}\exp(-\kappa^2\alpha^2t)$$
$$u(a,t)=0=E'+C'\frac{\sin\kappa a}{a}\exp(\text{''})\rightarrow E'=0,\ \kappa a=n\pi\quad(n=1,2,\dots)$$

so

$$u(\rho,t)=\sum_1^\infty C_n'\frac{\sin\frac{n\pi\rho}{a}}{\rho}\exp[-(n\pi\alpha/a)^2t]\qquad ①$$

Finally, $u(\rho,0)=f(\rho)=\sum_1^\infty C_n'\frac{\sin\frac{n\pi\rho}{a}}{\rho}\quad(0<\rho<a)$ ②

To guide us with the latter expansion, note that the St.-Liou. problem is

$$(\rho^2R')'+\kappa^2\rho^2R=0\quad(0<\rho<a)$$
$$R(0)\text{ bounded},\ R(a)=0\qquad ③$$

with eigenfunctions $\sin\frac{n\pi\rho}{a}/\rho$ and weight function ρ^2. Thus, the C_n''s in ② are computed as

$$C_n'=\frac{\langle f(\rho),\sin\frac{n\pi\rho}{a}/\rho\rangle}{\langle\sin\frac{n\pi\rho}{a}/\rho,\sin\frac{n\pi\rho}{a}/\rho\rangle}=\frac{\int_0^a f(\rho)(\sin\frac{n\pi\rho}{a}/\rho)\rho^2d\rho}{\int_0^a(\sin\frac{n\pi\rho}{a}/\rho)^2\rho^2d\rho}=\frac{\int_0^a\rho f(\rho)\sin\frac{n\pi\rho}{a}d\rho}{\int_0^a\sin^2\frac{n\pi\rho}{a}d\rho}$$

$$=\frac{2}{a}\int_0^a\rho f(\rho)\sin\frac{n\pi\rho}{a}d\rho.\qquad ④$$

The solution is given by ① and ④.

NOTE: We used the St.-Liou. problem ③ that is "built in" to assure us that the equality ② is indeed <u>possible</u> and then to show us how to compute the C_n''s. In this example we could have proceeded differently. Namely, multiply ② through by ρ and identify it as a half-range sine series. Then it follows that C_n' is given by ④, as before.

29. $\alpha^2 u_{xx} = u_t$

$$u(0,t) = u(L,t),\ u_x(0,t) = u_x(L,t),\ u(x,0) = f(x)$$

$u(x,t) = X(x)T(t)$ gives

$$u(x,t) = A + Bx + (C\cos kx + D\sin kx)e^{-k^2\alpha^2 t} \quad ①$$

$$u(0,t) - u(L,t) = 0 = \cancel{A} + C\exp(-k^2\alpha^2 t) - [\cancel{A} + BL + (C\cos kL + D\sin kL)\exp(-k^2\alpha^2 t)]$$

$$u_x(0,t) - u_x(L,t) = 0 = \cancel{B} + kD\exp('') - [\cancel{B} + (-kC\sin kL + kD\cos kL)\exp(\quad '')]$$

or,

$$-BL + [(1-\cos kL)C - (\sin kL)D]\exp(-k^2\alpha^2 t) = 0$$

$$[(k\sin kL)C + k(1-\cos kL)D]\exp(\ ''\ ) = 0$$

so $B=0$ and

$$\begin{pmatrix} 1-\cos kL & -\sin kL \\ k\sin kL & k(1-\cos kL) \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0} \quad ②$$

If we are to avoid the outcome $C=D=0$, we must set

$$\begin{vmatrix} 1-\cos kL & -\sin kL \\ k\sin kL & k(1-\cos kL) \end{vmatrix} = k[(1-\cos kL)^2 + \sin^2 kL] = 0 \quad ③$$

Solving ③, we disallow the root $k=0$ because $k \neq 0$ in ①, the $A+Bx$ terms already accounting for the $k=0$ case. ③ → $1-\cos kL = 0$ <u>and</u> $\sin kL = 0$. The roots of the first are $kL = 2\pi, 4\pi, \ldots$ and the roots of the second are $\pi, 2\pi, 3\pi, \ldots$ so the roots of <u>both</u> are $kL = 2n\pi$ $(n=1,2,\ldots)$. With that choice $(k = 2n\pi/L)$, (2) becomes $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \underset{\sim}{0}$, so the solution for C and D is C = arbitrary and D = arbitrary. Thus,

$$u(x,t) = A + \sum_1^\infty \left(C_n \cos\frac{n\pi x}{2L} + D_n \sin\frac{n\pi x}{2L}\right)e^{-(n\pi\alpha/2L)^2 t}. \quad ④$$

Finally,

$$u(x,0) = f(x) = A + \sum_1^\infty \left(C_n \cos\frac{n\pi x}{2L} + D_n \sin\frac{n\pi x}{2L}\right) \quad (0<x<L)$$

which is an eigenfunction expansion of $f(x)$ in terms of the eigenfunctions $1, \cos\pi x/2L, \sin\pi x/2L, \cos 2\pi x/2L, \sin 2\pi x/2L, \ldots$ of the (singular) St.-Liou. problem

$$X'' + k^2 X = 0 \quad (0<x<L)$$

$$X(0) - X(L) = 0,\ X'(0) - X'(L) = 0.$$

Thus,

$$A = \frac{\langle f,1\rangle}{\langle 1,1\rangle} = \frac{\int_0^L f(x)dx}{\int_0^L 1dx} = \frac{1}{L}\int_0^L f(x)\,dx \quad ⑤$$

$$C_n = \frac{\langle f, \cos n\pi x/2L\rangle}{\langle \cos\frac{n\pi x}{2L}, \cos\frac{n\pi x}{2L}\rangle} = \frac{\int_0^L f(x)\cos\frac{n\pi x}{2L}dx}{\int_0^L \cos^2\frac{n\pi x}{2L}dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}dx \quad ⑥$$

$$D_n = \frac{\langle f, \sin n\pi x/2L\rangle}{\langle \sin n\pi x/2L, \sin n\pi x/2L\rangle} = \frac{\int_0^L f(x)\sin n\pi x/2L\,dx}{\int_0^L \sin^2\frac{n\pi x}{2L}dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}dx \quad ⑦$$

so the solution is given by ④-⑦

30. (a) Using the "three-tier" solutions given in the exercise,

$$u(x,t) = (E' + F'e^{2x}) + e^x(C' + D'x)e^{-t} + e^x(A'\cos\omega x + B'\sin\omega x)e^{-k^2 t}$$

where E' is EI, F' is FI, and so on.

$u(0,t) = 50 = E' + F' + C'e^{-t} + A'e^{-k^2 t} \longrightarrow E' + F' = 50,\ C' = 0,\ A' = 0$ so

$u(x,t) = E' + (50 - E')e^{2x} + D'xe^{x}e^{-t} + B'e^{x}\sin\omega x\, e^{-k^2 t}$

$u(L,t) = 50 = E' + (50 - E')e^{2L} + D'Le^{L}e^{-t} + B'e^{L}\sin\omega L\, e^{-k^2 t} \longrightarrow E' = 50, D' = 0, \omega L = n\pi$

$(n = 1, 2, \ldots)$, so

$$u(x,t) = 50 + \sum_1^\infty B_n' e^{x} \sin\frac{n\pi x}{L}\, e^{-[(n\pi/L)^2 + 1]t} \qquad ①$$

Finally,

$$u(x,0) = 0 = 50 + \sum_1^\infty B_n' e^{x}\sin\frac{n\pi x}{L}$$

or,

$$-50 = \sum_1^\infty B_n' e^{x}\sin\frac{n\pi x}{L}, \quad (0 < x < L) \qquad ②$$

which is an eigenfunction expansion of -50 in terms of the eigenfunctions $e^{x}\sin\frac{n\pi x}{L}$ of the St.-Liou. problem

$$X'' - 2X' + k^2 X = 0 \quad (0 < x < L) \qquad ③$$

$$X(0) = 0,\ X(L) = 0.$$

To evaluate the B_n''s we need to determine the weight function. Write

$$\sigma X'' - 2\sigma X' + k^2 \sigma X = 0$$

where $-2\sigma = \sigma'$, so $\sigma(x) = e^{-2x}$. Thus the ODE can be written in the standard St.-Liou. form $(e^{-2x}X')' + k^2 e^{-2x} X = 0$, ④

so the weight function is e^{-2x}. Then

$$B_n' = \frac{\langle -50, e^{x}\sin\frac{n\pi x}{L}\rangle}{\langle e^{x}\sin\frac{n\pi x}{L}, e^{x}\sin\frac{n\pi x}{L}\rangle} = \frac{\int_0^L -50 e^{x}\sin\frac{n\pi x}{L}\, e^{-2x}\,dx}{\int_0^L e^{2x}\sin^2\frac{n\pi x}{L}\, e^{-2x}\,dx} =$$

$$= \frac{2}{L}(-50)\int_0^L e^{-x}\sin\frac{n\pi x}{L}\,dx = \text{etc.} \qquad ⑤$$

The solution is given by ①, where the B_n''s are given by ⑤.

NOTE: Alternatively to using the St.-Liou. theory to solve for the B_n''s in ②, we could multiply ② by e^{-x} and identify the result,

$$-50e^{-x} = \sum_1^\infty B_n' \sin\frac{n\pi x}{L}$$

as a half-range sine expansion. In that case,

$$B_n' = \frac{2}{L}\int_0^L (-50e^{-x})\sin\frac{n\pi x}{L}\,dx,$$

as in ⑤.

31(a) $(xJ_0')' + xJ_0 = 0, \quad \int_0^{z_n}(xJ_0')'\,dx + \int_0^{z_n} xJ_0\,dx = 0,$

$$xJ_0'(x)\Big|_0^{z_n} + \int_0^{z_n} xJ_0(x)\,dx = 0,$$

$$\int_0^{z_n} xJ_0(x)\,dx = 0 - z_n J_0'(z_n)$$

$$= z_n J_1(z_n)$$

(b) $P_n = -\frac{200}{c^2 J_1^2(z_n)}\int_0^c J_0(z_n \frac{r}{c})\, r\,dr = -\frac{200}{c^2 J_1^2(z_n)}\int_0^{z_n} J_0(\mu)\left(\frac{c}{z_n}\right)^2 \mu\,d\mu$

$$= -\frac{200}{z_n^2 J_1^2(z_n)}\, z_n J_1(z_n) = -\frac{200}{z_n J_1(z_n)}$$ verifies (83).

Section 18.4

1. $$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty}Fe^{-(x-\xi)^2/4\alpha^2t}\,d\xi\ \overset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=}\ \frac{F}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\mu^2}\,2\alpha\sqrt{t}\,d\mu$$
$$=\frac{F}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\mu^2}d\mu=\frac{F\sqrt{\pi}}{\sqrt{\pi}}=F.$$

2.(a) $$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}\int_{0}^{\infty}Fe^{-(x-\xi)^2/4\alpha^2t}\,d\xi\ \overset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=}\ \frac{F}{2\alpha\sqrt{\pi t}}\int_{-x/2\alpha\sqrt{t}}^{\infty}e^{-\mu^2}\,2\alpha\sqrt{t}\,d\mu$$
$$=\frac{F}{\sqrt{\pi}}\left(\int_{-x/2\alpha\sqrt{t}}^{0}e^{-\mu^2}d\mu+\int_{0}^{\infty}e^{-\mu^2}d\mu\right)=\frac{F}{\sqrt{\pi}}\left(\int_{0}^{x/2\alpha\sqrt{t}}e^{-\mu^2}d\mu+\frac{\sqrt{\pi}}{2}\right)$$
$$=\frac{F}{2}\left(\operatorname{erf}(x/2\alpha\sqrt{t})+1\right),\text{ as in (14).}$$

(b) $u(x,t)=\frac{F}{2}\left(1+\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right)$ where $\operatorname{erf}\psi=\frac{2}{\sqrt{\pi}}\int_0^{\psi}e^{-\xi^2}d\xi$

$$u_x=\frac{F}{2}\operatorname{erf}'\left(\frac{x}{2\alpha\sqrt{t}}\right)\frac{\partial}{\partial x}\left(\frac{x}{2\alpha\sqrt{t}}\right)=\frac{F}{2}\,\frac{2}{\sqrt{\pi}}e^{-(x/2\alpha\sqrt{t})^2}\frac{1}{2\alpha\sqrt{t}}=\frac{F}{2\alpha\sqrt{\pi t}}e^{-x^2/4\alpha^2t}$$
$$\alpha^2u_{xx}=\frac{\alpha^2F}{2\alpha\sqrt{\pi t}}e^{-x^2/4\alpha^2t}\left(\frac{-2x}{4\alpha^2t}\right)=-\frac{Fx}{4\alpha\sqrt{\pi}\,t^{3/2}}e^{-x^2/4\alpha^2t}$$
$$u_t=\frac{F}{2}\operatorname{erf}'\left(\frac{x}{2\alpha\sqrt{t}}\right)\frac{\partial}{\partial t}\left(\frac{x}{2\alpha\sqrt{t}}\right)=\frac{F}{2}\,\frac{2}{\sqrt{\pi}}e^{-x^2/4\alpha^2t}\,\frac{x}{2\alpha}\left(-\frac{1}{2}\right)t^{-3/2}=-\frac{Fx}{4\alpha\sqrt{\pi}\,t^{3/2}}e^{-x^2/4\alpha^2t}$$

so α^2u_{xx} does $=u_t$. Next, $u(x,0)=\frac{F}{2}(1+\operatorname{erf}(\infty))=\frac{F}{2}(1+1)=F.$ ✓

3. $$\int_{-\infty}^{\infty}K(\xi-x;t)\,d\xi=\int_{-\infty}^{\infty}\frac{e^{-(x-\xi)^2/4\alpha^2t}}{2\alpha\sqrt{\pi t}}\,d\xi\ \underset{\mu=(\xi-x)/2\alpha\sqrt{t}}{=}\ \int_{-\infty}^{\infty}\frac{e^{-\mu^2}}{2\alpha\sqrt{\pi t}}\,2\alpha\sqrt{t}\,d\mu$$
$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\mu^2}d\mu=1.\ ✓$$

4. $$\operatorname{erf}x=\frac{2}{\sqrt{\pi}}\int_0^{x}e^{-\xi^2}d\xi,\quad \operatorname{erf}(-x)=\frac{2}{\sqrt{\pi}}\int_0^{-x}e^{-\xi^2}d\xi\ \overset{\xi=-t}{=}\ \frac{2}{\sqrt{\pi}}\int_0^{x}e^{-t^2}(-dt)=-\operatorname{erf}(x).$$

5. $\alpha^2u_{xx}=u_t$, $u(x,0)=f(x)$ on $-\infty<x<\infty$.

Laplace: $$\alpha^2\int_0^{\infty}\frac{\partial^2u}{\partial x^2}e^{-st}dt=s\bar{u}(x,s)-u(x,0)$$
$$\alpha^2\frac{d^2}{dx^2}\int_0^{\infty}u(x,t)e^{-st}dt=s\bar{u}-f(x)$$
$$\alpha^2\frac{d^2}{dx^2}\bar{u}(x,s)-s\bar{u}(x,s)=-f(x)$$
$$\bar{u}_{xx}-\frac{s}{\alpha^2}\bar{u}=-\frac{1}{\alpha^2}f(x)$$

NOTE: Observe that the latter is a nonhomogeneous ODE, whereas the Fourier transform gave us the homogeneous ODE $\frac{d\hat{u}}{dt}+\alpha^2\omega^2\hat{u}=0$. Thus, although the Laplace transform will work, here, it is less convenient than the Fourier transform.

7. (a) $$u(x+\tau,t)=\int_{-\infty}^{\infty} f(\xi)K(\xi-(x+\tau);t)\,d\xi \overset{\mu=\xi-\tau}{=} \int_{-\infty}^{\infty} f(\mu+\tau)K(\mu-x;t)\,d\mu$$

$$=\int_{-\infty}^{\infty} f(\mu)K(\mu-x;t)\,d\mu$$ (because f is τ-periodic) $= u(x,t)$, so if f is τ-periodic then so is $u(x,t)$ a τ-periodic function of x.

(b) $$u(-x,t)=\int_{-\infty}^{\infty} f(\xi)K(\xi+x;t)\,d\xi \underset{\mu=-\xi}{=} \int_{\infty}^{-\infty} f(-\mu)K(-\mu+x;t)(-d\mu)$$

$$=\int_{-\infty}^{\infty} -f(\mu)\,K(\mu-x;t)\,d\mu$$ because f is odd and K is an even function of its first argument

$= -u(x,t)$, so if f is odd then $u(x,t)$ is an odd function of x.

(c) Same as in (b) but this sign is +.

8. (a) Add these equations:

$$\begin{array}{ll} \alpha^2 v_{xx} - v_t = 0 & v(x,0)=f(x) \\ \alpha^2 w_{xx} - w_t = -F(x,t) & w(x,0)=0 \\ \hline \alpha^2(v_{xx}+w_{xx}) - (v_t+w_t) = -F(x,t), & v(x,0)+w(x,0)=f(x) \end{array}$$

or, if $u(x,t)=v(x,t)+w(x,t)$,

$$\alpha^2 u_{xx} - u_t = -F(x,t)\ ,\checkmark \qquad u(x,0)=f(x).\ \checkmark$$

(b) See Answers to Selected Exercises.

(c) $\alpha^2 w_{xx} - w_t = -F(x)$, $w(x,0)=0$

Fourier transforming, $\alpha^2(i\omega)^2\hat{w} - \hat{w}_t = -\hat{F}(\omega)$,

$$\hat{w}_t + \alpha^2\omega^2\hat{w} = \hat{F}(\omega).$$

The latter differential equation is with respect to t, so $\hat{F}(\omega)$ is merely a constant. Thus,

$$\hat{w}(\omega,t) = Ae^{-\alpha^2\omega^2 t} + \frac{\hat{F}(\omega)}{\alpha^2\omega^2}.$$

Fourier transform of initial condition gives $\hat{w}(\omega,0)=0$, so

$$\hat{w}(\omega,0)=0=A+\hat{F}(\omega)/\alpha^2\omega^2,\quad A=-\hat{F}(\omega)/\alpha^2\omega^2,$$

so

$$\hat{w}(\omega,t)=\hat{F}(\omega)\,\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}$$

and, by convolution,

$$w(x,t) = F(x) * F^{-1}\left\{\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}\right\}$$

(d) To complete the solution we need the inverse of $(1-e^{-\alpha^2\omega^2 t})/\alpha^2\omega^2$. Define

$$\hat{g} \equiv (1-e^{-\alpha^2\omega^2 t})/\alpha^2\omega^2$$

and observe that d/dt gives a substantial simplification:

$$\hat{g}_t = e^{-\alpha^2\omega^2 t},$$

or,

$$\widehat{g_t} = e^{-\alpha^2\omega^2 t}, \qquad ①$$

since

$$\hat{g}_t(\omega,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} g(x,t)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \frac{\partial g}{\partial t}(x,t)e^{-i\omega x}\,dx = \widehat{g_t}.$$

Inverting ① by entry 6 in appendix D gives

$$g_t = \frac{e^{-x^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}} \qquad ②$$

Further, $g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega,t)\,e^{i\omega x}\,d\omega$, so

$$g(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\cancelto{0}{\omega,0})\,e^{i\omega x}\,d\omega = 0$$

so we can append to the differential equation ② the initial condition

$$g|_{t=0} = 0. \qquad ③$$

Thus, integrating ② from 0 to t and using ③ gives

$$g(x,t) = \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{2\alpha\sqrt{\pi\tau}}\,d\tau$$

Let $x^2/(4\alpha^2\tau) = \mu^2$. Then $\tau = x^2/(4\alpha^2\mu^2)$, $d\tau = -\frac{2x^2}{4\alpha^2}\mu^{-3}d\mu$

so

$$g(x,t) = \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{2\alpha\sqrt{\pi}}\,\frac{2\alpha\mu}{x}\left(-\frac{2x^2}{4\alpha^2}\mu^{-3}d\mu\right) = \frac{x}{2\alpha^2\sqrt{\pi}}\int_{x/2\alpha\sqrt{t}}^{\infty} \frac{\overset{u}{e^{-\mu^2}}}{\mu^2}\,\underset{dv}{d\mu}$$

$$= \frac{x}{2\alpha^2\sqrt{\pi}}\left\{-\frac{1}{\mu}e^{-\mu^2}\Big|_{x/2\alpha\sqrt{t}}^{\infty} - \int_{x/2\alpha\sqrt{t}}^{\infty}\left(-\frac{1}{\mu}\right)(-2\mu)e^{-\mu^2}d\mu\right\}$$

$$= \frac{x}{2\alpha^2\sqrt{\pi}}\left\{\frac{2\alpha\sqrt{t}}{x}e^{-x^2/4\alpha^2 t} - 2\int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2}d\mu\right\}$$

$$= \frac{1}{\alpha}\sqrt{\frac{t}{\pi}}\,e^{-x^2/4\alpha^2 t} - \frac{x}{2\alpha^2}\operatorname{erfc}(x/2\alpha\sqrt{t}).$$

Finally, $w(x,t) = F(x) * g(x,t) = \frac{1}{\alpha}\int_{-\infty}^{\infty} F(x-\xi)\Big[\sqrt{\frac{t}{\pi}}\,e^{-\xi^2/4\alpha^2 t}$

$$- \frac{x}{2\alpha}\operatorname{erfc}(\xi/2\alpha\sqrt{t})\Big]d\xi.$$

9. (a) The solution to $f_1(x) = -100 + 200[H(x) - H(x-L)]$ is

$$u_1(x,t) = -100 + 200\left[\frac{1}{2}\left(1+\operatorname{erf}\frac{x}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x-L}{2\alpha\sqrt{t}}\right)\right]$$

$$= 100\left[\operatorname{erf}\frac{x}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x-L}{2\alpha\sqrt{t}} - 1\right],$$

the solution to $f_2(x) = 200[H(x+2L) - H(x+L) + H(x-2L) - H(x-3L)]$ is

$$u_2(x,t) = 200\left[\frac{1}{2}\left(1+\operatorname{erf}\frac{x+2L}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x+L}{2\alpha\sqrt{t}}\right) + \frac{1}{2}\left(1+\operatorname{erf}\frac{x-2L}{2\alpha\sqrt{t}}\right) - \frac{1}{2}\left(1+\operatorname{erf}\frac{x-3L}{2\alpha\sqrt{t}}\right)\right]$$

$$= 100\left[\operatorname{erf}\frac{x+2L}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x+L}{2\alpha\sqrt{t}} + \operatorname{erf}\frac{x-2L}{2\alpha\sqrt{t}} - \operatorname{erf}\frac{x-3L}{2\alpha\sqrt{t}}\right],$$ and so on.

(b) From their graphs on page 990, observe that $f_1(x)$ agrees exactly with $f_{ext}(x)$ over $-L<x<2L$. The discrepancy occurs only over $x>2L$ and over $x<-L$, which regions are "far away" from the physical rod interval of $0<x<L$. Since it will take time for that misinformation to diffuse into $0<x<L$, it follows that for small t the solution to the f_1 problem should be quite accurate. Even more so for the f_1+f_2 problem since f_1+f_2 agrees with f_{ext} over $-3L<x<4L$, even more so for the $f_1+f_2+f_3$ problem, and so on.

(c) (9.1) gives $u(1,.1)=\frac{400}{\pi}\sum_1^\infty \frac{1}{2n-1}\sin\frac{(2n-1)\pi}{10}\, e^{-[(2n-1)\pi/10]^2(0.114)}$

To sum just the first term use the Maple commands

```
S:=sum(sin((2*i-1)*Pi/10)*exp(-.00114*(2*i-1)^2*Pi^2)/
 (2*i-1), i=1..1):
evalf(400*S/Pi);
```

and obtain 38.905
With $i=1..5$, obtain 99.391
With $i=1..10$, obtain 96.3460
With $i=1..20$, obtain 96.3764
With $i=1..30$, obtain 96.3764

So, for the results to settle down to 6 significant figures, say, we need around 20 terms of the series (9.1).

(9.5) gives $u(1,.1)=u_1(1,.1)+u_2(1,.1)+\cdots$

$$\approx u_1(1,.1) = 100\left[\operatorname{erf}\left(\frac{1}{2\sqrt{.114}}\right)-\operatorname{erf}\left(\frac{-9}{2\sqrt{.114}}\right)-1\right]$$

and the Maple command

```
evalf(100*(erf(1/(2*sqrt(.114)))-erf(-9/(2*sqrt(.114)))-1));
```

gives 96.3764

so even just one term of (9.5) gives excellent accuracy.

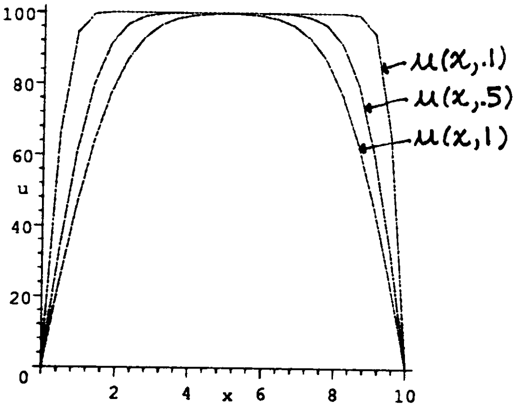
(d) $u_2, u_3, \ldots$ become negligible corrections to u_1, in (9.5), as $t\to 0$ because in that limit the arguments of the erfs in (9.5) $\to\infty$ and the erfs all approach the limiting value $\operatorname{erf}(\infty)=1$, in which case the erf pairs in (9.4) virtually cancel to zero. Thus, as a rule of thumb, let us ask that $L/(2\alpha\sqrt{t})\gg 1$. That is,

$$t \ll (L/2\alpha)^2,$$

which inequality is easily satisfied in the present case, where $L=10$, $\alpha=1.07$, and $t=0.1$. It would even be satisfied for $t=1$, say, which fact we use in part (e), where we use the approximate solution $u(x,t)\approx u_1(x,t)$ to generate some computer plots of the solution at $t=0.1$, 0.5, and 1.

(e) Maple:

```
> with(plots):
> p(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(.1)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(.1)))-1):
> q(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(.5)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(.5)))-1):
> r(x):=100*(erf(x/(2*sqrt(1.14)*sqrt(1.)))-erf((x-10)/(2*sqrt(1.14)
  *sqrt(1.)))-1):
> implicitplot({u=p(x),u=q(x),u=r(x)},x=0..10,u=0..100);
```



10. $\alpha^2 u_{xx} = u_t + V u_x \quad (-\infty<x<\infty, 0<t<\infty)$

$u(x,0) = f(x)$

Fourier transform: $\alpha^2(i\omega)^2\hat{u} = \hat{u}_t + i\omega V\hat{u}$,

$\hat{u}_t + (\alpha^2\omega^2 + i\omega V)\hat{u} = 0$,

$\hat{u}(\omega,t) = A e^{-(\alpha^2\omega^2 + i\omega V)t}$

$\hat{u}(\omega,0) = \hat{f}(\omega) = A$

so $\hat{u}(\omega,t) = \hat{f}(\omega)e^{-(\alpha^2\omega^2+i\omega V)t}$

From Appendix D,

Entry 6: $e^{-\alpha^2\omega^2 t} \rightarrow \frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-x^2/4\alpha^2 t}$

Entry 11 with $a=1$ and $b=-Vt$:

$$e^{-\alpha^2\omega^2 t} e^{-iVt\omega} \rightarrow \frac{1}{2(\alpha\sqrt{t})\sqrt{\pi}} e^{-(x-Vt)^2/4\alpha^2 t}$$

so $u(x,t) = f(x) * \frac{1}{2\alpha\sqrt{\pi t}} e^{-(x-Vt)^2/4\alpha^2 t}$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\exp[-(x-\xi-Vt)^2/4\alpha^2 t]\,d\xi,$$

which does indeed reduce to (18) if $V=0$.

11. Then (27) becomes $L\{u_t\} = s\bar{u} - u_0$, so (28) becomes

$$\alpha^2 \bar{u}_{xx} - s\,\bar{u} = -u_0,$$

$$\bar{u} = A e^{\sqrt{s}\,x/\alpha} + B e^{-\sqrt{s}\,x/\alpha} + u_0/s$$

u bdd as $x \to \infty \Rightarrow A = 0$ so

$$\bar{u}(x,s) = B e^{-\sqrt{s}\,x/\alpha} + u_0/s.$$

$u(0,t) = u_1 \longrightarrow \bar{u}(0,s) = u_1/s$, so

$$\bar{u}(0,s) = u_1/s = B + u_0/s \longrightarrow B = (u_1 - u_0)/s$$

and

$$\bar{u}(x,s) = (u_1 - u_0)\,\frac{e^{-\sqrt{s}\,x/\alpha}}{s} + u_0/s$$

From Appendix C, entry 21 gives $e^{-\sqrt{s}\,x/\alpha} \to \dfrac{x/2\alpha}{\sqrt{\pi}}\,\dfrac{e^{-x^2/4\alpha^2 t}}{t^{3/2}}$

entry 1 gives $1/s \to 1$,

so

$$u(x,t) = (u_1 - u_0)\, 1 * \frac{x}{2\alpha\sqrt{\pi}}\,\frac{e^{-x^2/4\alpha^2 t}}{t^{3/2}} + u_0 = u_0 + (u_1 - u_0)\frac{x}{2\alpha\sqrt{\pi}}\int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}}\,d\tau$$

$$= u_0 + (u_1 - u_0)\,\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right).$$

12. $\alpha^2 u_{xx} = u_t \quad (0 < x < \infty,\ 0 < t < \infty)$

$u(0,t) = u_0 \cos\omega t$, $u \to 0$ as $x \to \infty$.

We could include an initial condition but wont need one since we are after the steady-state response, as $t \to \infty$. Following the hint, consider

$$\alpha^2 v_{xx} = v_t \quad (0 < x < \infty,\ 0 < t < \infty)$$

$$v(0,t) = u_0 e^{i\omega t}, \quad v \to 0 \text{ as } x \to \infty.$$

Seeking $v(x,t) = X(x)\,e^{i\omega t}$ obtain $\alpha^2 X''\,\cancel{e^{i\omega t}} = i\omega X\,\cancel{e^{i\omega t}}$

$$X'' - \frac{i\omega}{\alpha^2}X = 0,$$

and since $\sqrt{i} = \pm(1+i)/\sqrt{2}$,

$$X(x) = A e^{\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x} + B e^{-\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x}$$

$v \to 0$ as $x \to \infty$ implies that $X(x) \to 0$ as $x \to \infty$ implies that $A = 0$, so

$$v(x,t) = X(x)\,e^{i\omega t} = B e^{-\frac{1+i}{\sqrt{2}}\frac{\sqrt{\omega}}{\alpha}x}\, e^{i\omega t}$$

$$v(0,t) = u_0 e^{i\omega t} = B e^{i\omega t} \longrightarrow B = u_0.$$

Then,

$$u(x,t) = \operatorname{Re} v(x,t) = \operatorname{Re}\left\{u_0 e^{-\sqrt{\frac{\omega}{2}}\frac{x}{\alpha}}\, e^{i\left(\omega t - \sqrt{\frac{\omega}{2}}\frac{x}{\alpha}\right)}\right\}$$

$$= u_0 e^{-rx}\cos(\omega t - rx)$$

where $r = \sqrt{\omega/2}\,/\alpha$.

13. (a) $u(x,t) = \frac{100x}{2\alpha\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}}\, d\tau$ Let $x^2/4\alpha^2\tau = \mu^2$, $\tau = \frac{x^2}{4\alpha^2}\frac{1}{\mu^2}$

$$= \frac{100x}{2\alpha\sqrt{\pi}} \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{\frac{x^3}{8\alpha^3}\frac{1}{\mu^3}} (-2)\frac{x^2}{4\alpha^2}\mu^{-3}\, d\mu = +\frac{200}{\sqrt{\pi}} \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2}\, d\mu$$

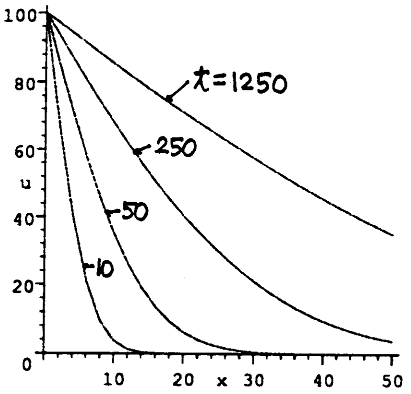
$$= 100 \operatorname{erfc} \frac{x}{2\alpha\sqrt{t}}$$

from (17).

(b) Better yet, let us obtain a plot of $u(x,t)$ versus x at representative t's, as given in Fig. 8. For example, over $0<x<50$, at $t=10, 50, 250, 1250$.

Maple:

```
> with(plots):
> implicitplot({u=100*erfc(x/(2*sqrt(1.14*10))),u=100*erfc(x/(2*sqrt
  (1.14*50))),u=100*erfc(x/(2*sqrt(1.14*250))),u=100*erfc(x/(2*sqrt(
  1.14*1250)))},x=0..50,u=0..100);
```



14. $\alpha^2 u_{xx} = u_t$ $(0<x<\infty,\ 0<t<\infty)$

$u(x,0)=0$, $u_x(0,t) = -Q$, $u \to 0$ as $x \to \infty$.

(a) Laplace: $\alpha^2 \bar{u}_{xx} = s\bar{u} - 0$

$$\bar{u}_{xx} - \frac{s}{\alpha^2}\bar{u} = 0$$

$$\bar{u}(x,s) = Ae^{\sqrt{s}\,x/\alpha} + Be^{-\sqrt{s}\,x/\alpha}$$

$u \to 0$ as $x \to \infty$ implies $\bar{u} \to 0$ as $x \to \infty$, so we need $A=0$. Thus,

$$\bar{u}(x,s) = Be^{-\sqrt{s}\,x/\alpha}.$$

Finally, $u_x(0,t) = -Q$ gives $\bar{u}_x(0,s) = -Q/s = B(-\sqrt{s}/\alpha)e^0$, so $B = \frac{\alpha Q}{s^{3/2}}$ and

$$\bar{u}(x,s) = \alpha Q \frac{e^{-\sqrt{s}\,x/\alpha}}{s^{3/2}} = \alpha Q \frac{1}{s} \frac{e^{-\sqrt{s}\,x/\alpha}}{\sqrt{s}}$$

Appendix C:

Entry 1: $1/s \to 1$

Entry 20: $\frac{e^{-\sqrt{s}\,x/\alpha}}{\sqrt{s}} \to \frac{e^{-x^2/4\alpha^2 t}}{\sqrt{\pi t}}$

so $u(x,t) = \alpha Q\, 1 * e^{-x^2/4\alpha^2 t}/\sqrt{\pi t} = \frac{\alpha Q}{\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\sqrt{\tau}}\, d\tau.$

(b) Let $x^2/4\alpha^2\tau = \mu^2$, $\tau = x^2/4\alpha^2\mu^2$. Then

$$u(x,t) = \frac{\alpha Q}{\sqrt{\pi}} \int_{\infty}^{x/2\alpha\sqrt{t}} \frac{e^{-\mu^2}}{\left(\frac{x}{2\alpha\mu}\right)} \left(-\frac{2x^2}{4\alpha^2\mu^3}\right) d\mu = \frac{Qx}{\sqrt{\pi}} \int_{x/2\alpha\sqrt{t}}^{\infty} \overset{U}{e^{-\mu^2}} \overset{dV}{\frac{d\mu}{\mu^2}}$$

$$= \frac{Qx}{\sqrt{\pi}} \left\{ -\frac{e^{-\mu^2}}{\mu} \Big|_{x/2\alpha\sqrt{t}}^{\infty} - \int_{x/2\alpha\sqrt{t}}^{\infty} \left(-\frac{1}{\mu}\right)(-2\mu) e^{-\mu^2} d\mu \right\}$$

$$= \frac{Qx}{\sqrt{\pi}} \left\{ 0 + \frac{e^{-x^2/4\alpha^2 t}}{x/2\alpha\sqrt{t}} - 2\int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} d\mu \right\} = 2\alpha Q\sqrt{\frac{t}{\pi}}\, e^{-x^2/4\alpha^2 t} - Qx \operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

Section 18.5

1. $$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_0^{\infty} 100 \left(e^{-(\xi - x)^2/4\alpha^2 t} - e^{-(\xi + x)^2/4\alpha^2 t}\right) d\xi$$

Let $(\xi - x)/2\alpha\sqrt{t} = \mu$, let $(\xi + x)/2\alpha\sqrt{t} = \nu$

$$= \frac{50}{\alpha\sqrt{\pi t}} \left[\int_{-x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} 2\alpha\sqrt{t}\, d\mu - \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\nu^2} 2\alpha\sqrt{t}\, d\nu \right]$$

$$= \frac{100}{\sqrt{\pi}} \int_{-x/2\alpha\sqrt{t}}^{x/2\alpha\sqrt{t}} e^{-\mu^2} d\mu = \frac{200}{\sqrt{\pi}} \int_0^{x/2\alpha\sqrt{t}} e^{-\mu^2} d\mu = 100 \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

2. (a) $$u(x,t) = \int_{-\infty}^{0} f_{ext}(\xi) K(\xi - x; t)\, d\xi + \int_0^{\infty} f_{ext}(\xi) K(\xi - x; t)\, d\xi$$

Let $\xi = -\mu$

$$= \int_{\infty}^{0} f_{ext}(-\mu) K(-\mu - x; t)(-d\mu) + \quad ''$$

$$= \int_0^{\infty} f_{ext}(\mu) K(\mu + x; t)\, d\mu \quad + \quad ''$$

now let $\mu = \xi$

because we are now using an even extension, and also K is an even function of its 1st argument

$$= \int_0^{\infty} f_{ext}(\xi) [K(\xi + x; t) + K(\xi - x, t)]\, d\xi$$

$$= \int_0^{\infty} f(\xi) \left(\frac{e^{-(\xi + x)^2/4\alpha^2 t} + e^{-(\xi - x)^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}} \right) d\xi$$

since $f_{ext} = f$ on $(0, \infty)$.

(c) $$u(x,t) = \frac{100}{2\alpha\sqrt{\pi t}} \left\{ \int_{x/2\alpha\sqrt{t}}^{\infty} e^{-\nu^2} 2\alpha\sqrt{t}\, d\nu + \int_{-x/2\alpha\sqrt{t}}^{\infty} e^{-\mu^2} 2\alpha\sqrt{t}\, d\mu \right\}$$

$$= \frac{100}{\sqrt{\pi}}\, 2\int_0^{\infty} e^{-\nu^2} d\nu = \frac{100}{\sqrt{\pi}}\, 2\, \frac{\sqrt{\pi}}{2} = 100,$$

which makes perfect sense: if $u(x,0) = 100$ then $u(x,t) = 100$.

3. Since (11) holds for all x, it also holds with x changed to $-x$. Thus,

$$E_1(x)+O_1(x) = E_2(x)+O_2(x) \quad ①$$
$$E_1(-x)+O_1(-x) = E_2(-x)+O_2(-x) \quad ②$$
or, $$E_1(x)-O_1(x) = E_2(x)-O_2(-x) \quad ③$$

Adding ① and ③ gives $2E_1(x)=2E_2(x)$, so $E_1(x)=E_2(x)$, and subtracting them gives $2O_1(x)=2O_2(x)$, so $O_1(x)=O_2(x)$.

4. $$F'(0)=\lim_{\Delta x\to 0}\frac{F(0+\Delta x)-F(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{F(\Delta x)-F(0)}{\Delta x}$$

Also, $$F'(0)=\lim_{\Delta x\to 0}\frac{F(0-\Delta x)-F(0)}{-\Delta x}=\lim_{\Delta x\to 0}\frac{F(\Delta x)-F(0)}{-\Delta x}$$

Thus, $F'(0)=-F'(0)$, $2F'(0)=0$, $F'(0)=0$.

7. (b) $$F_t(x,t)=\lim_{\Delta t\to 0}\frac{F(x,t+\Delta t)-F(x,t)}{\Delta t}$$

$$F_t(-x,t)=\lim_{\Delta t\to 0}\frac{F(-x,t+\Delta t)-F(x,t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{F(x,t+\Delta t)-F(x,t)}{\Delta t}=F_t(x,t)$$, so the latter is an even function of x.

8. (a) Yes (b) No, e^{-x} is not even (c) No, the coefficient of u_{xt}, namely 1, is not odd; further, the coefficient of u_x, namely 1, is not odd
(d) No, it is not linear, due to the u^2 term
(e) Yes (f) No, the coefficient $\cos x$ of u_x is not odd
(g) Yes (h) Yes (i) Yes (j) Yes; note that $\sin t$ is an even function of x, namely, a constant (k) Yes (l) Yes
(m) $(x^3u_x)_x-u_{tt}+u = x^3u_{xx}+3x^2u_x-u_{tt}+u$ hence, no, because the coefficient x^3 of u_{xx} is not even, nor is the coefficient $3x^2$ odd.

Section 18.6

1. (a) $$u_{xx}=(u_x)_x\approx\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}\approx\frac{\frac{u(x+2\Delta x,t)-u(x+\Delta x,t)}{\Delta x}-\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x}$$
$$=\frac{u(x+2\Delta x,t)-2u(x+\Delta x,t)+u(x,t)}{(\Delta x)^2}$$

and $$u_t\approx\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}$$

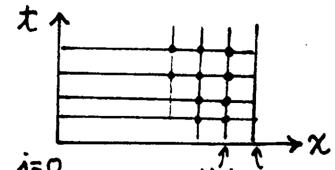
so $\alpha^2u_{xx}=u_t$ gives $$\alpha^2\frac{U_{j+2,k}-2U_{j+1,k}+U_{jk}}{(\Delta x)^2}=\frac{U_{j,k+1}-U_{jk}}{\Delta t}$$

so $$U_{j,k+1}=(1+r)U_{jk}-2rU_{j+1,k}+rU_{j+2,k}$$

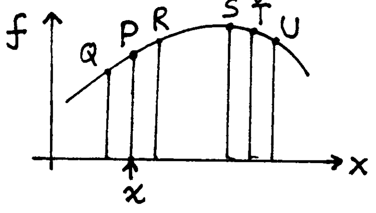
(b) $$u_{xx}=(u_x)_x\approx\frac{u_x(x,t)-u_x(x-\Delta x,t)}{\Delta x}\approx\frac{\frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}-\frac{u(x-\Delta x,t)-u(x-2\Delta x,t)}{\Delta x}}{\Delta x}$$
$$=\frac{u(x,t)-2u(x-\Delta x,t)+u(x-2\Delta x,t)}{(\Delta x)^2}$$

so $$\alpha^2 \frac{u(x,t)-2u(x-\Delta x,t)+u(x-2\Delta x,t)}{(\Delta x)^2} \approx \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}$$

so $$U_{j,k+1} = (1+r)U_{jk} - 2rU_{j-1,k} + rU_{j-2,k}.$$



(c) Two drawbacks come to mind. First, if we use (1.1), then at the grid points next to the right end ($j=N-1$) the $U_{j+2,k}$ is meaningless, in (1.1), since $j+2=N+1$ and there are not points at $N+1$. Similarly, if we use (1.2) then the $U_{j-2,k}$ term is meaningless when $j=1$, for $U_{1-2,k}$ is $U_{-1,k}$ is not defined. Also, we can expect The "double forward" formula (1.1) and the "double backward" formula (1.2) to be less accurate than the centered formula (8). Why? Look at it in this intuitive way: Suppose we seek $f''(x)$ (see sketch) knowing only the values of f at Q,P,R. We can fit a parabola through those 3 points and then take d^2/dx^2 of that parabolic funtion to evaluate f'' at x, approximatily. We could, alternatively, fit a parabola through S,T,U, say, as an approximation of f, and then take d^2/dx^2 to evaluate f'' at x, but surely we expect less accuracy using S,T,U than using Q,P,R, which are centered at the point x. Well, in using the "double-forward" formula the points S,T,U are not shifted as much as in the figure, but they are indeed shifted so as not to be centered at x. The foregoing argument has been intuitive; a rigorous case can be made using Taylor series.



2. $U_{j,k+1} = 0.16U_{j-1,k} + 0.68U_{jk} + 0.16U_{j+1,k}$

so

$U_{13} = 0.16(12)+0.68(7.3)+0.16(10.5) = 8.564$

$U_{23} = 0.16(7.3)+0.68(10.5)+0.16(21.4) = 11.732$

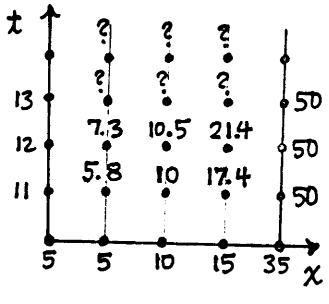
$U_{33} = 0.16(10.5)+0.68(21.4)+0.16(50) = 24.232$

and

$U_{14} = 0.16(13)+0.68(8.564)+0.16(11.732) = 9.7806$

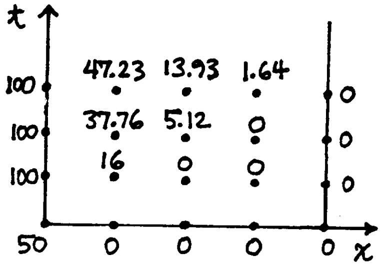
$U_{24} = 0.16(8.564)+0.68(11.732)+0.16(24.232) = 13.2251$

$U_{34} = 0.16(11.732)+0.68(24.232)+0.16(50) = 26.3549$



3. $r = \alpha^2 \Delta t/(\Delta x)^2 = 2/(2.5)^2 = 0.32$

$U_{j,k+1} = 0.32U_{j-1,k} + 0.36U_{jk} + 0.32U_{j+1,k}$



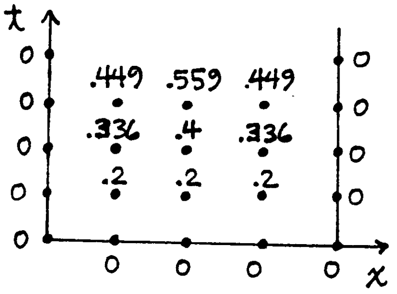
$U_{11} = .32(50)+0+0 = 16$
$U_{21} = 0+0+0 = 0$
$U_{31} = 0+0+0 = 0$

$U_{12} = .32(100)+.36(16)+0 = 37.76$
$U_{22} = .32(16)+0+0 = 5.12$
$U_{32} = 0+0+0 = 0$

$U_{13} = .32(100)+.36(37.76)+.32(5.12) = 47.232$
$U_{23} = .32(37.76)+.36(5.12)+0 = 13.926$
$U_{33} = .32(5.12)+0+0 = 1.638$

4. $$\alpha^2 \frac{U_{j-1,k}-2U_{jk}+U_{j+1,k}}{(\Delta x)^2} = \frac{U_{j,k+1}-U_{jk}}{\Delta t} + HU_{jk} - F_{jk}$$

multiplying by Δt, $r(U_{j-1,k}-2U_{jk}+U_{j+1,k}) = U_{j,k+1}-U_{jk}+H\Delta t\, U_{jk}-F_{jk}\Delta t$
so
$$U_{j,k+1} = rU_{j-1,k}+(1-2r-H\Delta t)U_{jk}+rU_{j+1,k}+F_{jk}\Delta t$$

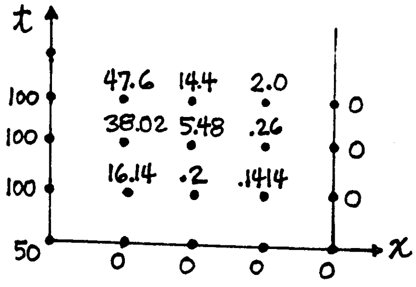


5. $r = (1)(0.02)/(0.25)^2 = 0.32$, $H=0$, $F(x,t)=10$, so
$$U_{j,k+1} = .32U_{j-1,k}+.36U_{jk}+.32U_{j+1,k}+.2$$

$U_{11} = 0+0+0+.2 = .2$
$U_{21} = 0+0+0+.2 = .2$
$U_{31} = 0+0+0+.2 = .2$

$U_{12} = .32(0)+.36(.2)+.32(.2)+.2 = .336$
$U_{22} = .32(.2)+.36(.2)+.32(.2)+.2 = .4$
$U_{32} = .32(.2)+.36(.2)+.32(0)+.2 = .336$ (Yes, there is symmetry about $x=0.5$.)

$U_{13} = .32(0)+.36(.336)+.32(.4)+.2 = .44896$
$U_{23} = .32(.336)+.36(.4)+.32(.336)+.2 = .55904$
$U_{33} = .32(.4)+.36(.336)+0+.2 = .44896$



6. $r=0.32$, $H=0$, $u(0,t)=100$, $u(x,0)=u(1,t)=0$,
$F(x,t) = 10\sin\pi x$, so

$$U_{j,k+1} = .32U_{j-1,k}+.36U_{jk}+.32U_{j+1,k}+.2\sin(j\pi/4)$$

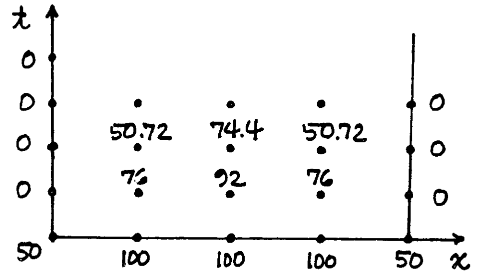
$U_{11} = .32(50)+0+0+(.7071)(.2) = 16.1414$
$U_{21} = 0+0+0+(1)(.2) = 0.2000$
$U_{31} = 0+0+0+(.7071)(.2) = 0.1414$

$U_{12} = .32(100)+.36(16.1414)+.32(.2)+(.7071)(.2) = 38.0163$
$U_{22} = .32(16.1414)+.36(.2)+.32(.1414)+(1)(.2) = 5.4825$
$U_{32} = .32(.2)+.36(.1414)+.32(0)+(.7071)(.2) = 0.2563$

$U_{13} = .32(100)+.36(38.0163)+.32(5.4825)+.2(.7071) = 47.5817$
$U_{23} = .32(38.0163)+.36(5.4825)+.32(.2563)+.2(1) = 14.4209$
$U_{33} = .32(5.4825)+.36(.2563)+0+.2(.7071) = 1.9881$

7. $r=0.32$, $H=4$, $u(0,t)=u(1,t)=0$, $u(x,0)=100$, $F(x,t)=0$, so

$$U_{j,k+1} = .32U_{j-1,k}+.28U_{jk}+.32U_{j+1,k}$$

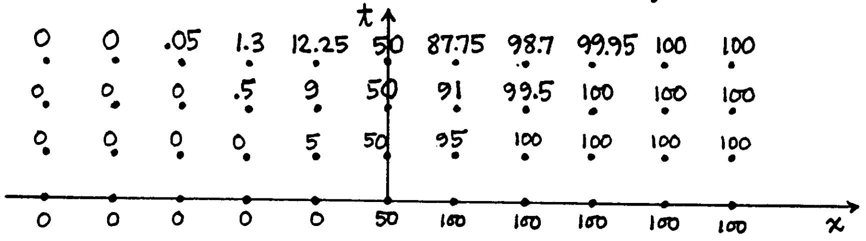


$U_{11} = .32(50)+.28(100)+.32(100) = 76$
$U_{21} = .32(100)+.28(100)+.32(100) = 92$
$U_{31} = .32(100)+.28(100)+.32(50) = 76$

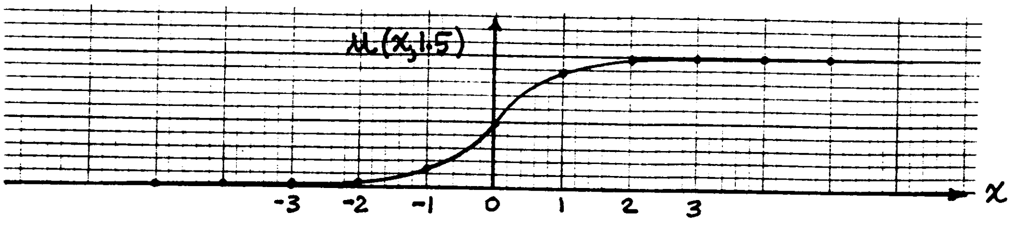
$U_{12} = .32(0)+.28(76)+.32(92) = 50.72$
$U_{22} = .32(76)+.28(92)+.32(76) = 74.4$
$U_{32} = .32(92)+.28(76)+.32(0) = 50.72$

$U_{13} = .32(0)+.28(50.72)+.32(74.4) = 38.0096$
$U_{23} = .32(50.72)+.28(74.4)+.32(50.72) = 53.2928$
$U_{33} = .32(74.4)+.28(50.72)+.32(0) = 38.0096$

8. $r=\alpha^2\Delta t/(\Delta x)^2 = 0.2(0.5)/(1)^2=0.1$, so $U_{j,k+1} = .1U_{j-1,k}+.8U_{jk}+.1U_{j+1,k}$



Let's plot the last one, at $t=1.5$



9. Let us do only the cases $\Delta t = 0.4$ (so $r = \alpha^2 \Delta t/(\Delta x)^2 = .4$) and $\Delta t = 0.6$ (so $r = .6$).

For $r = .4$, $U_{j,k+1} = .4U_{j-1,k} + .2U_{jk} + .4U_{j+1,k}$

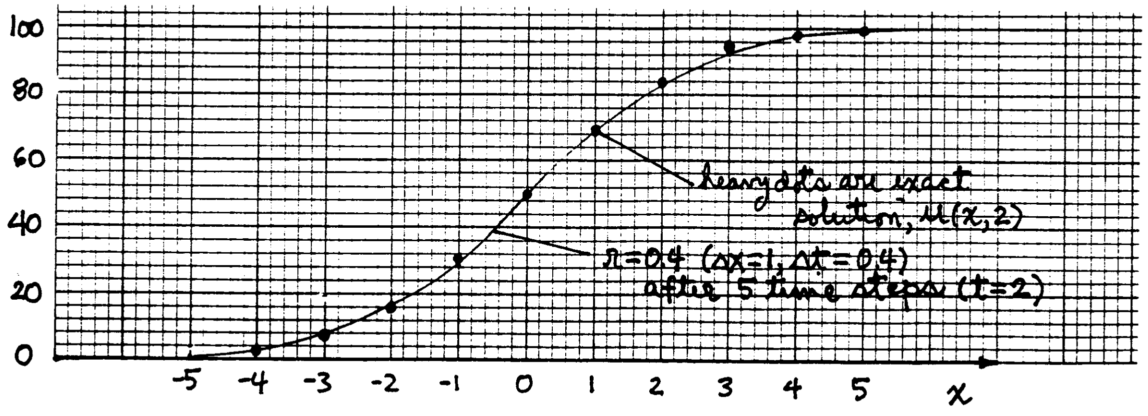
For $r = .6$, $U_{j,k+1} = .6U_{j-1,k} - .2U_{jk} + .6U_{j+1,k}$

Here are the results for these two cases: upper numbers for $r = 0.4$, lower for 0.6.

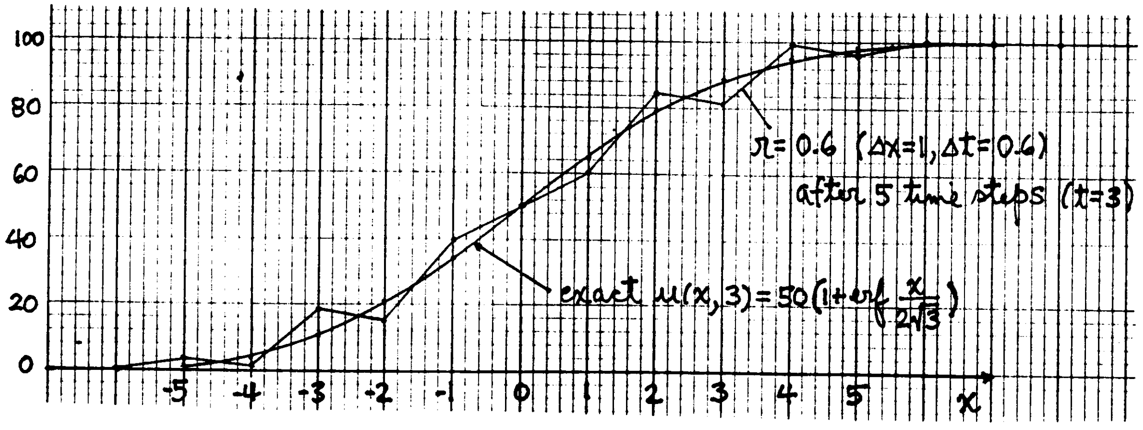
$t\uparrow$												
0 0	0.51 3.89	2.30 1.30	7.42 18.58	17.02 14.98	28.96 39.70	50 50	68.10 60.3	82.98 85.02	92.58 81.42	97.70 98.70	99.49 96.11	100 100
0 0	0 0	1.28 6.48	5.12 4.32	14.72 25.92	30.08 29.28	50 50	69.92 70.72	85.28 74.08	94.88 95.68	98.72 93.52	100 100	100 100
0 0	0 0	0 0	3.2 10.8	11.2 10.8	28 36	50 50	72 64	88.8 89.2	96.8 89.2	100 100	100 100	100 100
0 0	0 0	0 0	0 0	8 18	24 24	50 50	76 76	92 82	100 100	100 100	100 100	100 100
0 0	0 0	0 0	0 0	0 0	20 30	50 50	80 70	100 100	100 100	100 100	100 100	100 100
0	0	0	0	0	0	50	100	100	100	100	100	100
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6 x

Let us plot the final results (at $t = 5\times0.4 = 2$ for $r = .4$ and at $t = 5\times.6 = 3$ for $r = .6$) together with the exact solution, namely, $u(x,t) = 50(1 + \operatorname{erf}\frac{x}{2\alpha\sqrt{t}})$.

For $r = .4$:



For $r = .6$:



Of these two cases, the $r=0.6$ results reveal the anticipated instability. The $r=0.4$ results are stable but not very accurate since the grid is coarse. How can we tell it is coarse? Because most of the variation in u occurs over $-5<x<5$, so letting $\Delta x=1$ gives merely 10 subdivisions of that interval. Here are the comparisons:

x:	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$r=.4$ eval. of u:	0.51	2.30	7.42	17.02	28.96	50	68.10	82.98	92.58	97.70	99.49	100
Exact u:	0.62	2.28	6.68	15.87	30.85	50	69.15	84.13	93.32	97.72	99.38	100

10. $\Delta x=2,\ \Delta t=.5$

Left $(0<x<6)$: $r=1.8(.5)/4=0.225$ so $U_{j,k+1}=.225U_{j-1,k}+.55U_{jk}+.225U_{j+1,k}$ ①

Right $(3<x<12)$: $r=.2(.5)/4=0.025$ so $U_{j,k+1}=.025U_{j-1,k}+.95U_{jk}+.025U_{j+1,k}$ ②

But the latter finite-difference formulas do not hold at grid points at $x=6$. There, proceed as suggested in the exercise:

$$K_L\,\frac{U_{3k}-U_{2k}}{\Delta x}=K_R\,\frac{U_{4k}-U_{3k}}{\Delta x}$$

gives

$$U_{3k}=\frac{K_LU_{2k}+K_RU_{4k}}{K_L+K_R}=.893U_{2k}+.107U_{4k} \qquad ③$$

so the idea is to use ① to compute U_{1k} and U_{2k} using ①, and U_{4k} and U_{5k} using ②, then U_{3k} using ③.

$k=1$: $U_{11}=.225(50)+0+0=11.25$, $U_{21}=\cdots=U_{51}=0$

$k=2$: $U_{12}=.225(100)+.55(11.25)+0=28.69$,

$U_{22}=.225(11.25)+0+0=2.53$

$U_{42}=0+0+0=0$, $U_{52}=0+0+0=0$,

$U_{32}=.893(2.53)+.107(0)=2.26$

$k=3$: $U_{13}=.225(100)+.55(28.69)+.225(2.53)=38.85$,

$U_{23}=.225(28.69)+.55(2.53)+.225(2.26)=8.36$,

$U_{43}=.025(2.26)+0+0=.06$, $U_{53}=0+0+0=0$,

$U_{33}=.893(8.36)+.107(.06)=7.47$

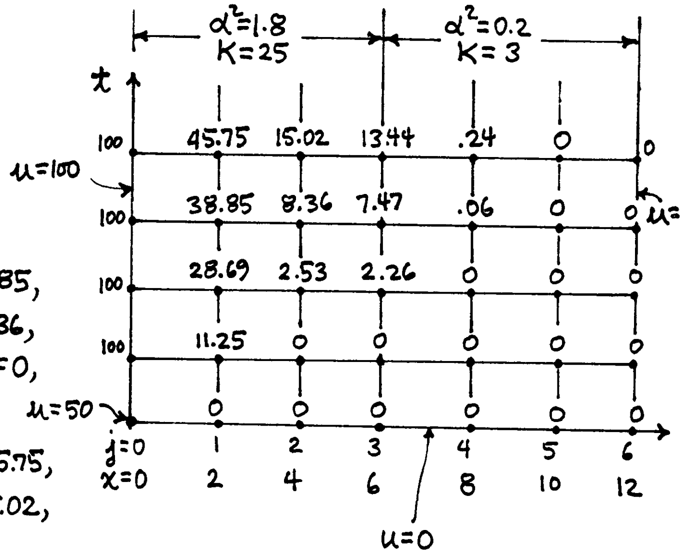
$k=4$: $U_{14}=.225(100)+.55(38.85)+.225(8.36)=45.75$,

$U_{24}=.225(38.85)+.55(8.36)+.225(7.47)=15.02$,

$U_{44}=.025(7.47)+.95(.06)+.025(0)=0.24$,

$U_{54}=0.25(.06)+0+0=0.00$,

$U_{34}=.893(15.02)+.107(.24)=13.44$



12. $u(x+\Delta x,t) = u(x,t) + u_x(x,t)\Delta x + \frac{1}{2}u_{xx}(x,t)(\Delta x)^2 + \frac{1}{6}u_{xxx}(x,t)(\Delta x)^3 + \cdots$

$u(x-\Delta x,t) = u(x,t) - u_x(x,t)\Delta x + \frac{1}{2}u_{xx}(x,t)(\Delta x)^2 - \frac{1}{6}u_{xxx}(x,t)(\Delta x)^3 + \cdots$

Addition gives $u(x+\Delta x,t) + u(x-\Delta x,t) = 2u(x,t) + u_{xx}(x,t)(\Delta x)^2 + O(\Delta x)^4$

Neglecting the $O(\Delta x)^4$ terms and solving for u_{xx} gives

$$u_{xx}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}$$

13. The final vector in (13.1) comes from the boundary conditions. That is, writing out (8) for an entire "time line":

$$\begin{aligned}
U_{1,k+1} &= rU_{0k} + (1-2r)U_{1k} + rU_{2k}\\
U_{2,k+1} &= rU_{1k} + (1-2r)U_{2k} + rU_{3k}\\
&\ \vdots\\
U_{N-2,k+1} &= rU_{N-3,k} + (1-2r)U_{N-2,k} + rU_{N-1,k}\\
U_{N-1,k+1} &= rU_{N-2,k} + (1-2r)U_{N-1,k} + rU_{Nk}
\end{aligned}$$

or,

Known from b.c.'s

$$\begin{array}{lll}
U_{1,k+1} = (1-2r)U_{1k} + rU_{2k} & & + rU_{0k}\\
U_{2,k+1} = rU_{1k} + (1-2r)U_{2k} + rU_{2k} & & \\
\ \vdots & & \\
U_{N-2,k+1} = & rU_{N-3,k} + (1-2r)U_{N-2,k} + rU_{N-1,k} & \\
U_{N-1,k+1} = & rU_{N-2,k} + (1-2r)U_{N-1,k} & + rU_{Nk}
\end{array}$$

which, in matrix form, gives (13.1). Continuing as suggested, arrive at (13.4):

$$\underset{\sim}{e}_{k+1} = \underset{\sim}{A}\underset{\sim}{e}_k + \underset{\sim}{b}_k.$$

Thus,

$$\begin{aligned}
\underset{\sim}{e}_1 &= \underset{\sim}{A}\underset{\sim}{e}_0 + \underset{\sim}{b}_0 \qquad \underset{\sim}{0} \text{ since } \underset{\sim}{b}_k = \underset{\sim}{c}_k - \underset{\sim}{c}_k^* \neq \underset{\sim}{0} \text{ only for } k=0\\
\underset{\sim}{e}_2 &= \underset{\sim}{A}\underset{\sim}{e}_1 + \underset{\sim}{b}_1 = \underset{\sim}{A}(\underset{\sim}{A}\underset{\sim}{e}_0 + \underset{\sim}{b}_0) + \underset{\sim}{b}_1 = \underset{\sim}{A}^2\underset{\sim}{e}_0 + \underset{\sim}{A}\underset{\sim}{b}_0 + \cancelto{\underset{\sim}{0}}{\underset{\sim}{b}_1}\\
\underset{\sim}{e}_3 &= \underset{\sim}{A}\underset{\sim}{e}_2 + \underset{\sim}{b}_2 = \underset{\sim}{A}(\underset{\sim}{A}^2\underset{\sim}{e}_0 + \underset{\sim}{A}\underset{\sim}{b}_0) + \underset{\sim}{b}_2 = \underset{\sim}{A}^3\underset{\sim}{e}_0 + \underset{\sim}{A}^2\underset{\sim}{b}_0 + \cancelto{\underset{\sim}{0}}{\underset{\sim}{b}_3}
\end{aligned}$$

and so on, which gives (13.5). Now, the $(N-1)\times(N-1)$ matrix $\underset{\sim}{A}$ is symmetric so it gives $N-1$ orthogonal — and hence LI — eigenvectors, which eigenvectors therefore comprise a basis. (We will not use the orthogonality but only need to be assured that they are LI and therefore do give a basis.) Putting (13.6,7) into (13.5) easily gives (13.8):

$$\begin{aligned}
\underset{\sim}{e}_k &= (\alpha_1\lambda_1^k + \beta_1\lambda_1^{k-1})\underset{\sim}{\Phi}_1 + \cdots + (\alpha_{N-1}\lambda_{N-1}^k + \beta_{N-1}\lambda_{N-1}^{k-1})\underset{\sim}{\Phi}_{N-1},\\
&= \lambda_1^{k-1}(\alpha_1\lambda_1 + \beta_1)\underset{\sim}{\Phi}_1 + \cdots + \lambda_{N-1}^{k-1}(\alpha_{N-1}\lambda_{N-1} + \beta_{N-1})\underset{\sim}{\Phi}_{N-1}, \qquad (13.8)
\end{aligned}$$

where the α's and β's, from (13.6,7) can be considered as known and arbitrary. If all of the λ's are less than or $=1$ in magnitude then (13.8) shows that $\underset{\sim}{e}_k \to \underset{\sim}{0}$ as $k\to\infty$. (That is not to say that the total roundoff error $\to 0$ since $\underset{\sim}{e}_k$ is only the roundoff error resulting from a single line of roundoffs, at $k=0$ Such "initial roundoffs" are actually being injected at each time line.) If any λ is greater than 1 in magnitude then $|\underset{\sim}{e}_k| \to \infty$ as $k\to\infty$ and we have instability.

Thus, for stability set $|\lambda_n| \le 1$ for each $n=1,2,\dots,N-1$. Or, using (13.11),

$$-1 \le 1-2r+2r\cos\frac{n\pi}{N} \le 1$$

↳ $1-2r+2r\cos\frac{n\pi}{N} \le 1$,

$2r(\cos\frac{n\pi}{N}-1) \le 0$, which is always satisfied (since $r>0$) and which is, therefore, uninformative

↓

gives $2r(\cos\frac{n\pi}{N}-1) \ge -2$

$$r(1-\cos\frac{n\pi}{N}) \le 1$$

$$r \le \frac{1}{1-\cos\frac{n\pi}{N}}$$

for $n=1,2,\dots,N-1$. Of these $N-1$ inequalities, the one with the smallest right-hand side supercedes all the others. The smallest right-hand side occurs when $n=N-1$, in which case we have

$$r \le \frac{1}{1-\cos(\frac{N-1}{N}\pi)}.$$

14. $u_t = -Au \quad (A>0)$

→ $\frac{U_{k+1}-U_k}{\Delta t} = -(1-\theta)AU_k - \theta AU_{k+1}$,

$$U_{k+1} = \frac{1-(1-\theta)A\Delta t}{1+\theta A\Delta t} U_k \qquad ①$$

With an initial roundoff error $U_0 - U_0^* \equiv e_0 \ne 0$, and without any subsequent roundoff error (i.e., using a perfect computer thereafter) we have

$$U^*_{k+1} = \frac{1-(1-\theta)A\Delta t}{1+\theta A\Delta t} U^*_k \qquad ②$$

and subtracting ② from ① gives

$$e_{k+1} = K e_k \quad , \quad K \equiv \frac{1-A(1-\theta)\Delta t}{1+A\theta\Delta t}.$$

Thus,

$$e_1 = Ke_0 \ , \ e_2 = Ke_1 = K^2 e_0 \ , \ \dots \ , \ e_k = K^k e_0$$

so the scheme is stable if and only if $|K| \le 1$; i.e.,

$$-1 \le \frac{1-A(1-\theta)\Delta t}{1+A\theta\Delta t} \le 1$$

↳ gives $1-A(1-\theta)\Delta t \le 1+A\theta\Delta t$

or, $-A\Delta t \le 0$, which is always satisfied and where, therefore, is uninformative

↓

gives

$-1-A\theta\Delta t \le 1-A(1-\theta)\Delta t$

or, $[A(1-\theta)-A\theta]\Delta t \le 2$

or, $A(1-2\theta)A\Delta t \le 2$. ✓

15. Crank-Nicolson scheme: $-rU_{j-1,k+1}+2(1+r)U_{j,k+1}-rU_{j+1,k+1} = rU_{j-1,k}+2(1-r)U_{jk}+rU_{j+1,k}$

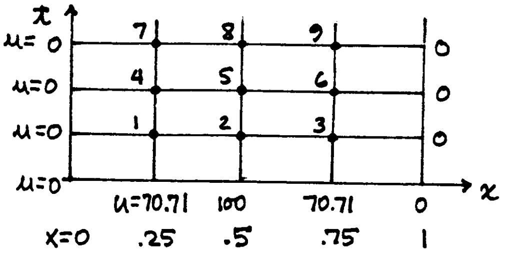
for $u_{xx} = u_t \quad (0<x<1)$

$u(0,t)=u(1,t)=0$, $u(x,0)=100\sin\pi x$

with $r = \alpha^2 \Delta t/(\Delta x)^2 = (1)(.1)/(.25)^2 = 1.6$.

With $r=1.6$ the method is given by

$$-1.6U_{j-1,k+1}+5.2U_{j,k+1}-1.6U_{j+1,k+1}$$
$$=1.6U_{j-1,k}-1.2U_{jk}+1.6U_{j+1,k}$$



For this hand calculation it will be simpler to denote the grid points as 1...9. Thus,

$0+5.2U_1-1.6U_2=0-1.2(70.71)+1.6(100)\ =75.15$ ①

$-1.6U_1+5.2U_2-1.6U_3=1.6(70.71)-1.2(100)+1.6(70.71)=106.27$ ②

$-1.6U_2+5.2U_3-0\ =1.6(100)-1.2(70.71)+0=75.15$ ③

$0+5.2U_4-1.6U_5=\ 0\ -1.2U_1+1.6U_2$ ④

$-1.6U_4+5.2U_5-1.6U_6=\ 1.6U_1-1.2U_2+1.6U_3$ ⑤

$-1.6U_5+5.2U_6-0\ =\ 1.6U_2-1.2U_3+0$ ⑥

$0+5.2U_7-1.6U_8=\ 0\ -1.2U_4+1.6U_5$ ⑦

$-1.6U_7+5.2U_8-1.6U_9=1.6U_4-1.2U_5+1.6U_6$ ⑧

$-1.6U_8+5.2U_9-0\ =\ 1.6U_5-1.2U_6+0$ ⑨

We can solve these one line at a time. That is, we can solve ①-③ for U_1,U_2,U_3. Then put those values into the RHS's (right-hand sides) of ④-⑥ and solve ④-⑥ for U_4,U_5,U_6. Put those values into the RHS's of ⑦-⑨ and solve those for U_7,U_8,U_9. Alternatively, we could solve ①-⑨ as a linear system for the unknowns $U_1,\dots,U_9$. Let us do that, using the Maple linsolve command:

```
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> linsolve(array([[5.2,-1.6,0,0,0,0,0,0,0],[-1.6,5.2,-1.6,0,0,0,0,0,
  0],[0,-1.6,5.2,0,0,0,0,0,0],[1.2,-1.6,0,5.2,-1.6,0,0,0,0],[-1.6,1.
  2,-1.6,-1.6,5.2,-1.6,0,0,0],[0,-1.6,1.2,0,-1.6,5.2,0,0,0],[0,0,0,1
  .2,-1.6,0,5.2,-1.6,0],[0,0,0,-1.6,1.2,-1.6,-1.6,5.2,-1.6],[0,0,0,0
  ,-1.6,1.2,0,-1.6,5.2]]),array([75.15,106.27,75.15,0,0,0,0,0,0]));
[25.58448905, 36.18083939, 25.58448905, 9.256512057, 13.09119158, 9.256512057,
  3.349285247, 4.736369514, 3.349285247]
```

u_1 u_9

Compare these results with the exact solution, which is $u(x,t)=100\sin\pi x\, e^{-\pi^2 t}$.

$u_1=u(.25,.1)=100\sin(\pi/4)\exp(-.9869)\approx 26.35$

$u_2=u(.5,.1)\approx 37.27$, $u_3=u(.75,.1)=26.35$, $u_4=u(.25,.2)\approx 9.82$,

$u_5=u(.5,.2)\approx 13.89$, $u_6=u(.75,.2)\approx 9.82$, $u_7=u(.25,.3)\approx 3.66$

$u_8=u(.5,.3)\approx 5.18$, $u_9=u(.75,.3)\approx 3.66$

so the Crank-Nicolson results look okay — considering how coarse the grid is.

NOTE: By symmetry about $x=0.5$, it is evident that $U_3=U_1$, $U_6=U_4$, and $U_9=U_7$. We could have used this fact to work with six equations instead of nine, but we chose to do the nine equations and use the symmetry of the results as a partial check on those results.

16. **SOR:** $U_{11}^{(0)} = 37.5$
$U_{21}^{(0)} = 50$
$U_{31}^{(0)} = 15$

Tentative G-S step: $U_{11}^{(1)} = 50$ so $\Delta U_{11}^{(0)} = 50 - 37.5 = 12.5$
$U_{21}^{(1)} = 66.25$ so $\Delta U_{21}^{(0)} = 66.25 - 50 = 16.25$
$U_{31}^{(1)} = 31.56$ so $\Delta U_{31}^{(0)} = 31.56 - 15 = 16.56$

Now an SOR step: $U_{11}^{(1)} = U_{11}^{(0)} + \omega \Delta U_{11}^{(0)} = 37.5 + 1.03(12.5) = 50.38$
$U_{21}^{(1)} = U_{21}^{(0)} + \omega \Delta U_{21}^{(0)} = 50 + 1.03(16.25) = \boxed{66.74}$
$U_{31}^{(1)} = U_{31}^{(0)} + \omega \Delta U_{31}^{(0)} = 15 + 1.03(16.56) = 32.06$

Tentative G-S step: $U_{11}^{(2)} = \frac{1}{4}(U_{21}^{(1)} + 150) = \frac{1}{4}(66.74 + 150) = 54.19$
so $\Delta U_{11}^{(1)} = 54.19 - 50.38 = 3.81$
$U_{21}^{(2)} = \frac{1}{4}(U_{11}^{(2)} + U_{31}^{(1)} + 200) = \frac{1}{4}(54.19 + 32.06 + 200) = 71.56$
so $\Delta U_{21}^{(1)} = 71.56 - 66.74 = 4.82$
$U_{31}^{(2)} = \frac{1}{4}(U_{31}^{(2)} + 60) = \frac{1}{4}(71.56 + 60) = 32.89$
so $\Delta U_{31}^{(1)} = 32.89 - 32.06 = 0.83$

Now an SOR step: $U_{11}^{(2)} = U_{11}^{(1)} + \omega \Delta U_{11}^{(1)} = 50.38 + 1.03(3.81) = 54.30$
$U_{21}^{(2)} = U_{21}^{(1)} + \omega \Delta U_{21}^{(1)} = 66.74 + 1.03(4.82) = \boxed{71.70}$
$U_{31}^{(2)} = U_{31}^{(1)} + \omega \Delta U_{31}^{(1)} = 32.06 + 1.03(0.83) = 32.91$

Jacoby: $U_{11}^{(2)} = \frac{1}{4}(63.13 + 150) = 53.28$
$U_{21}^{(2)} = \frac{1}{4}(U_{11}^{(1)} + U_{31}^{(1)} + 200) = \frac{1}{4}(50 + 27.5 + 200) = \boxed{69.38}$
$U_{31}^{(2)} = \frac{1}{4}(U_{21}^{(1)} + 60) = \frac{1}{4}(63.13 + 60) = 30.78$
Further,
$U_{21}^{(3)} = \frac{1}{4}(U_{11}^{(2)} + U_{31}^{(2)} + 200) = \frac{1}{4}(53.28 + 30.78 + 200) = 71.02$

Gauss-Seidel: $U_{11}^{(2)} = \frac{1}{4}(66.25 + 150) = 54.06$
$U_{21}^{(2)} = \frac{1}{4}(54.06 + 31.56 + 200) = \boxed{71.41}$
$U_{31}^{(2)} = \frac{1}{4}(71.41 + 60) = 32.85$
Further,
$U_{11}^{(3)} = \frac{1}{4}(71.41 + 150) = 55.35$
$U_{21}^{(3)} = \frac{1}{4}(55.35 + 32.85 + 200) = 72.05$

18. $\underset{\sim}{c} = \sum_{1}^{N-1} c_j \underset{\sim}{\Phi}_j$

$\underset{\sim}{U}^{(0)}_{k+1} = \beta \sum c_j \underset{\sim}{\Phi}_j$ from (31).

$$\underset{\sim}{U}^{(1)}_{k+1} = \beta \underset{\sim}{c} - \beta \underset{\sim}{A}' \underset{\sim}{U}^{(0)}_{k+1} = \beta \sum c_j \underset{\sim}{\Phi}_j - \beta^2 \sum \lambda_j c_j \underset{\sim}{\Phi}_j = \beta \sum (1-\beta\lambda_j) c_j \underset{\sim}{\Phi}_j$$

$$\begin{aligned}\underset{\sim}{U}^{(2)}_{k+1} = \beta \underset{\sim}{c} - \beta \underset{\sim}{A}' \underset{\sim}{U}^{(1)}_{k+1} &= \beta \sum c_j \underset{\sim}{\Phi}_j - \beta \underset{\sim}{A}' \beta \sum (1-\beta\lambda_j) c_j \underset{\sim}{\Phi}_j \\ &= \quad '' \quad - \beta^2 \sum (1-\beta\lambda_j) c_j \lambda_j \underset{\sim}{\Phi}_j \\ &= \beta \sum (1-\beta\lambda_j + \beta^2\lambda_j^2) c_j \underset{\sim}{\Phi}_j\end{aligned}$$

$\vdots$

$$\underset{\sim}{U}^{(p)}_{k+1} = \beta \sum_{1}^{N-1} \underbrace{(1-\beta\lambda_j + \beta^2\lambda_j^2 - \cdots + (-1)^p \beta^p \lambda_j^p)}_{*} c_j \underset{\sim}{\Phi}_j$$

As $p \to \infty$, $*$ becomes a geometric series, which converges to $1/(1+\beta\lambda_j)$ if $|\beta\lambda_j| < 1$ and diverges otherwise. Since λ_j's are the eigenvalues of $\underset{\sim}{A}'$ and $\underset{\sim}{A}'$ is of the type in Exercise 7 of Section 11.2, with "a"="c"$=-r$ and "b"$=0$, then

$$\lambda_j = \text{``}b + 2\sqrt{ac} \cos \frac{j\pi}{n+1}\text{''} = 2r \cos \frac{j\pi}{N}$$

so

$$|\beta\lambda_j| = \frac{2r}{2(1+r)} \left|\cos \frac{j\pi}{N}\right| < \frac{r}{1+r} < 1$$

for each $j = 1, \ldots, N-1$ and for every positive value of r. Thus,

$$\lim_{p\to\infty} \underset{\sim}{U}^{(p)}_{k+1} = \beta \sum_{1}^{N-1} \frac{1}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j$$

and we need to show that the latter satisfies (26). Recalling that $\underset{\sim}{A} = 2(1+r)\underset{\sim}{I} + \underset{\sim}{A}'$ we have

$$\begin{aligned}\left(2(1+r)\underset{\sim}{I} + \underset{\sim}{A}'\right) \beta \sum_{1}^{N-1} \frac{1}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j &= \sum_{1}^{N-1} \frac{c_j}{1+\beta\lambda_j} \underset{\sim}{\Phi}_j + \sum_{1}^{N-1} \frac{\beta c_j \lambda_j}{1+\beta\lambda_j} \underset{\sim}{\Phi}_j \\ &= \sum_{1}^{N-1} \frac{1+\beta\lambda_j}{1+\beta\lambda_j} c_j \underset{\sim}{\Phi}_j = \sum_{1}^{N-1} c_j \underset{\sim}{\Phi}_j = \underset{\sim}{c}. \checkmark\end{aligned}$$

CHAPTER 19

Section 19.1

1. In (2), put $f=0$ and $\tau=\sigma(l-x)$ so, in place of (2), we have

$$\sigma g[l-(x+\Delta x)]\sin\theta(x+\Delta x,t)-\sigma g(l-x)\sin\theta(x,t)=\sigma\Delta s\, y_{tt}(x+\beta\Delta x,t).$$

Then, as before, $\sin\theta\approx\tan\theta\approx y_x$ and $\Delta s\approx\Delta x$, so

$$\sigma g[l-(x+\Delta x)]y_x(x+\Delta x,t)-\sigma g(l-x)y_x(x,t)=\sigma\Delta x\, y_{tt}(x+\beta\Delta x,t)$$

or,

$$g\frac{[l-(x+\Delta x)]y_x(x+\Delta x,t)-(l-x)y_x(x,t)}{\Delta x}=y_{tt}(x+\beta\Delta x,t)$$

and letting $\Delta x\to 0$ gives $g((l-x)y_x)_x=y_{tt}$. ✓

2. Newton's 2nd law: $[S(x+\Delta x,t)-S(x,t)]A=\sigma\Delta x\,\overset{\text{acceleration of mass center}}{u_{tt}}(x+\frac{\Delta x}{2},t)$

Dividing by Δx and letting $\Delta x\to 0$ gives $A\frac{\partial S}{\partial x}=\sigma\frac{\partial^2 u}{\partial t^2}$. ①

Also,

$$S=E\epsilon=E\,\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}\to E\frac{\partial u}{\partial x}\quad\text{as } \Delta x\to 0$$

so

$$S=E\frac{\partial u}{\partial x} \qquad ②$$

Putting ② into ① gives the wave equation $\overset{c^2}{\boxed{\frac{EA}{\sigma}}}u_{xx}=u_{tt}$.

Alternatively $\partial/\partial x$ of ① gives $AS_{xx}=\sigma u_{ttx}$ ③

and $\partial^2/\partial t^2$ of ② gives $S_{tt}=Eu_{xtt}$ ④

Assuming $u_{ttx}=u_{xtt}$, ③ and ④ give

$$\frac{A}{\sigma}s_{xx}=\frac{1}{E}s_{tt}\ ,\text{or,}\quad \frac{EA}{\sigma}s_{xx}=s_{tt}.$$

3. (a) Curl of (3.1) gives $\nabla\times(\nabla\times\mathbf{H})=\epsilon_0\nabla\times\frac{\partial\mathbf{E}}{\partial t}$ or, using the identity in the HINT,

$$\nabla\underbrace{(\nabla\cdot\mathbf{H})}_{0\text{ by }(3.3)}-\nabla^2\mathbf{H}=\epsilon_0\frac{\partial}{\partial t}\underbrace{\nabla\times\mathbf{E}}_{-\mu_0\frac{\partial\mathbf{H}}{\partial t}\text{ by }(3.2)}$$

so $\nabla^2\mathbf{H}=\epsilon_0\mu_0\frac{\partial^2\mathbf{H}}{\partial t^2}$.

Similarly, we can take curl of (3.2) and then use the given identity and (3.4) and obtain

$$\nabla^2\mathbf{E}=\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

These give (5), vector wave equations on $\mathbf{E}$ and $\mathbf{H}$.

(b)
$$\left.\begin{aligned}c^2\nabla^2E_x&=\partial^2E_x/\partial t^2\\ c^2\nabla^2E_y&=\partial^2E_y/\partial t^2\\ c^2\nabla^2E_z&=\partial^2E_z/\partial t^2\end{aligned}\right\}$$
Thus, each of the components of $\mathbf{E}$ satisfies the same wave equation; similarly for $\mathbf{H}$.

4. (a) A given charge moves according to
$$x = x(t)$$
$$y = y[x(t), t]$$
$$z = z[x(t), t]$$
so its velocity is $= \dot{x}\hat{\imath} + (y_x\dot{x} + y_t)\hat{\jmath} + (z_x\dot{x} + z_t)\hat{k}$

so $\underset{\sim}{\mathrm{Vel}} \approx U\hat{\imath} + (Uy_x + y_t)\hat{\jmath} + (Uz_x + z_t)\hat{k}$.

Thus, the force on the charge is $Q\underset{\sim}{\mathrm{Vel}} \times \underset{\sim}{B}$. In a wire segment of length Δx the charge is $Q = qA\Delta x$, so

$$\Delta \underset{\sim}{F} \approx qA\Delta x\,[U\hat{\imath} + (Uy_x + y_t)\hat{\jmath} + (Uz_x + z_t)\hat{k}\,] \times \underset{\sim}{B}$$
$$= (\text{etc.})\hat{\imath} + qA\Delta x\{[(Uz_x + z_t)B_1 - UB_3]\hat{\jmath} + [UB_2 - (Uy_x + y_t)B_1]\hat{k}\}$$
$$= (\text{etc.})\hat{\imath} + \Delta x\{[(Iz_x + qAz_t)B_1 - IB_3]\hat{\jmath} + [IB_2 - (Iy_x + qAy_t)B_1]\hat{k}\}$$

(b) With these additional lateral force terms (i.e., the y and z components of $\Delta\underset{\sim}{F}$), the y and z equations of motion become
$$\tau y_{xx} - IB_3 + (Iz_x + qAz_t)B_1 = \sigma y_{tt}$$
$$\tau z_{xx} + IB_2 - (Iy_x + qAy_t)B_1 = \sigma z_{tt}$$

5. Neglecting the second-order term uu_x relative to the first-order terms u_t and $-g\eta_x$, and neglecting the second-order term $u\eta$ relative to the first-order terms uh and $-\eta_t$, gives these approximate equations:
$$u_t = -g\eta_x \quad ①$$
$$(uh)_x = -\eta_t. \quad ②$$
If we seek a PDE on η alone then we need to eliminate the other dependent variable u from ① and ②. First re-express them as
$$hu_t = -gh\eta_x \quad \text{or, } (hu)_t = -gh\eta_x \quad [\text{since } h = h(x)]$$
$$(hu)_x = -\eta_t$$
Then $\partial/\partial x$ of the former and $\partial/\partial t$ of the latter gives
$$(hu)_{tx} = -g(h\eta_x)_x$$
$$(hu)_{xt} = -\eta_{tt}$$
so $\quad g(h\eta_x)_x = \eta_{tt}$

or, if $h(x) =$ constant, $\quad \overset{\swarrow c^2}{\boxed{gh}}\,\eta_{xx} = \eta_{tt}$.

Section 19.2

1. With $L = 10$, $c = 12$, $f_0 = 1$, (20) becomes

$$y(x,t) = \frac{8}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\cos 1.2n\pi t$$

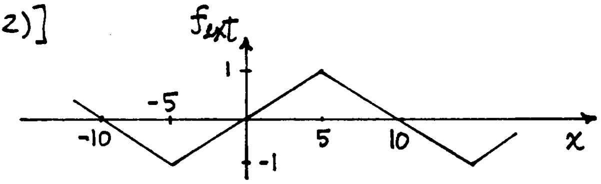
(a) To evaluate $y(5,1) = \frac{8}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi 5}{10}\cos 1.2n\pi$, let us use the maple command

```
evalf((8/Pi^2)* sum(sin(i*Pi/2)*sin(i*Pi/2)*cos(1.2*i*Pi)/i^2,
        i=1..10));
```

and obtain -0.58949. Changing the upper summation limit from 10 to 20 gives -0.59978. Summing to 40 gives -0.59997, and summing to 80 gives -0.60000.

Alternatively, (30) gives (exactly)

$$\begin{aligned} y(5,1) &= \tfrac{1}{2}[f_{ext}(5-12)+f_{ext}(5+12)] \\ &= \tfrac{1}{2}[f_{ext}(-7)+f_{ext}(17)] \\ &= \tfrac{1}{2}[f_{ext}(-7)+f_{ext}(-3)] \\ &= \tfrac{1}{2}[-f_{ext}(7)-f_{ext}(3)] \\ &= \tfrac{1}{2}(-.6-.6) = -0.6 \end{aligned}$$



For the remaining parts let us just use (30).

(b) $y(5,2) = \frac{1}{2}[f_{ext}(5-24)+f_{ext}(5+24)] = \frac{1}{2}[f_{ext}(-19)+f_{ext}(29)]$
$= \frac{1}{2}[f(1)+f(9)] = \frac{1}{2}(.2+.2) = 0.2$

(c) $y(5,3) = \frac{1}{2}[f_{ext}(5-36)+f_{ext}(5+36)] = \frac{1}{2}[f_{ext}(-31)+f_{ext}(41)]$
$= \frac{1}{2}[f(9)+f(1)]$ (because f_{ext} has period 20, so we can add or subtract integer multiples of 20 to the arg of f_{ext} without changing its value)
$= \frac{1}{2}(.2+.2) = 0.2$

(d) $y(5,4) = \frac{1}{2}[f_{ext}(5-48)+f_{ext}(5+48)] = \frac{1}{2}[f_{ext}(-43)+f_{ext}(53)]$
$= \frac{1}{2}[f_{ext}(-3)+f_{ext}(-7)]$ (because f_{ext} has period 20)
$= \frac{1}{2}[-f_{ext}(3)-f_{ext}(7)]$ (because f_{ext} is odd)
$= \frac{1}{2}(-f(3)-f(7)) = \frac{1}{2}(-.6-.6) = -0.6$

(e) $y(5,6) = \frac{1}{2}[f_{ext}(5-72)+f_{ext}(5+72)] = \frac{1}{2}[f_{ext}(-67)+f_{ext}(77)]$
$= \frac{1}{2}[f_{ext}(-7)+f_{ext}(-3)]$
$= \frac{1}{2}[-f(7)-f(3)] = \frac{1}{2}(-.6-.6) = -0.6$

2\. (a) $f(x)=0$, $g(x)=50\sin(\pi x/L)$. Then (18a) gives $R_n=0$ and (18b) gives

$$S_n = \frac{2}{n\pi c}\int_0^L 50\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{100}{n\pi c}\,\frac{L\sin n\pi}{\pi(n-1)(n+1)}$$ using the Maple int command. Now,

$\sin n\pi = 0$ for $n=1,2,\dots$ so it is tempting to conclude that $S_n=0$ for $n=1,2,\dots$. However, the denominator also $=0$ when $n=1$. For $n=1$ use l'Hôpital's rule and obtain $S_1 = -\frac{100}{\pi c}\frac{L}{\pi}\left.\frac{\pi\cos n\pi}{n+1}\right|_{n=1} = \frac{50L}{\pi c}$, so (16) gives

$$y(x,t) = \frac{50L}{\pi c}\sin\frac{\pi x}{L}\sin\frac{\pi c t}{L}.$$

Alternatively, we could have seen from (17b),

$$50\sin\frac{\pi x}{L} = \sum_1^\infty \frac{n\pi c}{L}S_n\sin\frac{n\pi x}{L}$$

that $S_n=0$ for all $n\neq 1$ and that $50 = \frac{\pi c}{L}S_1$, so $S_1 = 50L/\pi c$, as above. We will use that idea, simply "matching terms" in (b) and (c) below.

(b)

$$g(x) = 3\sin\frac{\pi x}{L} - 5\sin\frac{4\pi x}{L} = \sum_1^\infty \frac{n\pi c}{L}S_n\sin\frac{n\pi x}{L}$$

so S_n's $=0$ except for $n=1$ and $n=4$: $3=\frac{\pi c}{L}S_1$ gives $S_1 = 3L/\pi c$
$-5 = \frac{4\pi c}{L}S_4$ gives $S_4 = -5L/4\pi c$

so (16) gives

$$y(x,t) = \frac{3L}{\pi c}\sin\frac{\pi x}{L}\sin\frac{\pi c t}{L} - \frac{5L}{4\pi c}\sin\frac{4\pi x}{L}\sin\frac{4\pi c t}{L}$$

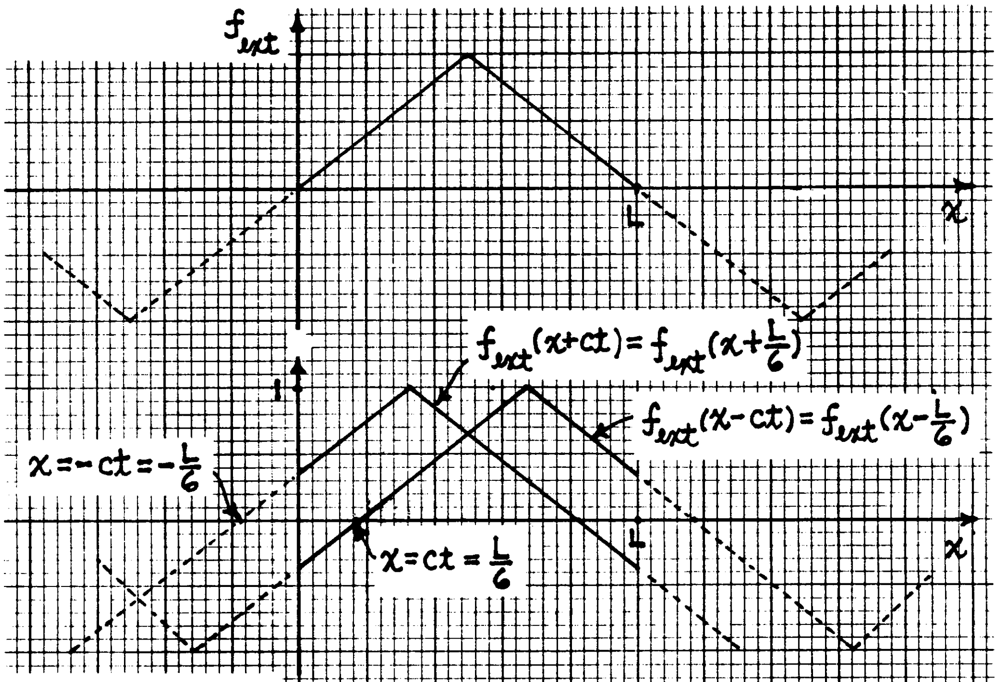
(c) $g(x) = \sin\frac{2\pi x}{L} + \sin\frac{3\pi x}{L} + 4\sin\frac{8\pi x}{L} = \sum_1^\infty \frac{n\pi c}{L}S_n\sin\frac{n\pi x}{L}$

gives $1=\frac{2\pi c}{L}S_2$, $1=\frac{3\pi c}{L}S_3$, $4=\frac{8\pi c}{L}S_8$ or $S_2 = L/2\pi c$, $S_3 = L/3\pi c$, $S_8 = 4L/8\pi c$ with all other S_n's $=0$, so (16) gives

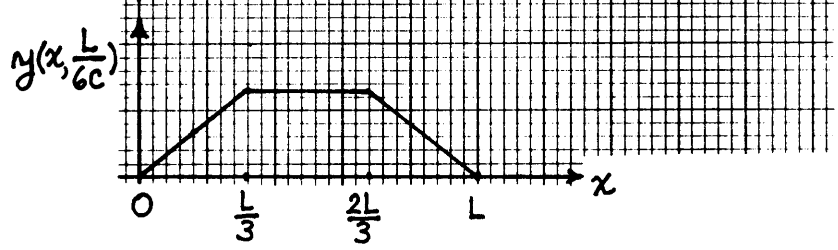
$$y(x,t) = \frac{L}{2\pi c}\sin\frac{2\pi x}{L}\sin\frac{2\pi c t}{L} + \frac{L}{3\pi c}\sin\frac{3\pi x}{L}\sin\frac{3\pi c t}{L}$$

$$+ \frac{L}{2\pi c}\sin\frac{8\pi x}{L}\sin\frac{8\pi c t}{L}$$

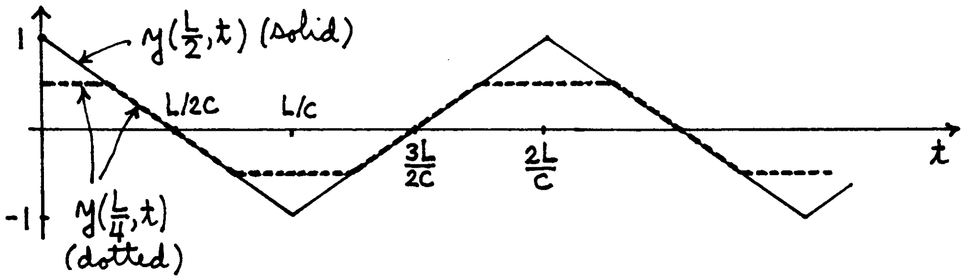
3. (a)



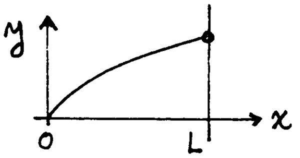
Now add the two graphs and multiply by ½:



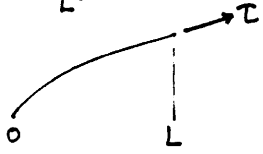
4.



5. (a) Suppose $y_x(L,t) \neq 0$, as sketched:



Then the tension at $x = L$ (which is the same as the force exerted on the string by the wire) is oriented thus:



The vertical component of that force will be due to friction between the string and the wire. But since the wire is assumed frictionless that vertical force component must be zero: thus, the string's slope must be zero where it attaches to the wire.

(b) Let us start with (8):

$$y(x,t) = (A+Bx)(H+It) + (D\cos\kappa x + E\sin\kappa x)(J\cos\kappa ct + K\sin\kappa ct)$$

$$y(0,t) = 0 = A(H+It) + D(J\cos\kappa ct + K\sin\kappa ct), \text{ so } A=D=0 \text{ and}$$

$$y(x,t) = Bx(H+It) + E\sin\kappa x(J\cos\kappa ct + K\sin\kappa ct)$$

$$= x(H'+I't) + \sin\kappa x(J'\cos\kappa ct + K'\sin\kappa ct)$$

$$y_x(L,t) = 0 = H'+I't + \kappa\cos\kappa L \left(\quad '' \quad \right) \rightarrow H'=0, I'=0, \kappa L = \frac{n\pi}{2} \text{ (n odd)}$$

so $$y(x,t) = \sum_{1,3,\dots}^{\infty} \sin\frac{n\pi x}{2L}\left(J_n'\cos\frac{n\pi ct}{2L} + K_n'\sin\frac{n\pi ct}{2L}\right) \qquad ①$$

Finally, $$y(x,0) = f(x) = \sum_{1,3,\dots}^{\infty} J_n'\sin\frac{n\pi x}{2L} \qquad (0<x<L) \qquad ②$$

and $$y_t(x,0) = 0 = \sum_{1,3,\dots}^{\infty} \frac{n\pi c}{2L}K_n'\sin\frac{n\pi x}{2L}. \qquad (0<x<L) \qquad ③$$

② and ③ are QRS expansions. By inspection, ③ gives $K_n'=0$ for $n=1,3,\dots$, and ② gives

$$J_n' = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx. \qquad ④$$

Thus,

$$y(x,t) = \sum_{1,3,\dots}^{\infty} J_n'\sin\frac{n\pi x}{2L}\cos\frac{n\pi ct}{2L}$$

with the J_n''s given by ④.

(c) $$y(x,t) = (A+Bx)(H+It) + (D\cos\kappa x + E\sin\kappa x)(J\cos\kappa ct + K\sin\kappa ct)$$

$$y_x(0,t) = 0 = B(H+It) + \kappa E(J\cos\kappa ct + K\sin\kappa ct) \rightarrow B=E=0$$

so $$y(x,t) = A(H+It) + D\cos\kappa x(J\cos\kappa ct + K\sin\kappa ct)$$

$$= H'+I't + \cos\kappa x(J'\cos\kappa ct + K'\sin\kappa ct)$$

$$y_x(L,t) = 0 = -\kappa\sin\kappa L\left(\quad '' \quad '' \quad \right) \rightarrow \sin\kappa L = 0, \ \kappa L = n\pi \ (n=1,2,..)$$

so $$y(x,t) = H'+I't + \sum_1^{\infty}\cos\frac{n\pi x}{L}\left(J_n'\cos\frac{n\pi ct}{L} + K_n'\sin\frac{n\pi ct}{L}\right) \qquad ①$$

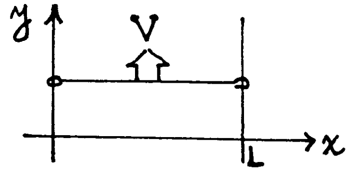
Finally,

$$y(x,0) = 0 = H' + \sum_1^{\infty} J_n'\cos\frac{n\pi x}{L} \qquad (0<x<L) \qquad ②$$

$$y_t(x,0) = V = I' + \sum_1^{\infty}\frac{n\pi c}{L}K_n'\cos\frac{n\pi x}{L}. \qquad (0<x<L) \qquad ③$$

② and ③ are HRC expansions. ② gives $H'=0$ and $J_n'=0$ and ③ gives $I'=V$ and $K_n'=0$ (either by inspection or by the HRC formulas). Thus, ① reduces to

$$y(x,t) = Vt.$$



With hindsight, at least, we could have anticipated this simple result. The string is looped around vertical frictionless wires at $x=0$ and at $x=L$ and given an initial upward velocity, so it "just keeps going".

(d) $y(x,t) = (A+Bx)(H+It) + (D\cos kx + E\sin kx)(J\cos kct + K\sin kct)$

$y_x(0,t) = 0 = B(H+It) + kE\,(J\cos kct + K\sin kct) \rightarrow B=E=0$, so

$y(x,t) = H' + I't + \cos kx\,(J'\cos kct + K'\sin kct)$

$y(L,t) = 0 = H' + I't + \cos kL\,(\quad '' \qquad '' \quad) \rightarrow H'=I'=0,\ kL = \frac{n\pi}{2}\ (n=1,3,\dots)$

so $$y(x,t) = \sum_{1,3,\dots}^{\infty} \cos\frac{n\pi x}{2L}\left(J_n'\cos\frac{n\pi ct}{2L} + K_n'\sin\frac{n\pi ct}{2L}\right) \qquad ①$$

Finally, $$y(x,0) = 0 = \sum_{1,3,\dots}^{\infty} J_n'\cos\frac{n\pi x}{2L} \quad (0<x<L) \qquad ②$$

and $$y_t(x,0) = g(x) = \sum_{1,3,\dots}^{\infty} \frac{n\pi c}{2L}K_n'\cos\frac{n\pi x}{2L}. \quad (0<x<L) \qquad ③$$

② and ③ are QRS expansions. ② gives $J_n'=0$ and ③ gives

$$\frac{n\pi c}{2L}K_n' = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{2L}\,dx\,,$$

so $$y(x,t) = \sum_{1,3,\dots}^{\infty} K_n'\cos\frac{n\pi x}{2L}\sin\frac{n\pi ct}{2L}\,,$$

where $$K_n' = \frac{4}{n\pi c}\int_0^L g(x)\cos\frac{n\pi x}{2L}\,dx.$$

6. $c^2 y_{xx} = y_{tt} + a y_t$. $y(x,t) = X(x)T(t)$ gives $\frac{X''}{X} = \frac{T''+aT'}{c^2 T} = -k^2$

so $X'' + k^2 X = 0$, $\quad X = \begin{cases} A\cos kx + B\sin kx\,, & k\neq 0 \\ C + Dx, & k=0 \end{cases}$

$T'' + aT' + k^2c^2T = 0$,

To solve for T seek $T = e^{\lambda t}$. That step gives $\lambda = (-a \pm \sqrt{a^2-4k^2c^2})/2$. Anticipating (from the X story) that k will $= n\pi/L$, it follows from the given inequality $0<a<2\pi c/L$ that $a^2-4k^2c^2 < \left(\frac{2\pi c}{L}\right)^2 - 4\left(\frac{n\pi c}{L}\right)^2 \leq 0$, so let us re-express λ as $\lambda = (-a \pm i\sqrt{4k^2c^2-a^2})/2$. Thus,

$$T(t) = \begin{cases} e^{-at/2}\left(E\cos\sqrt{k^2c^2-(a/2)^2}\,t + F\sin\sqrt{k^2c^2-(a/2)^2}\,t\right), & k\neq 0 \\ G + He^{-at}. & , k=0 \end{cases}$$

Note that the point to distinguishing the $k=0$ case had to do with the X equation, namely, $A\cos kx + B\sin kx$ is not the general solution to $X''+k^2X=0$ if $k=0$ since it fails to account for the $\underline{x}$ part of the solution $C+Dx$. However, in view of the $y(0,t) = y(L,t) = 0$ b.c.'s we foresee a bleak future for the x term and expect D to turn out to be zero. In that case we could have saved ourselves a bit of writing by not distinguishing the $k=0$ case. However, at this stage the student may not have available that foresight and confidence so let us include it, as above. Thus, we have

$$y(x,t) = (C+Dx)(G+He^{-at}) + (A\cos kx + B\sin kx)(e^{-at/2})(E\cos\Gamma t + F\sin\Gamma t)$$

where Γ is shorthand for $\sqrt{k^2c^2-(a/2)^2}$

The b.c.'s:

$$y(0,t)=0=C(G+He^{-at})+Ae^{-at/2}(E\cos\Gamma t+F\sin\Gamma t)\rightarrow C=A=0\text{ , so}$$

$$y(x,t)=x(G'+H'e^{-at})+\sin kx\; e^{-at/2}(E'\cos\Gamma t+F'\sin\Gamma t)$$

$$y(L,t)=0=L(G'+H'e^{-at})+\sin kL\; e^{-at/2}(\quad " \qquad " \quad " \quad)\rightarrow G'=H'=0,\text{ and}$$

$$k=n\pi/L \quad (n=1,2,\dots)$$

so $y(x,t)=\sum_1^\infty \sin\frac{n\pi x}{L}\, e^{-at/2}(E_n'\cos\omega_n t+F_n'\sin\omega_n t)$

or,

$$y(x,t)=e^{-at/2}\sum_1^\infty \sin\frac{n\pi x}{L}(E_n'\cos\omega_n t+F_n'\sin\omega_n t), \qquad ①$$

where

$$\omega_n=\sqrt{(n\pi c/L)^2-(a/2)^2}. \qquad ②$$

Finally,

$$y(x,0)=f(x)=\sum_1^\infty E_n'\sin\frac{n\pi x}{L} \qquad (0<x<L)$$

and

$$y_t(x,0)=0=-\frac{a}{2}\sum_1^\infty E_n'\sin\frac{n\pi x}{L}+\sum_1^\infty \omega_n F_n'\sin\frac{n\pi x}{L}=\sum_1^\infty\left(\omega_n F_n'-\frac{a}{2}E_n'\right)\sin\frac{n\pi x}{L} \quad (0<x<L)$$

Both of these are HRS expansions, so

$$E_n'=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad ③$$

$$\omega_n F_n'-\frac{a}{2}E_n'=0,\text{ or, } F_n'=\frac{a}{2\omega_n}E_n'. \qquad ④$$

The solution is given by ①-④. If we drop the ay_t term in (6.1) (i.e, if we set $a=0$) then the solution simplifies to

$$y(x,t)=\sum_1^\infty E_n'\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\;,\quad E_n'=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Thus, the effects of the ay_t term are threefold:

(i) The oscillation decays, with time, due to the $e^{-at/2}$ factor.

(ii) The frequency of oscillation diminishes from $n\pi c/L$ to $\sqrt{(n\pi c/L)^2-(a/2)^2}$
NOTE: We assumed that the damping is such that $a<2\pi c/L$. That implied that for every vibrational mode (i.e., each product within ①) the damping is subcritical. If we increase a sufficiently, we can get the first N modes to be overdamped (possibly with the Nth mode critically damped), with all the modes $N>n$ underdamped.

(iii) As a increases from 0, F_n' increases in magnitude so that a phase shift is introduced and increases with a.

All three of these results "make sense" physically.

7. $c^2 y_{xx} - by = y_{tt}$. $y = XT$ gives $c^2\frac{X''}{X} - b = \frac{T''}{T}$. Whether we write

$c^2X''/X - b = T''/T = \text{const.} = -k^2$, or $c^2X''/X = T''/T + b = \text{const.} = -k^2$ won't matter. Let us choose $\frac{X''}{X} = \frac{1}{c^2}\left(\frac{T''}{T}+b\right) = -k^2$.

$X''+k^2X=0$, $T''+(k^2c^2+b)T=0$. (Actually, even now we can see that the lateral spring will cause an increase in the modal frequencies of vibration since the temporal frequency $=\sqrt{k^2c^2+\underline{b}}$.) Thus,

$$y(x,t) = \underbrace{(A+Bx)(C\cos\sqrt{b}t + D\sin\sqrt{b}t)}_{k=0} + \underbrace{(E\cos kx + F\sin kx)(G\cos\sqrt{\Gamma}t + H\sin\sqrt{\Gamma}t)}_{k\neq 0}$$

where $\sqrt{\Gamma} \equiv \sqrt{k^2c^2+b^2}$. As in Exercise 6 we expect the $y(0,t)=y(L,t)=0$ bc's to cause B to =0 in which case there's really no need to distinguish the k=0 case. But, let us proceed with the form given above. First, the bc's:

$y(0,t) = A(C\cos\sqrt{b}t + D\sin\sqrt{b}t) + E(G\cos\sqrt{\Gamma}t + H\sin\sqrt{\Gamma}t) \to A=E=0$ so

$y(x,t) = x(C'\cos\sqrt{b}t+D'\sin\sqrt{b}t) + \sin kx(G'\cos\sqrt{\Gamma}t+H'\sin\sqrt{\Gamma}t)$

$y(L,t) = L(\quad " \qquad " \quad) + \sin kL(\quad " \qquad " \quad) \to C'=D'=0, k=n\pi/L$ for $n=1,2,\dots$

Thus the k=0 term does drop out, as we expected, and we have

$$y(x,t)=\sum_1^\infty \sin\frac{n\pi x}{L}(G_n'\cos\omega_n t + H_n'\sin\omega_n t) \qquad ①$$

where

$$\omega_n \equiv \sqrt{(n\pi c/L)^2+b}\,. \qquad ②$$

Finally,

$$y(x,0)=f(x)=\sum_1^\infty G_n'\sin\frac{n\pi x}{L} \quad (0<x<L)$$

and

$$y_t(x,0) = 0 = \sum_1^\infty \omega_n H_n' \sin\frac{n\pi x}{L}. \quad (0<x<L)$$

Both are HRS expansions so

$$G_n' = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad ③$$

$$H_n'=0, \qquad ④$$

and the solution is given by ①-④. We see that the only effect of the lateral spring is to increase the frequencies, from $n\pi c/L$ to $\sqrt{(n\pi c/L)^2+b}$

8. $c^2y_{xx}=y_{tt}+g$. $y(x,t)=XT$ gives $c^2\frac{X''}{X}=\frac{T''}{T}+\frac{g}{XT}$ which can't be separated due to the g/XT term. Thus, let us try again with $y(x,t)=y_p(x)+X(x)T(t)$ where $y_p(x)$ will be a particular solution whose job it will be to handle the g forcing term. That step gives (8.4) and the choice

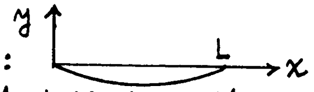
$$c^2y_p''=g\,, \quad \text{so } y_p(x)=\frac{g}{2c}x^2+Ax+B.$$

We could impose the b.c.'s on $y_p(x)$ and then also on the full solution $y(x,t)$ or just wait and impose it on the full solution; it won't matter. Let us impose them immediately on $y_p(x)$:

$$\left.\begin{aligned} y_p(0)&=0=B \\ y_p(L)&=0=gL^2/2c+AL+B \end{aligned}\right\} \text{ gives } B=0,\ A=-gL/2c,$$

so $$y_p(x)=\frac{gx}{2c}(x-L)\,, \qquad ①$$

which, physically, is the static "sag" of the string:



(Actually it will be a catenary (see index) but for small deflections the catenary is approximated as a parabola.) Proceeding,

$$y(x,t)=y_p(x)+(C+Dx)(E+Ft)+(G\cos kx+H\sin kx)(I\cos kct+J\sin kct)$$
$$y(0,t)=0=0+C(E+Ft)+G(I\cos kct+J\sin kct)\to C=G=0$$

so $$y(x,t)=y_p(x)+x(E'+F't)+\sin kx(I'\cos kct+J'\sin kct)$$
$$y(L,t)=0=0+L(\ \text{"}\ \ \text{"}\ )+\sin kL(\ \text{"}\qquad\qquad \text{"}\ )\to E'=F'=0,\ k=n\pi/L$$
for $n=1,2,\dots$

so $$y(x,t)=y_p(x)+\sum_1^\infty \sin\frac{n\pi x}{L}\left(I_n'\cos\frac{n\pi ct}{L}+J_n'\sin\frac{n\pi ct}{L}\right) \qquad ②$$

Finally,

$$y(x,0)=f(x)=y_p(x)+\sum_1^\infty I_n'\sin\frac{n\pi x}{L}\ ,\text{or,}\ f(x)-y_p(x)=\sum_1^\infty I_n'\sin\frac{n\pi x}{L}\quad (0<x<L)$$

and

$$y_t(x,0)=0=0+\sum_1^\infty \frac{n\pi c}{L}J_n'\sin\frac{n\pi x}{L}\,. \quad (0<x<L)$$

These are HRS expansions that give

$$I_n'=\frac{2}{L}\int_0^L[f(x)-y_p(x)]\sin\frac{n\pi x}{L}\,dx \qquad ③$$

$$J_n'=0 \qquad ④$$

so the solution is given by ①-④. Notice that setting $f(x)=0$ does not simplify the solution since, in physical terms, $y(x,0)=f(x)=0$ is a displacement <u>relative to</u> the static sag configuration $y_p(x)$. The only $f(x)$ that does give a simplification would be $y(x,0)=f(x)=y_p(x)=gx(x-L)/2c$ since that would, by ③, give $I_n'=0$ so that $y(x,t)$ simply reduces to

$$y(x,t)=y_p(x).$$

9. (a) Putting (9.2) and (9.3) into (9.1) gives

$$-c^2\sum\left(\frac{n\pi}{L}\right)^2 h_n\sin\frac{n\pi x}{L}=\sum h_n''\sin\frac{n\pi x}{L}+\sum F_n\sin\frac{n\pi x}{L}$$

so, equating coefficients of $\sin n\pi x/L$ harmonics,

$$h_n''+(n\pi c/L)^2h_n=-F_n(t). \qquad ①$$

What are the initial conditions for ①? Well,

$$y(x,0)=f(x)=\sum_1^\infty f_n\sin\frac{n\pi x}{L}=\sum_1^\infty h_n(0)\sin\frac{n\pi x}{L}\longrightarrow h_n(0)=f_n\,,$$

and

$$y_t(x,0) = 0 = \sum_1^\infty h_n'(0) \sin\frac{n\pi x}{L} \rightarrow h_n'(0) = 0$$

Let us solve $h_n'' + (n\pi c/L)^2 h_n = -F_n(t)$; $h_n(0) = f_n$, $h_n'(0) = 0$ by Laplace transform: $s^2\bar{h}_n - sf_n + (n\pi c/L)^2 \bar{h}_n = -\bar{F}_n$
so

$$\bar{h}_n(s) = \frac{s}{s^2+(n\pi c/L)^2} f_n - \bar{F}_n(s)\frac{1}{s^2+(n\pi c/L)^2}$$

$$h_n(t) = f_n \cos\frac{n\pi ct}{L} - F_n(t) * \frac{1}{n\pi c/L}\sin\frac{n\pi ct}{L}$$

$$= f_n \cos\frac{n\pi ct}{L} - \frac{L}{n\pi c}\int_0^t F_n(\tau)\sin\frac{n\pi c}{L}(t-\tau)\,d\tau$$

so $$y(x,t) = \sum_1^\infty \left[f_n \cos\omega_n t + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(\tau-t)\,d\tau\right]\sin\frac{n\pi x}{L}. \quad ①$$

(c) Then (9.4) and (9.5) give $F_n(t) = \frac{2}{L}\int_0^L F_0 \sin\Omega t \sin\frac{n\pi x}{L}dx = \begin{cases}\frac{4F_0}{n\pi}\sin\Omega t, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$

and $f_n = \frac{2}{L}\int_0^L 0\,dx = 0$

so ① gives

$$y(x,t) = \sum_{1,3,\ldots}^\infty \left(\frac{1}{\omega_n}\frac{4F_0}{n\pi}\int_0^t \sin\Omega\tau \sin\omega_n(\tau-t)\,d\tau\right)\sin\frac{n\pi x}{L}$$

$$= \frac{4F_0}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n\omega_n}\frac{\omega_n\sin\Omega t - \Omega\sin\omega_n t}{\Omega^2-\omega_n^2}\sin\frac{n\pi x}{L} \quad ②$$

(d) If $\Omega = \omega_k$ for some odd k then ② has 0/0 as one of its coefficients. We can nevertheless extract the correct result using l'Hôpital's rule:

$$\lim_{\Omega\to\omega_k}\frac{\omega_k\sin\Omega t - \Omega\sin\omega_k t}{\Omega^2-\omega_k^2} = \lim_{\Omega\to\omega_k}\frac{\omega_k t\cos\Omega t - \sin\omega_k t}{2\Omega}$$

$$= \frac{\omega_k t\cos\omega_k t - \sin\omega_k t}{2\omega_k}$$

so ② becomes

$$y(x,t) = \frac{4F_0}{\pi}\sum_{\substack{n=1,3,\ldots\\ n\neq k}}^\infty \frac{1}{n\omega_n}\frac{\omega_n\sin\Omega t - \Omega\sin\omega_n t}{\Omega^2-\omega_n^2}\sin\frac{n\pi x}{L}$$

$$+ \frac{4F_0}{\pi}\frac{1}{k\omega_k}\frac{\omega_k t\cos\omega_k t - \sin\omega_k t}{2\omega_k}\sin\frac{k\pi x}{L}$$

and we see that the kth mode exhibits resonance due to the $t\cos\omega_k t$ term, as is not surprising since the system is being forced at the ω_k natural frequency.

10. $y(x,t) = z(x,t) + (1-\frac{x}{L})p(t) + \frac{x}{L}q(t)$

so $y(0,t) = p(t) = z(0,t) + p(t)$ gives $z(0,t)=0$

$y(L,t) = q(t) = z(L,t) + q(t)$ gives $z(L,t)=0$

$y(x,0) = f(x) = z(x,0) + (1-\frac{x}{L})p(0) + \frac{x}{L}q(0)$ gives $z(x,0) = f(x) - (1-\frac{x}{L})p(0) - \frac{x}{L}q(0)$

$y_t(x,0) = 0 = z_t(x,0) + (1-\frac{x}{L})p'(0) + \frac{x}{L}q'(0)$ gives $z_t(x,0) = -(1-\frac{x}{L})p'(0) - \frac{x}{L}q'(0)$

$y_{xx}(x,t) = z_{xx}(x,t)$

$y_{tt}(x,t) = z_{tt}(x,t) + (1-\frac{x}{L})p''(t) + \frac{x}{L}q''(t)$

so the z problem is

$$c^2 z_{xx} = z_{tt} + \overset{F(x,t)}{\left[(1-\frac{x}{L})p''(t) + \frac{x}{L}q''(t)\right]}$$

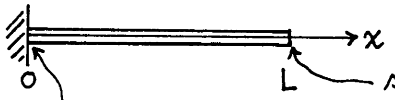
$$z(0,t)=0, \quad z(L,t)=0$$
$$z(x,0) = f(x) - (1-\frac{x}{L})p(0) - \frac{x}{L}q(0)$$
$$z_t(x,0) = -(1-\frac{x}{L})p'(0) - \frac{x}{L}q'(0)$$

which problem is of the form of (9.1) in Exercise 9 and can therefore be solved by the method of Exercise 9. (Actually, we took $y_t(x,0)=0$ in Exercise 9, but we could just as well have used $y_t(x,0)=g(x)$ with hardly any complication. In fact, in that case we would have $h_n'(0)=g_n$ instead of $h_n'(0)=0$, and

$$y(x,t) = \sum_1^\infty \left[f_n \cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(\tau - t)\,d\tau\right]\sin\frac{n\pi x}{L}$$

11. That is a major misuse of the idea of superposition, which involves the superposition of the responses to various inputs. There is no basis for superimposing different parts of the operator.

12. (a)



stress-free, so $S(L,t) = E\frac{\partial u(L,t)}{\partial x} = 0$, so $u_x(L,t)=0$

no displacement, so $u(0,t)=0$

Initially, $S(x,0) = S_0 = F_0/A = E u_x(x,0)$

Itegrating the latter on x, from 0 to x, gives

$$S_0 x = E u(x,0) - E\underset{\rightarrow 0}{u(0,0)} \quad \text{so} \quad u(x,0) = \frac{S_0}{E}x. \quad u_t(x,0)=0$$

Because initially the rod is in static equilibrium, it is stationary

(b) $u(x,t) = (B + Cx)(G + Ht) + (I\cos kx + J\sin kx)(M\cos kct + N\sin kct)$

$u(0,t)=0 \Rightarrow B=I=0$ so

$u(x,t) = x(G'+H't) + \sin kx(M'\cos kct + N'\sin kct)$

$u_x(L,t)=0 = G'+H't + k\cos kL($ " " $) \rightarrow G'=H'=0,\ k=n\pi/2L$ (n odd)

so $u(x,t) = \sum_{1,3,\dots}^{\infty} \sin\frac{n\pi x}{2L}\left(M_n'\cos\frac{n\pi ct}{2L} + N_n'\sin\frac{n\pi ct}{2L}\right)$

Initial conditions:

$$u(x,0) = \frac{S_0}{E}x = \sum_{1,3,\dots}^{\infty} M_n' \sin\frac{n\pi x}{2L} \qquad (0<x<L)$$

$$u_t(x,0) = 0 = \sum_{1,3,\dots}^{\infty} N_n' \frac{n\pi c}{2L}\sin\frac{n\pi x}{2L}. \qquad (0<x<L)$$

Both are QRS expansions, so

$$M_n' = \frac{2}{L}\int_0^L \frac{S_0}{E}x\sin\frac{n\pi x}{2L}\,dx = \frac{4S_0L}{n^2\pi^2E}\left(2\sin\frac{n\pi}{2} - n\pi\cancelto{0 \text{ for } n \text{ odd}}{\cos\frac{n\pi}{2}}\right)$$

$$N_n' = 0$$

so

$$u(x,t) = \frac{8S_0L}{\pi^2E}\sum_{1,3,\dots}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin\frac{n\pi x}{2L}\cos\frac{n\pi ct}{2L}$$

(c) $$S(0,t) = Eu_x(0,t) = \frac{4S_0}{\pi}\sum_{1,3,\dots}^{\infty}\frac{\sin(n\pi/2)}{n}\cos\frac{n\pi ct}{2L}$$

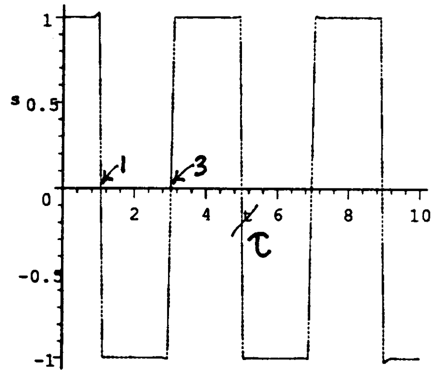
Although it wasn't asked for, let us plot $S(0,t)$ versus $ct/L \equiv \tau$. Since the cosines are $\cos \pi\tau/2, \cos 3\pi\tau/2, \dots$ we see that the fundamental period T is defined by $\pi T/2 = 2\pi$, i.e., $T=4$. Let us plot a couple of periods, say from $\tau=0$ to $\tau=10$. Now,

$$S(0,t) = S_0\frac{4}{\pi}\sum_{1}^{\infty}\frac{\sin((2n-1)\pi/2)}{2n-1}\cos\frac{(2n-1)\pi\tau}{2}$$

so let us use the Maple commands

stands for tau

```
> with(plots):
> f:=sum(sin((2*i-1)*Pi/2)*cos((2*i-1)*Pi*t/2)/(2*i-1),i=1..500):
> implicitplot(s=4*f/Pi,t=0..10,s=-2..2,numpoints=5000);
```



Actually, we omitted the S_0 factor, so instead of S varying between +1 and -1 it varies between $+S_0$ and $-S_0$; i.e., scale the ordinate by S_0. We see that $S(0,t)$ is a square wave [as one could have predicted analytically, with the help of $(30)^0$].

13. $$\frac{dI}{dt} = \frac{d}{dt}\int_0^L (w_t^2 + c^2 w_x^2)\,dx = \int_0^L (2w_t w_{tt} + 2c^2 w_x w_{xt})\,dx \quad \text{by Leibniz}$$
$$= 2c^2\int_0^L (w_t w_{xx} + w_x w_{xt})\,dx \quad \text{by PDE } c^2 w_{xx} = w_{tt}$$
$$= 2c^2 \int_0^L \frac{\partial}{\partial x}(w_t w_x)\,dx$$
$$= 2c^2 w_t w_x \Big|_0^L = 0 \quad \text{because } w(0,t)=0 \Rightarrow w_t(0,t)=0 \text{ and, similarly, } w_t(L,t)=0$$

Thus, $E(t) =$ constant. Further, $w(x,0)=0 \Rightarrow w_x(x,0)=0$, so
$$E(0) = \int_0^L [\underbrace{w_t(x,0)}_{0 \text{ by initial condition}}{}^2 + c^2 \overset{0}{\cancel{w_x(x,0)}}{}^2]\,dx = 0, \quad \text{so } E(t) = \text{const.} = 0$$

Now, $E(t) = \int_0^L (w_t^2 + c^2 w_x^2)\,dx = 0 \Rightarrow w_t(x,t)=0$ and $w_x(x,t)=0$ because the integrand is the sum of squares. These imply that $w(x,t) =$ constant, which constant must be zero by any of the conditions $w(0,t)=0$, $w(L,t)=0$, $w(x,0)=0$. $w(x,t) = y_1(x,t) - y_2(x,t) = 0$ implies $y_1(x,t) = y_2(x,t)$ for any solutions y_1 and y_2. Hence, the solution is unique.
NOTE: This argument does not prove existence (i.e., that a solution exists); it shows that if one does exist then it is unique. But generally we can demonstrate existence by actually finding a solution. Note also how each of these was used: the PDE $c^2 w_{xx} = w_{tt}$, the b.c.'s $w(0,t)=0$, $w(L,t)=0$ and the initial conditions $w(x,0)=0$, $w_t(x,0)=0$.

14. (a) $y = XT$ gives $$\frac{X''''}{X} = -\frac{1}{EI/\sigma}\frac{T''}{T} = +k^4,$$ the + sign so that T is oscillatory

Let us call $EI/\sigma \equiv \alpha^4$

and the 4th power because we will soon take 1/4 roots of it. Thus, the PDE was indeed separable.
$$X'''' - k^4 X = 0, \quad X = \begin{cases} B\cos kx + C\sin kx + D\cosh kx + E\sinh kx, & k \neq 0 \\ F + Gx + Hx^2 + Ix^3, & k = 0 \end{cases}$$
$$T'' + k^4\alpha^4 T = 0, \quad T = \begin{cases} J\cos k^2\alpha^2 t + M\sin k^2\alpha^2 t, & k \neq 0 \\ P + Qt, & k = 0 \end{cases}$$
We don't expect the "extra" terms Gx, Hx^2, Ix^3, Qt to survive, but let us retain them (i.e., let us distinguish the $k=0$ case) just in case. Thus,
$$y(x,t) = (F + Gx + Hx^2 + Ix^3)(P + Qt)$$
$$+ (B\cos kx + C\sin kx + D\cosh kx + E\sinh kx)(J\cos k^2\alpha^2 t + M\sin k^2\alpha^2 t)$$
$$y(0,t) = 0 = F(P + Qt) + (B + D)(J\cos k^2\alpha^2 t + M\sin k^2\alpha^2 t) \rightarrow F = 0,\ B + D = 0$$
$$y_x(0,t) = 0 = G(\ \text{"}\ ) + (kC + kE)(\quad \text{"} \qquad \text{"} \quad) \rightarrow G = 0,\ C + E = 0$$
Thus far,
$$y(x,t) = (Hx^2 + Ix^3)(P + Qt)$$
$$+ [B(\cos kx - \cosh kx) + C(\sin kx - \sinh kx)](J\cos k^2\alpha^2 t + M\sin k^2\alpha^2 t).$$

Normally, in Chapters 18 and 19, we apply the b.c.'s before the initial conditions, but let us jump, nevertheless, to the condition $y_t(x,0)=0$ because it will knock more terms out.

$$y_t(x,0)=0=(Hx^2+Ix^3)Q+[B(\cos Kx-\cosh Kx)+C(\sin Kx-\sinh Kx)]K^2\alpha^2 M$$

implies $Q=0$ and $M=0$. Letting HP be H', IP be I', BJ be B', and CJ be C', we have

$$y(x,t)=H'x^2+I'x^3+[B'(\cos Kx-\cosh Kx)+C'(\sin Kx-\sinh Kx)]\cos K^2\alpha^2 t.$$

Now,

$$y_{xx}(L,t)=0=2H'+6I'L+K^2[-B'(\cos KL+\cosh KL)-C'(\sin KL+\sinh KL)]\cos K^2\alpha^2 t$$
$$y_{xxx}(L,t)=0=\quad 6I'+K^3[B'(\sin KL-\sinh KL)-C'(\cos KL+\cosh KL)]\cos K^2\alpha^2 t$$

implies $\left.\begin{matrix}2H'+6I'L=0\\ 6I'=0\end{matrix}\right\}\rightarrow H'=I'=0$

and $$\begin{pmatrix}\cos KL+\cosh KL & \sin KL+\sinh KL\\ -\sin KL+\sinh KL & \cos KL+\cosh KL\end{pmatrix}\begin{pmatrix}B'\\ C'\end{pmatrix}=\underset{\sim}{0} \qquad ①$$

implies

$$\begin{vmatrix}\cos KL+\cosh KL & \sin KL+\sinh KL\\ -\sin KL+\sinh KL & \cos KL+\cosh KL\end{vmatrix}=\cosh^2 KL-\sinh^2 KL+2\cos KL\cosh KL+1$$
$$=2(\cos KL\cosh KL+1)=0$$

so the characteristic equation for the K's is

$$\cos KL\cosh KL+1=0. \qquad ②$$

Denote the roots of $\cos z\cosh z+1=0$ as z_n. Then $K=z_n/L$. Now put that into ① and solve ① for the nontrivial solutions for B', C' that are thereby guaranteed. The two scalar equations in ① are then redundant so we can discard the first, say, and solve the second for B' in terms of C':

$$B'=\frac{\cos z_n+\cosh z_n}{\sin z_n-\sinh z_n}C'.$$

With these results,

$$y(x,t)=\sum_1^\infty C_n'\underbrace{\left[\sin\frac{z_n x}{L}-\sinh\frac{z_n x}{L}+\left(\frac{\cos z_n+\cosh z_n}{\sin z_n-\sinh z_n}\right)\left(\cos\frac{z_n x}{L}-\cosh\frac{z_n x}{L}\right)\right]}_{X_n(x)}\cos\omega_n t,$$

where the z_n's are the (presumably countably infinite) roots of

$$\cos z\cosh z+1=0$$

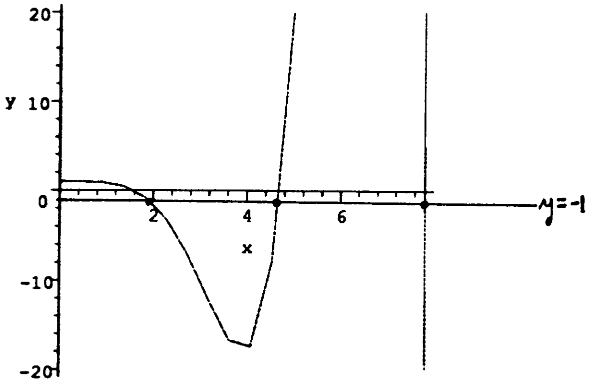
and $$\omega_n=\frac{z_n^2}{L^2}\alpha^2=\sqrt{\frac{EI}{\sigma}}\left(\frac{z_n}{L}\right)^2.$$

NOTE: As an additional computer assignment one might ask the student to determine the first few eigenfrequencies and graphs of the corresponding mode shapes. Using Maple,

First, let us plot cosx coshx to see where it = -1 (and let us proceed under the assumption that all roots are to be found on the real axis):

```
> with(plots):
> implicitplot(y=cos(x)*cosh(x),x=0..10,y=-20..20,numpoints=500);
```

The first 3 roots are ≈ 1.9, 4.6, 7.8, so now we can find those roots with suitable search intervals.



The roots:

```
> z1:=fsolve(cos(x)*cosh(x)=-1,x=0..2);
```

$z1 := 1.875104069$ so $\omega_1 = \sqrt{\frac{EI}{\sigma}}\left(\frac{1.875104069}{L}\right)^2$

```
> z2:=fsolve(cos(x)*cosh(x)=-1,x=4..5);
```

$z2 := 4.694091133$ so $\omega_2 = \sqrt{\frac{EI}{\sigma}}\left(\frac{4.694091133}{L}\right)^2$

```
> z3:=fsolve(cos(x)*cosh(x)=-1,x=7..8);
```

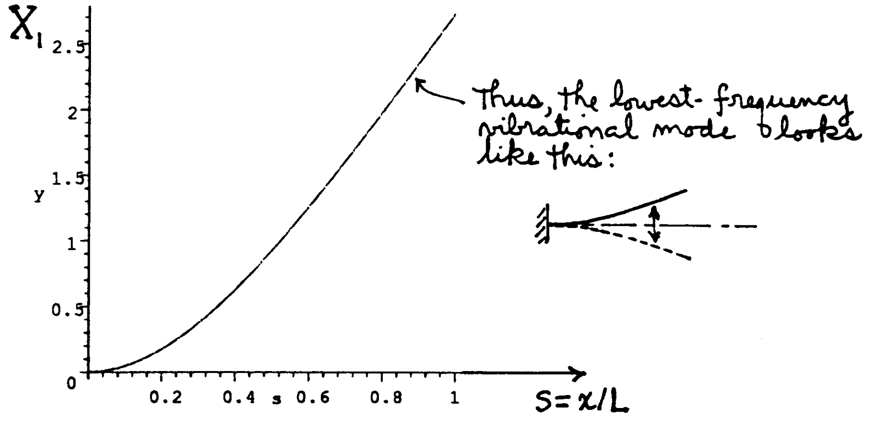
$z3 := 7.854757438$ so $\omega_3 = \sqrt{\frac{EI}{\sigma}}\left(\frac{7.854757438}{L}\right)^2$

Now plot X_1 versus $s = x/L$, from $s = 0$ to $s = 1$:

```
> X1:=sin(z1*s)-sinh(z1*s)+(cos(z1)+cosh(z1))/(sin(z1)-sinh(z1))*(co
  s(z1*s)-cosh(z1*s));
```

$$X1 := \sin(1.875104069\,s) - \sinh(1.875104069\,s) - 1.362220557\cos(1.875104069\,s) + 1.362220557\cosh(1.875104069\,s)$$

```
> implicitplot(y=X1,s=0..1,y=-10..10);
```



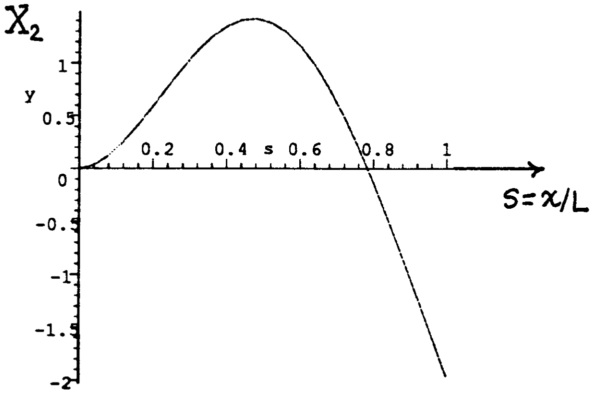
Thus, the lowest-frequency vibrational mode looks like this:

Similarly for X_2 and X_3:

```
> X2:=sin(z2*s)-sinh(z2*s)+(cos(z2)+cosh(z2))/(sin(z2)-sinh(z2))*(co
  s(z2*s)-cosh(z2*s));
```

$$X2 := \sin(4.694091133\, s) - \sinh(4.694091133\, s) - .9818675391 \cos(4.694091133\, s)$$
$$+ .9818675391 \cosh(4.694091133\, s)$$

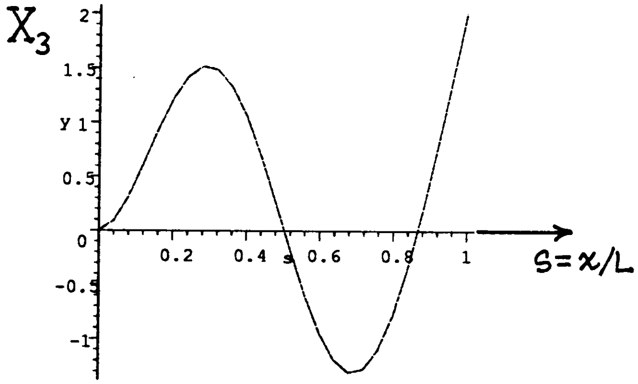
```
> implicitplot(y=X2,s=0..1,y=-10..10);
```



```
> X3:=sin(z3*s)-sinh(z3*s)+(cos(z3)+cosh(z3))/(sin(z3)-sinh(z3))*(co
  s(z3*s)-cosh(z3*s));
```

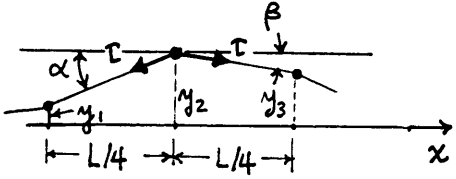
$$X3 := \sin(7.854757438\, s) - \sinh(7.854757438\, s) - 1.000776106 \cos(7.854757438\, s)$$
$$+ 1.000776106 \cosh(7.854757438\, s)$$

```
> implicitplot(y=X3,s=0..1,y=-10..10);
```



15. The masses are $\sigma L/4$, so Newton's 2nd law (in y direction) gives, for the 2nd mass, say,

$$(\sigma L/4)\ddot{y}_2 = -\tau \sin\alpha - \tau \sin\beta$$
$$\approx -\tau \tan\alpha - \tau \tan\beta$$
$$= -\tau \frac{y_2 - y_1}{L/4} - \tau \frac{y_2 - y_3}{L/4}$$



so

$$\ddot{y}_2 + \left(\frac{16\tau}{\sigma L^2}\right)(-y_1 + 2y_2 - y_3) = 0$$

Similarly for the other masses, yielding (15.2,3).

(b) Putting $y(t)=\underset{\sim}{\eta}\sin(\omega t+\phi)$ into $\ddot{\underset{\sim}{y}}+\underset{\sim}{A}\underset{\sim}{y}=\underset{\sim}{0}$ gives $-\omega^2\underset{\sim}{\eta}\cancel{\sin(\omega t+\phi)}+\underset{\sim}{A}\underset{\sim}{\eta}\cancel{\sin(\omega t)}=\underset{\sim}{0}$ so $\underset{\sim}{A}\underset{\sim}{\eta}=\omega^2\underset{\sim}{\eta}$ or,

$$\underset{\sim}{B}\underset{\sim}{\eta}=\lambda\underset{\sim}{\eta}$$

where $\lambda=(\sigma/\tau)(L/N)^2\omega^2$.

(c)
```
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> B:=array([[3,-1,0,0],[-1,2,-1,0],[0,-1,2,-1],[0,0,-1,3]]);
```

$$B:=\begin{bmatrix}3&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&3\end{bmatrix}$$

```
> eigenvects(B);
```

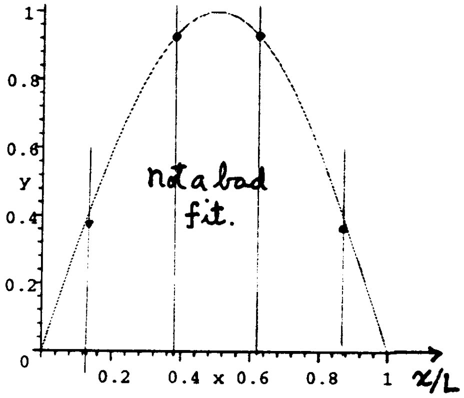
$$[4,1,\{[-1,1,-1,1]\}],[2+\sqrt{2},1,\{[1,1-\sqrt{2},1-\sqrt{2},1]\}],$$
$$[2-\sqrt{2},1,\{[1,1+\sqrt{2},1+\sqrt{2},1]\}],[2,1,\{[-1,-1,1,1]\}]$$

so $\lambda_1=2-\sqrt{2}=.5858$, $\lambda_2=2$, $\lambda_3=2+\sqrt{2}=3.4142$, $\lambda_4=4$. Thus, we have the following comparison of the eigenfrequencies – from the lumped-parameter model and from the continuous model:

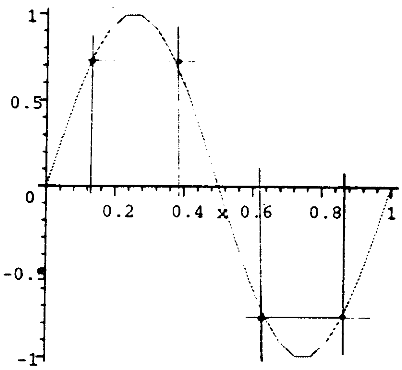
Lumped-Parameter	Continuous	
$\omega_1=\frac{4}{L}\sqrt{\frac{\tau}{\sigma}}\sqrt{.5858}=\frac{3.061}{L}\sqrt{\frac{\tau}{\sigma}}$	$\omega_1=\frac{\pi}{L}\sqrt{\frac{\tau}{\sigma}}=\frac{3.142}{L}\sqrt{\frac{\tau}{\sigma}}$	Infinite number of eigenfrequencies
$\omega_2=\frac{4}{L}\sqrt{\frac{\tau}{\sigma}}\sqrt{2}=\frac{5.657}{L}\sqrt{\frac{\tau}{\sigma}}$	$\omega_2=\frac{2\pi}{L}\sqrt{\frac{\tau}{\sigma}}=\frac{6.283}{L}\sqrt{\frac{\tau}{\sigma}}$	
$\omega_3=\frac{4}{L}\sqrt{\frac{\tau}{\sigma}}\sqrt{3.4142}=\frac{7.391}{L}\sqrt{\frac{\tau}{\sigma}}$	$\omega_3=\frac{3\pi}{L}\sqrt{\frac{\tau}{\sigma}}=\frac{9.425}{L}\sqrt{\frac{\tau}{\sigma}}$	
$\omega_4=\frac{4}{L}\sqrt{\frac{\tau}{\sigma}}\sqrt{4}=\frac{8}{L}\sqrt{\frac{\tau}{\sigma}}$	$\omega_4=\frac{4\pi}{L}\sqrt{\frac{\tau}{\sigma}}=\frac{12.566}{L}\sqrt{\frac{\tau}{\sigma}}$	
	$\omega_5=\frac{5\pi}{L}\sqrt{\frac{\tau}{\sigma}}=\frac{15.708}{L}\sqrt{\frac{\tau}{\sigma}}$	
	$\vdots$	

We see that with $N=4$ we obtain approximations to the first 4 eigenfrequencies. The first is only a few percent off, the second is worse, the third is still worse, and the fourth is the worst. The reason the approximations grow worse is that the mode shape approximations (by the eigenvectors) inevitably grow worse, as we shall see below, where we plot the eigenvector $\underset{\sim}{e}_1$ with $\sin(\pi x/L)$, $\underset{\sim}{e}_2$ with $\sin(2\pi x/L)$, etc. Note that we are comparing shapes not amplitudes, so we will scale the $\underset{\sim}{e}_j$'s to get the nicest (in an "eyeball" sense) possible match.

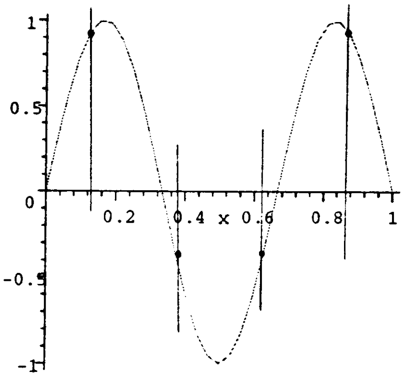
$\underset{\sim}{e}_1 = \alpha(1, 2.414, 2.414, 1)$
I'll choose α so the middle two values are exact, say.



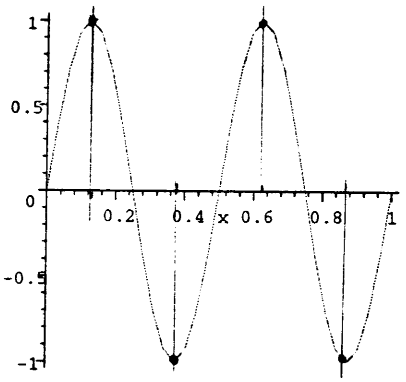
$\underset{\sim}{e}_2 = \alpha(-1, -1, 1, 1)$
I'll chose α so the first and fourth are exact, say.



$\underset{\sim}{e}_3 = \alpha(1, -.414, -.414, 1)$
I'll choose α so the first and fourth are exact, say.



$\underset{\sim}{e}_4 = \alpha(-1, 1, -1, 1)$
so $\alpha \approx -1$



(d) With N=10 we obtain these results, using Maple as above. (Actually, to obtain just a list of the eigenvalues, say to 6 places, we can use the command

```
evalf(eigenvals(B), 6);
```

Even if we use the eigenvects command, the results, being exact, are in an awkward form, such as $2-\frac{1}{2}\sqrt{10+2\sqrt{5}}$ rather than 0.0978870. Thus, we suggest

```
evalf(eigenvects(B), 6);
```

rather than just eigenvects(B);. Our results are as follows.

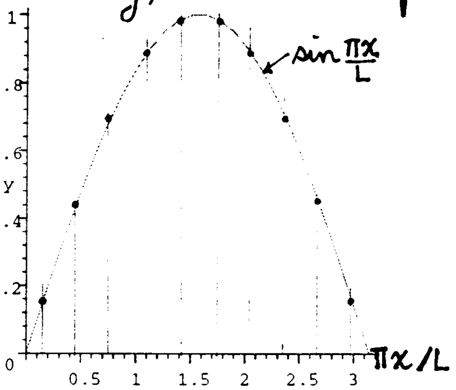
```
> B:=array([[3,-1,0,0,0,0,0,0,0,0],[-1,2,-1,0,0,0,0,0,0,0],[0,-1,2,-
  1,0,0,0,0,0,0],[0,0,-1,2,-1,0,0,0,0,0],[0,0,0,-1,2,-1,0,0,0,0],[0,
  0,0,0,-1,2,-1,0,0,0],[0,0,0,0,0,-1,2,-1,0,0],[0,0,0,0,0,0,-1,2,-1,
  0],[0,0,0,0,0,0,0,-1,2,-1],[0,0,0,0,0,0,0,0,-1,3]]);
```

$$B := \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

```
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> evalf(eigenvects(B),4);
[3.618, 1., {[-1., .6180, 0, -.6180, 1., -1., .6180, 0, -.6180, 1.]}],
    [1.382, 1., {[-1., -1.618, 0, 1.618, 1., -1., -1.618, 0, 1.618, 1.]}],
    [3.902, 1., {[1., -.902, .720, -.44, .15, .15, -.44, .720, -.902, 1.]}],
    [.098, 1., {[1., 2.902, 4.520, 5.70, 6.32, 6.32, 5.70, 4.520, 2.902, 1.]}],
    [3.176, 1., {[1., -.176, -.788, 1.13, -.48, -.48, 1.13, -.788, -.176, 1.]}],
    [.824, 1., {[1., 2.176, 1.557, -.35, -1.97, -1.97, -.35, 1.557, 2.176, 1.]}],
    [2.618, 1., {[-1., -.382, 1.236, -.382, -1., 1., .382, -1.236, .382, 1.]}],
    [.382, 1., {[-1., -2.618, -3.236, -2.618, -1., 1., 2.618, 3.236, 2.618, 1.]}],
    [4., 1., {[-1., 1., -1., 1., -1., 1., -1., 1., -1., 1.]}], [2., 1., {[1., 1., -1., -1., 1., 1., -1., -1., 1., 1.]}]
```

<u>Lumped Parameter</u>	<u>Continuous</u>
$\omega_1 = (10/L)\sqrt{\tau/\sigma}\sqrt{.0979} = 3.13\sqrt{\tau/\sigma}/L$	$\omega_1 = \pi\sqrt{\tau/\sigma}/L = 3.14\sqrt{\tau/\sigma}/L$
$\omega_2 = (10/L)\sqrt{\tau/\sigma}\sqrt{.382} = 6.18\sqrt{\tau/\sigma}/L$	$\omega_2 = 2\pi\sqrt{\tau/\sigma}/L = 6.28\sqrt{\tau/\sigma}/L$
$\omega_3 = (10/L)\sqrt{\tau/\sigma}\sqrt{.824} = 9.08\sqrt{\tau/\sigma}/L$	$\omega_3 = 3\pi\sqrt{\tau/\sigma}/L = 9.42\sqrt{\tau/\sigma}/L$
$\omega_4 = \cdots \sqrt{1.382} = 11.76$ "	$\omega_4 = \cdots = 12.57$ "
$\omega_5 = \cdots \sqrt{2} = 14.14$ "	$\omega_5 = \cdots = 15.71$ "
$\omega_6 = \cdots \sqrt{2.618} = 16.18$ "	$\omega_6 = \cdots = 18.85$ "
$\omega_7 = \cdots \sqrt{3.176} = 17.82$ "	$\omega_7 = \cdots = 21.99$ "
$\omega_8 = \cdots \sqrt{3.618} = 19.02$ "	$\omega_8 = \cdots = 25.13$ "
$\omega_9 = \cdots \sqrt{3.902} = 19.75$ "	$\omega_9 = \cdots = 28.27$ "
$\omega_{10} = \cdots \sqrt{4} = 20.00$ "	$\omega_{10} = \cdots = 31.42$ "
	$\omega_{11} = \cdots$ etc

We see that the lumped-parameter model gives a good approximation for the low modes (since the 10 beads give a good approximation to the continuous string for the low frequency mode shapes, but a poorer and poorer one for the higher modes). Finally, let us also compare the eigenvector with the exact mode shape, but only for the first mode.

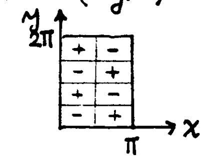


Section 19.3

1. Let us go right to (16a), for brevity: $w(x,y,t)=\sum\sum H_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t$

$\omega_{mn}=\pi c\sqrt{(m/a)^2+(n/b)^2}$

(a) $w(x,y,0)=f(x,y)=8\sin 2x\sin 2y=\sum\sum H_{mn}\sin mx\sin(ny/2)$ $(a=\pi,\ b=2\pi)$
so $H_{24}=8$, all other H_{mn}'s $=0$, and
$w(x,y,t)=8\sin 2x\sin 2y\cos 2\sqrt{2}t$.
Nodal lines are $x=\pi/2$, $y=\pi/2,\pi,3\pi/2$



(b) $w(x,y,0)=f(x,y)=-5\sin x\sin 4y=\sum\sum H_{mn}\sin(mx/2)\sin ny$ $(a=2\pi,\ b=\pi)$
so $H_{24}=-5$, all others $=0$, so
$w(x,y,t)=-5\sin x\sin 4y\cos\sqrt{17}t$.
Nodal lines are $x=\pi$, $y=\pi/4,\pi/2,3\pi/4$.

(c) $w(x,y,0)=f(x,y)=\sin 3x\sin y-\sin x\sin 3y=\sum\sum H_{mn}\sin mx\sin ny$ $(a=b=\pi)$
so $H_{31}=1$, $H_{13}=-1$, all others $=0$, and
$w(x,y,t)=\sin 3x\sin y\cos\sqrt{10}t-\sin x\sin 3y\cos\sqrt{10}t$
$=(\sin 3x\sin y-\sin x\sin 3y)\cos\sqrt{10}t$
For the nodal curve(s) we set $\sin 3x\sin y-\sin x\sin 3y=0$, ①
or, $\frac{\sin 3x}{\sin x}=\frac{\sin 3y}{\sin y}$, $\frac{3\sin x-4\sin^3 x}{\sin x}=\frac{3\sin y-4\sin^3 y}{\sin y}$,

$3-4\sin^2 x=3-4\sin^2 y$, so $\sin x=\pm\sin y=\sin(\pm y)$.

From the graph we see that $\sin A=\sin B$ implies that either B and A are a multiple of 2π apart or A and B average to an odd multiple of π/2:

$$\left.\begin{aligned}&B=A+2k\pi\\ \text{or }&(B+A)/2=(2k+1)\pi/2\end{aligned}\right\}②$$

where $k=0,\pm1,\pm2,\dots$. Putting $B=\pm y$ and $A=x$ into ② gives

$$\pm y = x + 2k\pi$$
$$\text{or } \pm y = -x + (2k+1)\pi$$

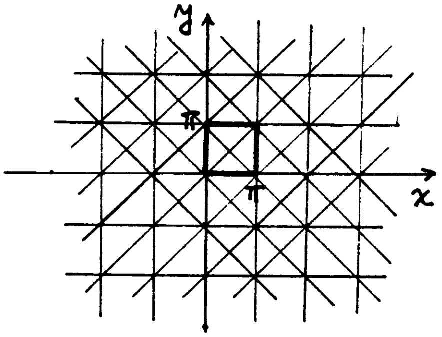
Together, these are equivalent to

$$y = \pm x + p\pi \quad (p = 0, \pm 1, \pm 2, \ldots) \qquad ③$$

However, note further that our division of ① by $\sin x$ and $\sin y$ is justified only if $\sin x \neq 0$ and $\sin y \neq 0$. In fact, if $\sin x = 0$ or $\sin y = 0$ then ① is satisfied. Thus, besides the nodal lines ③ we also have these:

$$x = 0, \pm\pi, \pm 2\pi, \ldots \qquad ④$$
$$\text{and} \quad y = 0, \pm\pi, \pm 2\pi, \ldots . \qquad ⑤$$

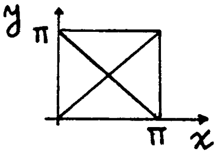
The question, now, is: how many of the lines ③-⑤ fall within the given domain. The answer is: just the lines $y = x$ and $y = \pi - x$, as seen at the right.



If, instead, we chose to use Maple, then the commands

```
with(plots):
implicitplot(sin(3*x)*sin(y)
 - sin(x)*sin(3*y)=0, x=0..Pi,
 y=0..Pi, grid=[50,100]);
```

would give this plot:



NOTE: Below ① we used the identity $\sin 3A = 3\sin A - 4\sin^3 A$. The latter is but one of a whole family of identities that are often useful, namely:

$\sin^2 A = \frac{1}{2}(-\cos 2A + 1).$

$\sin^3 A = \frac{1}{4}(-\sin 3A + 3\sin A).$

$\sin^4 A = \frac{1}{8}(\cos 4A - 4\cos 2A + \frac{6}{2}).$

$\sin^5 A = \frac{1}{16}(\sin 5A - 5\sin 3A + 10\sin A).$

$\sin^6 A = \frac{1}{32}(-\cos 6A + 6\cos 4A - 15\cos 2A + \frac{20}{2}).$

$\sin^7 A = \frac{1}{64}(-\sin 7A + 7\sin 5A - 21\sin 3A + 35\sin A).$

$\vdots$

$\cos^2 A = \frac{1}{2}(\cos 2A + 1).$

$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A).$

$\cos^4 A = \frac{1}{8}(\cos 4A + 4\cos 2A + \frac{6}{2}).$

$\cos^5 A = \frac{1}{16}(\cos 5A + 5\cos 3A + 10\cos A).$

$\cos^6 A = \frac{1}{32}(\cos 6A + 6\cos 4A + 15\cos 2A + \frac{20}{2}).$

$\cos^7 A = \frac{1}{64}(\cos 7A + 7\cos 5A + 21\cos 3A + 35\cos A).$

$\vdots$

In the reverse direction we can express

$\sin 2A = 2\sin A\cos A$

$\sin 3A = 3\sin A - 4\sin^3 A.$

$\sin 4A = \cos A(4\sin A - 8\sin^3 A).$

$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A.$

$\sin 6A = \cos A(6\sin A - 32\sin^3 A + 32\sin^5 A).$

$\sin 7A = 7\sin A - 56\sin^3 A + 112\sin^5 A - 64\sin^7 A.$

$\vdots$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 3A = 4\cos^3 A - 3\cos A.$

$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1.$

$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A.$

$\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1.$

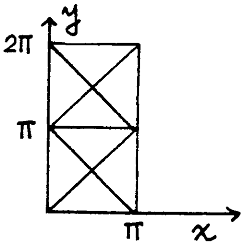
$\cos 7A = 64\cos^7 A - 112\cos^5 A + 56\cos^3 A - 7\cos A.$

$\vdots$

(d) $w(x,y,0)=f(x,y)=\sin 3x\sin y-\sin x\sin 3y=\sum\sum H_{mn}\sin mx\sin(ny/2)$ (for $a=\pi, b=2\pi$), so $H_{32}=1, H_{16}=-1$, all others $=0$, and

$$w(x,y,t)=\sin 3x\sin y\cos\sqrt{10}t-\sin x\sin 3y\cos\sqrt{10}t$$
$$=(\sin 3x\sin y-\sin x\sin 3y)\cos\sqrt{10}t$$

so the nodal lines are the same as in part (c), above. But since the domain is now $0<x<\pi$, $0<y<2\pi$, we find, within that domain, the nodal lines shown at the right.



(e) $w(x,y,0)=f(x,y)=1.05\sin 3x\sin y-\sin x\sin 3y=\sum\sum H_{mn}\sin mx\sin(ny/2)$ (for $a=\pi, b=2\pi$), so $H_{32}=1.05, H_{16}=-1$, all others $=0$, and

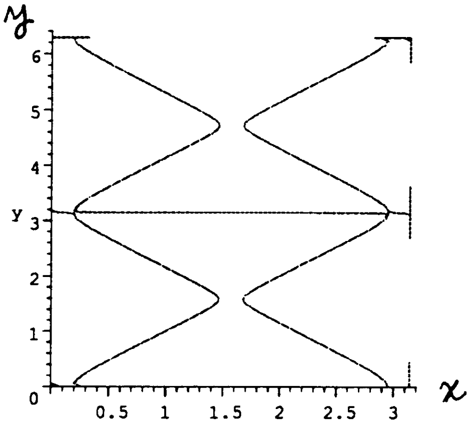
$$w(x,y,t)=1.05\sin 3x\sin y\cos\sqrt{10}t-\sin x\sin 3y\cos\sqrt{10}t$$
$$=(1.05\sin 3x\sin y-\sin x\sin 3y)\cos\sqrt{10}t$$

and the nodal lines are given implicitly by

$$1.05\sin 3x\sin y-\sin x\sin 3y=0.$$

Maple:

```
> with(plots):
> implicitplot(1.05*sin(3*x)*sin(y)=sin(x)*sin(3*y),x=0..Pi,y=0..2*P
  i,grid=[50,100]);
```



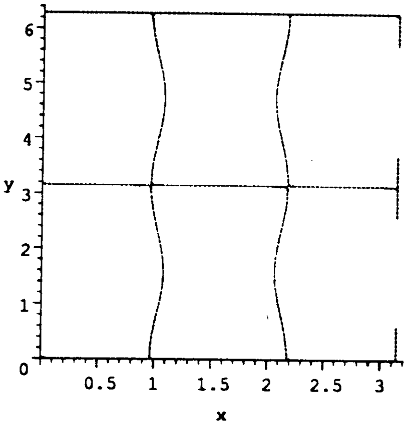
(f) Same as in (e) with the 1.05 changed to 10. The resulting nodal pattern is shown at the right. The results of (e) and (f) make sense since if we let $A\to\infty$ in

$$w(x,y,t)=(A\sin 3x\sin y-\sin x\sin 3y)\cos\sqrt{10}t$$

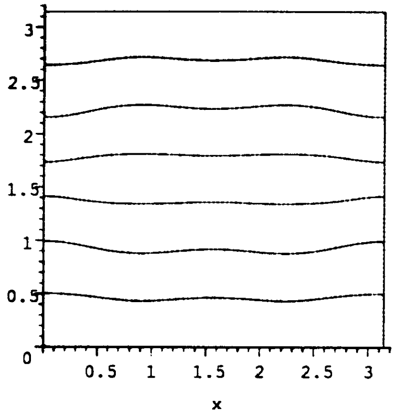
then

$$w(x,y,t)\sim A\sin 3x\sin y\cos\sqrt{10}t,$$

with the nodal lines $x=\pi/3, 2\pi/3$ and $y=\pi$.



(g) $w(x,y,t) = (8\sin x \sin 7y + \sin 5x \sin 5y)\cos\sqrt{50}\,t$ with the nodal pattern defined implicitly by $8\sin x \sin 7y + \sin 5x \sin 5y = 0$, as shown at the right.



(h) $w(x,y,0) = f(x,y) = 8\sin 2x \sin 7y + \sin 5x \sin 5y = \sum\sum H_{mn} \sin mx \sin ny \quad (a=b=\pi)$
so $H_{27} = 8$, $H_{55} = 1$, all others $= 0$, so

$$w(x,y,t) = 8\sin 2x \sin 7y \cos\sqrt{53}\,t + \sin 5x \sin 5y \cos\sqrt{50}\,t.$$

Since $\cos\sqrt{53}\,t$ and $\cos\sqrt{50}\,t$ are linearly independent, $w=0$ for all t requires that each coefficient is zero:

$$\sin 2x \sin 7y = 0 \quad ①$$
$$\sin 5x \sin 5y = 0 \quad ②$$

The solution set of ① is the set of lines

$$x = \frac{\pi}{2}, \; y = \frac{\pi}{7}, \frac{2\pi}{7}, \dots, \frac{5\pi}{7}, \frac{6\pi}{7} \qquad ③$$

(plus others that do not pass through the domain) and the solution set of ② is the set of lines

$$x = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \quad y = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5} \qquad ④$$

The solution sets ③ and ④ are disjoint, so there are no nodal lines in the given domain, as can also be found using Maple:

```
with(plots):
implicitplot((sin(2*x)*sin(7*y))^2+(sin(5*x)*
    sin(5*y))^2=0, x=0..Pi, y=0..Pi, grid=[100,100]);
```

4. (a) See part (b), with $f=0$.
(b) Using the text analysis, through (13), we have

$$w(x,y,t) = \sum_1^\infty \sum_1^\infty \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right),$$

where

$$\omega_{mn} = \pi c\sqrt{(m/a)^2 + (n/b)^2}.$$

Then,

$$w(x,y,0) = f(x,y) = \sum_1^\infty \sum_1^\infty A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

and

$$w_t(x,y,0) = g(x,y) = \sum_1^\infty \sum_1^\infty \omega_{mn} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

so

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

5. (b) Thus far, $w(r,t) = (A+B\ln r)(D+Et)$
$+ [FJ_0(kr)+GY_0(kr)][H\cos kct + I\sin kct]$.

w bdd as $r\to 0 \Rightarrow B=0$ and $G=0$. w bdd as $t\to\infty \Rightarrow E=0$, so

$$w(r,t) = AD + FJ_0(kr)(H\cos kct + I\sin kct)$$
$$= A' + J_0(kr)(H'\cos kct + I'\sin kct).$$
$$w(a,t)=0 = A' + J_0(ka)(\quad " \qquad " \quad) \Rightarrow A'=0 \text{ and } J_0(ka)=0.$$

Denoting the roots of $J_0(z)=0$ as $z=z_n$ $(n=1,2,\dots)$, gives $ka = z_n$, so

$$w(r,t) = \sum_{n=1}^{\infty} J_0\left(z_n\frac{r}{a}\right)(H'_n\cos\omega_n t + I'_n\sin\omega_n t) \qquad ①$$

where

$$\omega_n = z_n c/a.$$

(c) $w(r,0) = f(r) = \sum_1^\infty H'_n J_0\left(z_n\frac{r}{a}\right) \qquad (0<r<a) \qquad ②$

and

$w_t(r,0) = g(r) = \sum_1^\infty \omega_n I'_n J_0\left(z_n\frac{r}{a}\right) \qquad (0<r<a) \qquad ③$

Since the weight function in the Sturm-Liouville equation

$$rR''+R'+k^2\underline{r}R=0$$

is r, the eigenfunction expansions ② and ③ give

$$H'_n = \frac{\langle f,J_0\rangle}{\langle J_0,J_0\rangle} = \frac{\int_0^a f(r)J_0(z_n r/a)\,r\,dr}{\int_0^a J_0^2(z_n r/a)\,r\,dr} = \frac{2}{a^2J_1^2(z_n)}\int_0^a f(r)J_0\left(z_n\frac{r}{a}\right)r\,dr \qquad ④$$

and

$$I'_n = \frac{1}{\omega_n}\frac{\langle g,J_0\rangle}{\langle J_0,J_0\rangle} = \text{etc.} = \frac{2}{\omega_n a^2 J_1^2(z_n)}\int_0^a g(r)J_0\left(z_n\frac{r}{a}\right)r\,dr, \qquad ⑤$$

as in Example 5 on page 971.

(d) $w = R\Theta T$ in (5.1) gives

$$\frac{R''+\frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{c^2}\frac{T''}{T} = -k^2$$

so $T''+k^2c^2T=0 \qquad ⑥$

and $\dfrac{R''+\frac{1}{r}R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} = -k^2,$

$$\frac{r^2R''+rR'+k^2r^2R}{R} = -\frac{\Theta''}{\Theta} = \alpha^2,$$

so

$$\Theta''+\alpha^2\Theta=0, \qquad ⑦$$
$$r^2R''+rR'+(k^2r^2-\alpha^2)R=0. \qquad ⑧$$

6. (a) $u(x,y,t) = X(x)Y(y)T(t)$ gives $\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$
so

$$T' + k^2\alpha^2 T = 0$$

and $\frac{X''}{X} = -\frac{Y''}{Y} - k^2 = -\beta^2$ so $X'' + \beta^2 X = 0,$
$Y'' + (k^2-\beta^2)Y = 0.$

Not bothering to distinguish the special cases where $\beta=0$ and where $k=\beta$ since we do not expect the additional terms thereby obtained to survive the application of the homogeneous b.c.'s, we have

$$u(x,y,t) = (A\cos\beta x + B\sin\beta x)(C\cos\sqrt{k^2-\beta^2}\,y + D\sin\sqrt{k^2-\beta^2}\,y)\cancel{E}e^{-k^2\alpha^2 t}$$

$$u(0,y,t) = 0 = \quad A\,(\quad " \quad\quad " \quad)e^{-k^2\alpha^2 t}$$

absorb E into C and D

so $A=0$, and

$$u(x,y,t) = \sin\beta x\,(C'\cos\sqrt{k^2-\beta^2}\,y + D'\sin\sqrt{k^2-\beta^2}\,y)e^{-k^2\alpha^2 t}$$

$u(x,0,t) = 0 = (\sin\beta x)(C'\ )\,e^{-k^2\alpha^2 t}$ so $C'=0$ and

$$u(x,y,t) = D'\sin\beta x \sin\sqrt{k^2-\beta^2}\,y \exp(-k^2\alpha^2 t)$$

$u(a,y,t) = 0 = D'\sin\beta a \quad " \quad\quad " \quad \Rightarrow \beta a = m\pi \ (m=1,2,\ldots)$

$u(x,b,t) = 0 = D'\sin\beta x \sin\sqrt{k^2-\beta^2}\,b \quad " \quad \Rightarrow \sqrt{k^2-\beta^2}\,b = n\pi \ (n=1,2,\ldots)$

so $k^2-\beta^2 = (n\pi/b)^2$

$k = \sqrt{(m\pi/a)^2+(n\pi/b)^2} \equiv k_{mn}$

so $u(x,y,t) = \sum_1^\infty\sum_1^\infty D'_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-k_{mn}^2\alpha^2 t).$

Then $u(x,y,0) = f(x,y) = \sum_1^\infty\sum_1^\infty D'_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$
and, as in (17)-(22),

$$D'_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

(c) If $f(x,y) = 100$, then

$$D'_{mn} = \frac{4}{ab}\int_0^b 100\left(-\frac{\cos\frac{m\pi x}{a}}{m\pi/a}\right)\Bigg|_0^a \sin\frac{n\pi y}{b}\,dy$$

$$= \frac{400}{bm\pi}(1-\cos m\pi)\int_0^b \sin\frac{n\pi y}{b}\,dy$$

$$= \frac{400}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi) = \frac{1600}{mn\pi^2}$$ if m and n are odd, and 0 otherwise

so $u(x,y,t) = \frac{1600}{\pi^2}\sum_{m=1,3,\ldots}^\infty\ \sum_{n=1,3,\ldots}^\infty \frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-k_{mn}^2\alpha^2 t)$

Section 19.4

1. (b) $u_{xx} - 4u_{xy} - 5u_{yy} = 0$. $\xi = x - y, \eta = 5x + y$

$\partial/\partial x = (\partial/\partial\xi)(1) + (\partial/\partial\eta)(5)$

$\partial/\partial y = (\partial/\partial\xi)(-1) + (\partial/\partial\eta)(1)$

so $(\frac{\partial}{\partial\xi} + 5\frac{\partial}{\partial\eta})(u_\xi + 5u_\eta) - 4(-\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta})(u_\xi + 5u_\eta) - 5(-\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta})(-u_\xi + u_\eta) = 0$

$\cancel{u_{\xi\xi}} + 5u_{\xi\eta} + 5u_{\xi\eta} + \cancel{25u_{\eta\eta}} + \cancel{4u_{\xi\xi}} + 20u_{\xi\eta} - 4u_{\xi\eta} - \cancel{20u_{\eta\eta}} - \cancel{5u_{\xi\xi}} + 5u_{\xi\eta} + 5u_{\xi\eta} - \cancel{5u_{\eta\eta}} = 0$

so $36u_{\xi\eta} = 0$

$u_{\xi\eta} = 0$, $u = F(\xi) + G(\eta) = F(x - y) + G(5x + y)$

The PDE was hyperbolic because $A = 1, B = -2, C = -5$ so $B^2 - AC = 9 > 0$.

(c) $u_{xx} + 6u_{xy} + 8u_{yy} = 0$. $\xi = 4x - y, \eta = 2x - y$

$\partial/\partial x = (\partial/\partial\xi)(4) + (\partial/\partial\eta)(2)$

$\partial/\partial y = (\partial/\partial\xi)(-1) + (\partial/\partial\eta)(-1)$

so $(4\frac{\partial}{\partial\xi} + 2\frac{\partial}{\partial\eta})(4u_\xi + 2u_\eta) + 6(-\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta})(4u_\xi + 2u_\eta) + 8(-\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta})(-u_\xi - u_\eta) = 0$

$\cancel{16u_{\xi\xi}} + 16u_{\xi\eta} + \cancel{4u_{\eta\eta}} - \cancel{24u_{\xi\xi}} - 36u_{\xi\eta} - \cancel{12u_{\eta\eta}} + \cancel{8u_{\xi\xi}} + 16u_{\xi\eta} + \cancel{8u_{\eta\eta}} = 0$

so $-4u_{\xi\eta} = 0$, $u_{\xi\eta} = 0$, $u = F(\xi) + G(\eta) = F(4x - y) + G(2x - y)$.

The PDE was hyperbolic because $A = 1, B = 3, C = 8$ so $B^2 - AC = 9 - 8 = 1 > 0$.

3. (b) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$. $A = 1, B = -1, C = -3$ so $B^2 - AC = 4 > 0$, so the PDE is hyperbolic and the method will work.

$\xi = ax + by, \eta = cx + dy$

gives $\partial/\partial x = a\,\partial/\partial\xi + c\,\partial/\partial\eta$ and $\partial/\partial y = b\,\partial/\partial\xi + d\,\partial/\partial\eta$. Thus, the PDE becomes

$(a\frac{\partial}{\partial\xi} + c\frac{\partial}{\partial\eta})(au_\xi + cu_\eta) - 2(b\frac{\partial}{\partial\xi} + d\frac{\partial}{\partial\eta})(au_\xi + cu_\eta) - 3(b\frac{\partial}{\partial\xi} + d\frac{\partial}{\partial\eta})(bu_\xi + du_\eta) = 0$

$$(\underbrace{a^2 - 2ab - 3b^2}_{P})u_{\xi\xi} + (\underbrace{2ac - 2bc - 2ad - 6bd}_{Q})u_{\xi\eta} + (\underbrace{c^2 - 2cd - 3d^2}_{R})u_{\eta\eta} = 0$$

Plan: Choose a, b, c, d so $P = R = 0$ (and $Q \neq 0$).

$P = 0$ gives $a/b = 3, -1$ so $a = 3b, -b$. Letting $b = 1$, say, $a = 3$ or -1.

$R = 0$ " $c/d =$ " " $c = 3d, -d$. " $d = 1$, say, $c = 3$ or -1.

If we choose $a = 3$ and $c = 3$ then $\xi = 3x + y$ and $\eta = 3x + y$ are redundant and we find that $Q = 0$. Thus, choose $a = 3, c = -1$ (or visa versa; same result) so $-12u_{\xi\eta} = 0$, $u_{\xi\eta} = 0$,

$u = F(\xi) + G(\eta) = F(3x + y) + G(-x + y)$

or, equivalently, $u = F(3x + y) + G(y - x)$.

(c) Proceeding as in (b),

$$\left(a\frac{\partial}{\partial\xi}+c\frac{\partial}{\partial\eta}\right)(au_\xi+cu_\eta)-10\left(b\frac{\partial}{\partial\xi}+d\frac{\partial}{\partial\eta}\right)(au_\xi+cu_\eta)+9\left(b\frac{\partial}{\partial\xi}+d\frac{\partial}{\partial\eta}\right)(bu_\xi+du_\eta)=0$$

$$(\underbrace{a^2-10ab+9b^2}_{P})u_{\xi\xi}+(\underbrace{2ac-10bc-10ad+18bd}_{Q})u_{\xi\eta}+(\underbrace{c^2-10cd+9d^2}_{R})u_{\eta\eta}=0$$

$P=0 \Rightarrow a=b$ or $a=9b$. Let $b=1$. $a=1$ or 9.

$R=0 \Rightarrow c=d$ or $c=9d$. Let $d=1$. $c=1$ or 9.

Choosing $a=1$ and $c=9$ (or visa versa) gives $\xi=x+y, \eta=9x+y$ and

$$-73u_{\xi\eta}=0,\ u_{\xi\eta}=0,\ u=F(\xi)+G(\eta)=F(x+y)+G(9x+y).$$

4. $\xi=x-ct, \eta=t$ give $\partial/\partial x=\partial/\partial\xi$ and $\partial/\partial t=-c\,\partial/\partial\xi+\partial/\partial\eta$, so

$$(-c\cancel{u_\xi}+u_\eta)+c(\cancel{u_\xi})=0,$$
$$u_\eta=0,$$
$$u=F(\xi)=F(x-ct)$$
$$u(x,0)=f(x)=F(x)$$

so $u(x,t)=f(x-ct)$.

NOTE: The choice $\xi=x-ct$ is crucial, but the choice $\eta=t$ is arbitrary (as long as the Jacobian of the transformation is nonzero so we can go back and forth between x,t and ξ,η) and simple for convenience. Instead, suppose we choose $\eta=2x+t$, say. Then the PDE becomes

$$\cancel{u_\xi(-c)}+u_\eta(1)+c[\cancel{u_\xi(1)}+u_\eta(2)]=0,$$
$$(2c+1)u_\eta=0,\ u_\eta=0,\ u=F(\xi)=F(x-ct),$$

as before. Or, let us not even specify $\eta(x,t)$ at all. Then we obtain

$$\cancel{u_\xi(-c)}+u_\eta\eta_t+c[\cancel{u_\xi(1)}+u_\eta\eta_x]=0,$$

or,

$$(\eta_t+c\eta_x)u_\eta=0.$$

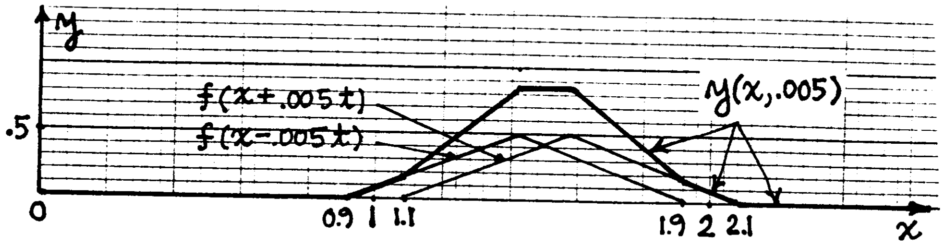
But the coefficient is in fact the Jacobian

$$J(x,t)=\begin{vmatrix}\xi_x & \xi_t\\ \eta_x & \eta_t\end{vmatrix}=\begin{vmatrix}1 & -c\\ \eta_x & \eta_t\end{vmatrix}=\eta_t+c\eta_x.$$

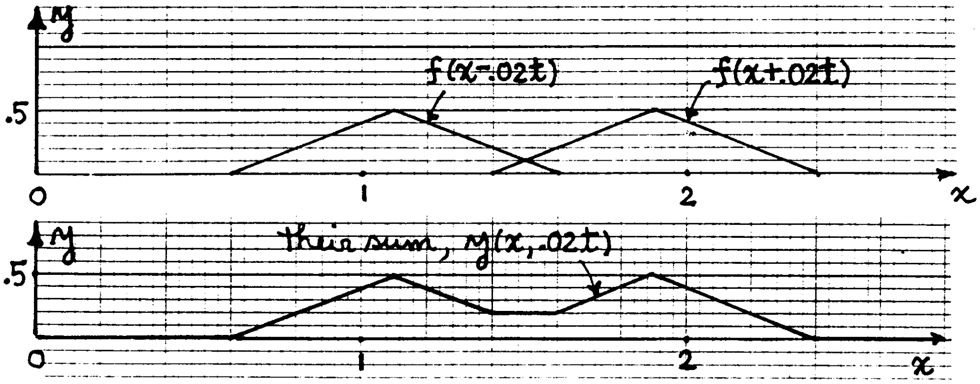
So, <u>any</u> $\eta(x,t)$ such that the Jacobian $\neq 0$ gives $u_\eta=0$ and $u=F(x-ct)$.

5. $ct=20(.005)=.1$ and $ct=20(.02)=.4$

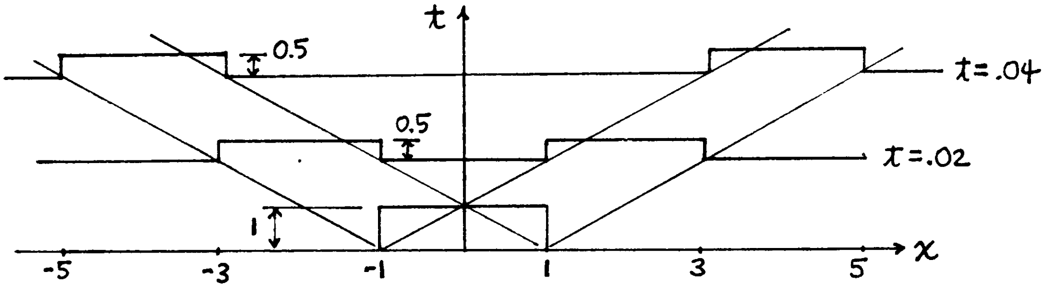
$t=0.005$:



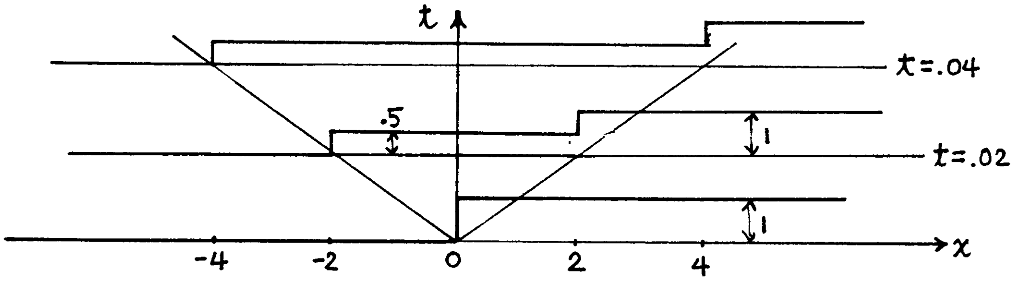
$t=.02$:



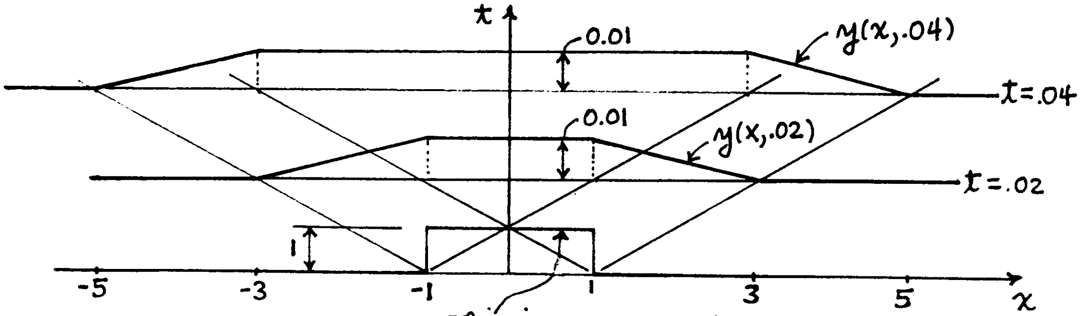
6. (a)



(b)

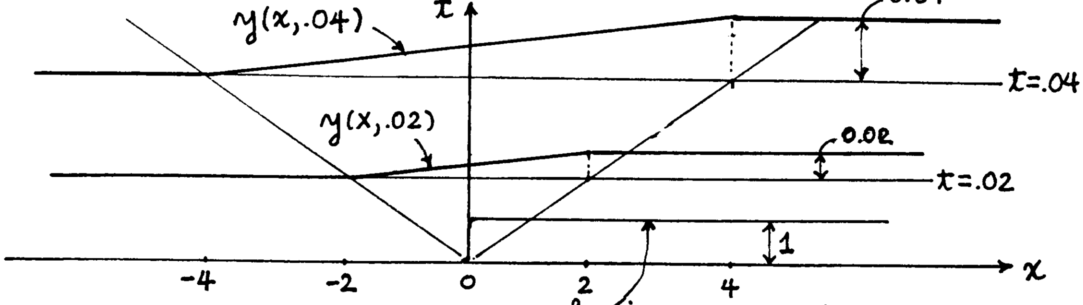


(c) Recall that if $f(x)=0$ then $y(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)d\mu$.



This is $y_t(x,0)$, not $y(x,0)$; $y(x,0)$ is 0

(d)



This is $y_t(x,0)$ not $y(x,0)$; $y(x,0)=0$

8. **NOTE: This solution is outlined in the Answers to Selected Exercises, but it is a valuable problem, so let us go through it here too.**

$y(x,t) = F(x-ct) + G(x+ct)$

$$*\left\{\begin{array}{l} y(x,0) = F(x) + G(x) = 0 \\ y_t(x,0) = -cF'(x) + cG'(x) = 0, \text{ or, } F(x) - G(x) = A \end{array}\right\} \text{Solving, } F(x) = A/2,\ G(x) = -A/2$$

so $y(x,t) = A/2 - A/2 = 0$. This does satisfy the two initial conditions and also the PDE, but not the condition $y(0,t) = h(t)$. What is wrong? The point is that * holds only over $0 < x < \infty$. Setting $A = 0$, without loss, what $F(x) = A/2 = 0$ and $G(x) = -A/2 = 0$ really say is

$$F(\text{arg}) = 0 \text{ and } G(\text{arg}) = 0 \text{ for arg} > 0.$$

What we have learned about

$$y(x,t) = F(x-ct) + G(x+ct),$$

then, is that $F(x-ct) = 0$ for $x > ct$ (region I below) and $G(x+ct) = 0$ for $x > -ct$ (regions I, II, III). Thus, we have learned that

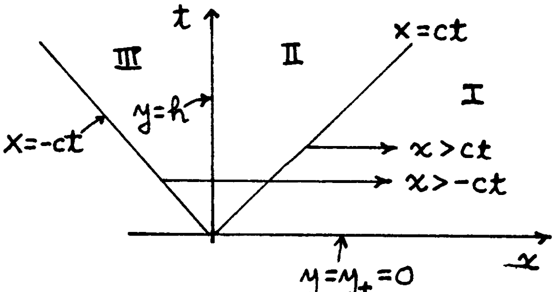
$$y(x,t) = 0 \text{ in I},$$
$$y(x,t) = F(x-ct) \text{ in II}.$$

To find $F(x-ct)$ in II impose

$$y(0,t) = h(t) = F(-ct)$$

or, $\quad F(\text{"arg"}) = h(-\frac{\text{arg}}{c})$,

so in II $\quad y(x,t) = F(x-ct) = h(\frac{ct-x}{c}) = h(t-\frac{x}{c})$.



Thus,

$$y(x,t) = \left\{\begin{array}{ll} 0, & \text{I} \\ h(t-\frac{x}{c}), & \text{II} \end{array}\right. = H(t-\tfrac{x}{c})\, h(t-\tfrac{x}{c})$$

9. $c^2 y_{xx} = y_{tt} \quad (0<x<\infty,\ 0<t<\infty)$, $\ y(x,0) = y_t(x,0) = 0$, $y(0,t) = h(t)$

Laplace transforming on t gives $\quad c^2 \bar{y}_{xx} = s^2 \bar{y} - 0 - 0$

so

$$\bar{y}(x,s) = Ae^{-sx/c} + Be^{sx/c}$$

$$\lim_{x\to\infty} y(x,t) = 0 \Rightarrow \lim_{x\to\infty} \bar{y}(x,s) = 0 \Rightarrow B = 0, \text{ so } \bar{y}(x,s) = Ae^{-sx/c}$$

Then, L.T. of $y(0,t) = h(t)$ gives $\bar{y}(0,s) = \bar{h}(s) = Ae^{0}$ so $A = \bar{h}(s)$.

Thus, $\quad \bar{y}(x,s) = \bar{h}(s)\, e^{-sx/c}$.

Then, entry 30 of Appendix C (with $a = x/c$) gives

$$y(x,t) = H(t-\tfrac{x}{c})\, h(t-\tfrac{x}{c}),$$

as in Exercise 8.

10. $\frac{c^2}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) = \frac{\partial^2 u}{\partial t^2}$ gives $\frac{c^2}{\rho^2}(2\rho u_\rho + \rho^2 u_{\rho\rho}) = u_{tt}$, $\ c^2(2u_\rho + \rho u_{\rho\rho}) = (\rho u)_{tt}$

or, $\quad c^2(\rho u)_{\rho\rho} = (\rho u)_{tt}$, so general solution is, using (9),

$$\rho u(\rho,t) = F(\rho - ct) + G(\rho + ct) \quad \text{or,} \quad u(\rho,t) = \frac{1}{\rho}[F(\rho - ct) + G(\rho + ct)].$$

11. From our knowledge of diffusion (Ch.18), it seems unlikely, but let us see. If $u(x,t) = F(x-at)$ then $u_{xx} = F''(x-at)$ and $u_t = -aF'(x-at)$ so we need F and a to satisfy

$$\alpha^2 F''(x-at) = -aF'(x-at)$$

or, with $x - at \equiv \xi$, say,

$$F''(\xi) + \frac{a}{\alpha^2}F'(\xi) = 0$$

so $$F(x-at) = A + B\exp\left[-\frac{a}{\alpha^2}(x-at)\right] \qquad ①$$

Finally,

$$u(x,0) = f(x) = A + Be^{-\frac{a}{\alpha^2}x}, \qquad ②$$

so travelling wave solutions will occur, but only if the initial condition is of the form ① for some constants (positive, negative, or zero) A, B, and a. In that case the solution is

$$u(x,t) = F(x-at) = A + B\exp\left[-\frac{a}{\alpha^2}(x-at)\right].$$

12. NOTE: This problem is longer than the typical exercise and involves more thought and ideas. Thus, it would be good for a homework "project" or for in-class discussion. The motivation is that thus far we have considered the one-dimensional medium to be homogeneous. As a "first problem" for inhomogeneous media (e.g., where the string density varies with x), it is sensible to consider a density that has a single step discontinuity, say at x = 0, for in principle we can consider any density distribution to be approximated by a stepwise varying function with many small steps.

The following solution is from M. D. Greenberg's "Foundations of Applied Mathematics", Prentice Hall, 1978, pp 544-546.

Example 26.10. *A More Complicated Case.* Suppose that we have an infinite string under a tension T and that the density (mass per unit length) is discontinuous at $x = 0$—that is, $\rho(x) = \rho_1$ for $x < 0$ and ρ_2 for $x > 0$. Given some initial rightward moving wave $F(x - a_1 t)$ (Fig. 26.13), where $a_1 = \sqrt{T/\rho_1}$, what happens when this wave reaches the density discontinuity? Turning to the x, t plane, we anticipate a situation as shown in Fig. 26.14, where we have assumed that $\rho_2 > \rho_1$ for definiteness (and hence $a_2 = \sqrt{T/\rho_2} < a_1 = \sqrt{T/\rho_1}$). That is, besides the incoming wave I, we allow for a reflected wave II, and a transmitted wave III. Why? Consider two special cases. If $\rho_2 = \infty$, then the right-hand string is *rigid* and $y(0, t) = 0$ for all t. Next, the left half of the string, with $y(0, t) = 0$, behaves as in Example 26.8, and there will be a reflection (and inversion) of the incoming wave with no transmission

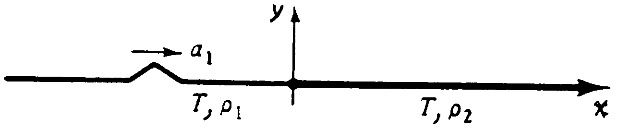


Figure 26.13. Density with a step discontinuity.

NOTE: Here we use ρ in place of σ and a in place of c.

at all due to the infinite density of the right-hand string. If, on the other hand, $\rho_2 = \rho_1$, then, of course, "nothing happens"; that is, there is 100% transmission with no reflection. For the general case, then, it is reasonable to allow for both transmitted and reflected waves. So we seek

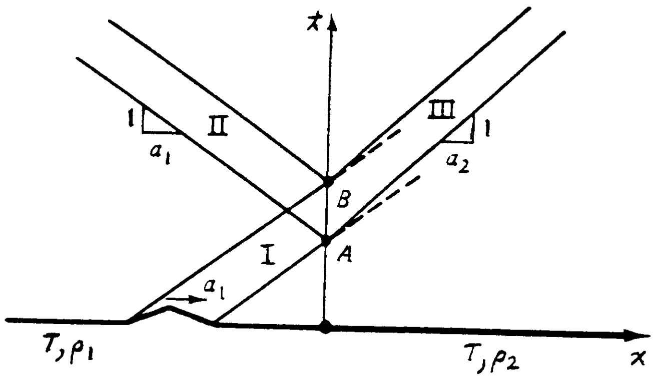


Figure 26.14. Reflection and transmission.

$$y_1 = F(x - a_1 t) + g(x + a_1 t) \quad \text{in } x < 0 \tag{26.95a}$$

and

$$y_2 = h(x - a_2 t) \quad \text{in } x > 0, \tag{26.95b}$$

where F, g, h are nonzero only in channels I, II, III, respectively. It remains to determine the form of g and h, a step that is accomplished by blending y_1 and y_2 suitably along the t axis or, more specifically, along AB.

Obviously one condition that needs to be enforced is

$$y_1(0, t) = y_2(0, t), \tag{26.96a}$$

so that the string does not break. We might also write $y_{1,t}(0, t) = y_{2,t}(0, t)$, but this result is already implied by (26.96a). Expecting that a second condition is needed, since we have two unknown functions, g and h, consider a "freebody" diagram of an infinitesimal element of the string located at $x = 0$. Suppose that there is a kink (Fig. 26.15). Then there is a *finite* vertical force on the element, even as $\Delta x \to 0$, which implies an infinite vertical acceleration at $x = 0$. If it persists for any finite amount of time, infinite vertical velocities will result. To avoid this situation, we require that the slope be *continuous* at $x = 0$:

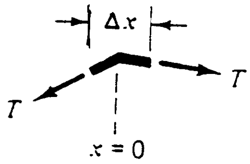


Figure 26.15. Suppose there is a kink at $x = 0$.

$$y_{1x}(0, t) = y_{2x}(0, t), \tag{26.96b}$$

and this is our second condition. Putting (26.95) into (26.96),

$$F(-a_1 t) + g(a_1 t) = h(-a_2 t) \tag{26.97a}$$

and

$$F'(-a_1 t) + g'(a_1 t) = h'(-a_2 t), \tag{26.97b}$$

where (as usual) primes denote differentiation with respect to the argument, whatever that may be. Solving for g and h in terms of F, we obtain (Exercise 26.18)

$$y_1(x, t) = F(x - a_1 t) + \left(\frac{a_2 - a_1}{a_2 + a_1}\right)F(-x - a_1 t) \tag{26.98a}$$

$$y_2(x, t) = \left(\frac{2a_2}{a_2 + a_1}\right)F\left[\frac{a_1}{a_2}(x - a_2 t)\right]. \tag{26.98b}$$

As a partial check, consider the two special cases mentioned above. If $\rho_1 = \rho_2$, then $a_1 = a_2$; thus the "reflection coefficient" $(a_2 - a_1)/(a_2 + a_1) = 0$ and the "transmission coefficient" $2a_2/(a_2 + a_1) = 1$, which, of course, is correct. If $\rho_2 = \infty$ instead, then $a_2 = 0$; as a result; the reflection coefficient is -1 (the minus sign denoting the inversion) and the transmission coefficient is 0, which is correct.

More generally, (26.98) indicates that we have a partial transmission and a partial reflection. The transmitted wave is always right side up, since $2a_2/(a_2 + a_1) \geq 0$, whereas the reflected wave is inverted if $a_2 < a_1$ $(\rho_2 > \rho_1)$ and right side up if $a_2 > a_1$ $(\rho_2 < \rho_1)$.

Suppose instead that $\rho(x)$ varies *continuously*, as in Fig. 26.16 say. Without even becoming involved in the calculations, we can anticipate the general form of the solution if we approximate $\rho(x)$ by a number of finite steps (dotted lines) and then recall the preceding solution for the single step. For instance, as the pulse moves through the channel between the C and D characteristics, it will be deformed by the sundry reflections. In fact, the disturbance spreads throughout the region between the E and C characteristics as well. At time t_0, for example, the disturbance extends from A to B and is probably of the general form shown in Fig. 26.17, where the dashed line denotes the result if the density had remained constant, for reference.[10]

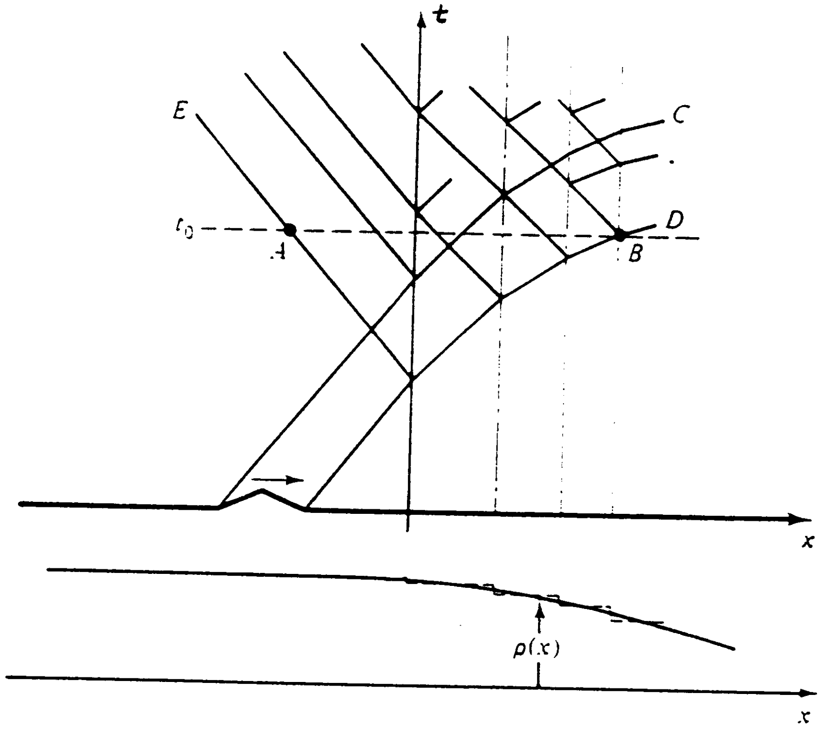


Figure 26.16. Continuous variation in $\rho(x)$.

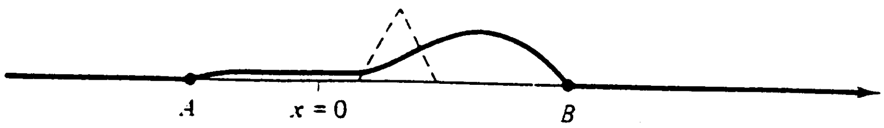


Figure 26.17. The resulting distortion.

13. $$y(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$$

If f and g are even, then

$$y(-x,t) = \frac{f(-x-ct)+f(-x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(\xi)d\xi \quad \swarrow \text{Let } \xi = -\mu \qquad ①$$

$$= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x+ct}^{x-ct} g(-\mu)(-d\mu)$$

$$= \quad '' \quad + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)d\mu = y(x,t)\text{, so } y \text{ is an even}$$

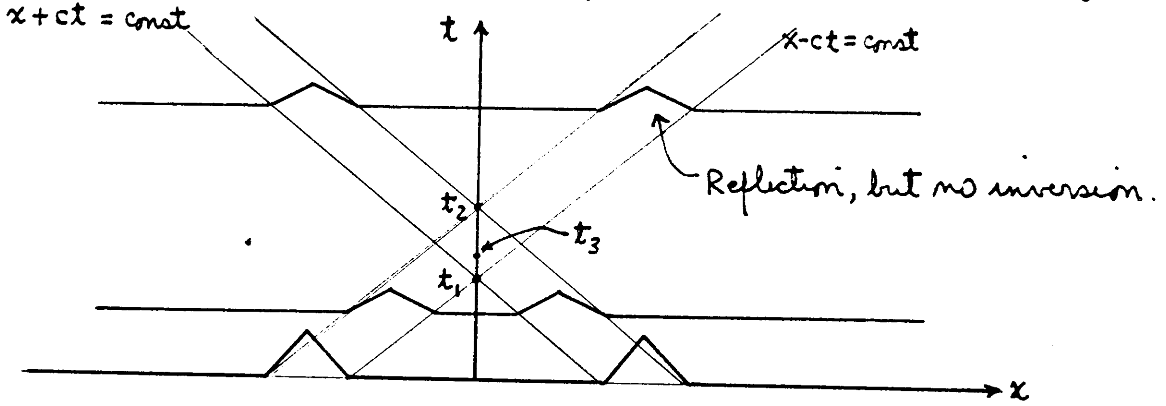
function of x. If, instead, f and g are odd, then ① gives

$$y(-x,t) = \frac{-f(x+ct)-f(x-ct)}{2} + \frac{1}{2c}\int_{x+ct}^{x-ct} \underbrace{g(-\mu)}_{\leftarrow\, g(-\mu)=-g(\mu)}(-d\mu)$$

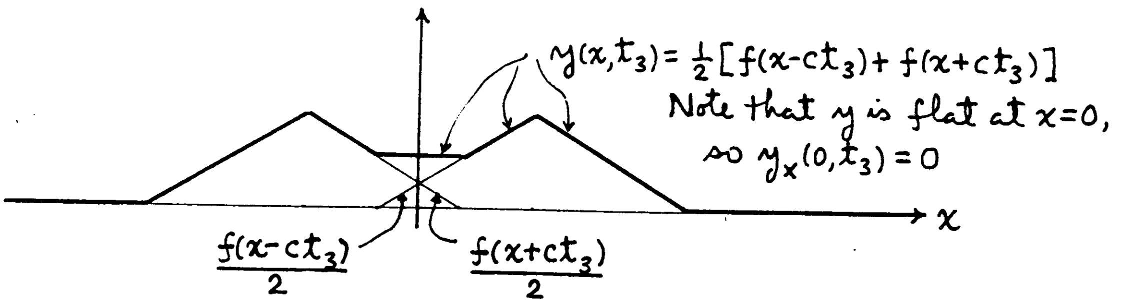
$$= -\frac{f(x+ct)+f(x-ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)d\mu = -y(x,t)\text{, so } y \text{ is an}$$

odd function of x.

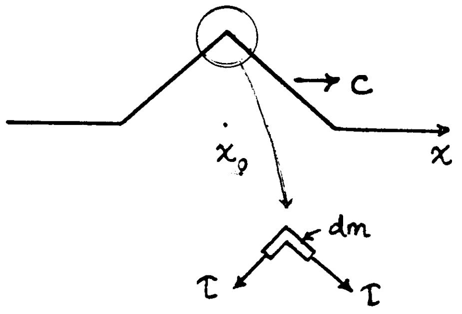
14. This time we need an even extension of $f(x)$, to satisfy the condition $y_x(0,t)=0$.



To show how the $y_x(0,t)=0$ condition is satisfied over $t_1<t<t_2$, when the left- and right-running waves are overlapping, let us show the two half-pulses and their sum at a typical time such as t_3.



15. The following discussion will be heuristic Consider a traveling wave with kinks in it, as shown at the right, and consider the free-body diagram of an infinitesimal element, of mass dm, at x_0. A finite tension force τ acts on each end, as shown, as well as a weight force (not shown because the weight force $\rightarrow 0$ as $dm \rightarrow 0$ while the τ forces remain unchanged). Thus, in the vertical direction we have a finite vertical force acting on an arbitrarily small dm giving, according to Newton's second law, an infinite vertical acceleration (so not only does y_{xx} fail to exist at x_0, so does y_{tt}). However, the key point is that the infinite vertical force persists only for zero time because the kink is translating with speed c.



16. (a) $c^2 y_{xx} = y_{tt}$

$$c^2 \frac{Y_{i-1,j} - 2Y_{ij} + Y_{i+1,j}}{(\Delta x)^2} = \frac{Y_{i,j-1} - 2Y_{ij} + Y_{i,j+1}}{(\Delta t)^2}$$

so, with $r = c\Delta t/\Delta x$,

$$\begin{aligned} Y_{i,j+1} &= r^2(Y_{i-1,j} - 2Y_{ij} + Y_{i+1,j}) - Y_{i,j-1} + 2Y_{ij} \\ &= r^2 Y_{i-1,j} + 2(1-r^2)Y_{ij} + r^2 Y_{i+1,j} - Y_{i,j-1} \quad ♀ \end{aligned}$$

(b) At $j = 0$, ♀ will include $Y_{i,-1}$, whereas the j index has the values $0, 1, 2, \dots$, not -1. To compute these terms use a backward difference quotient for y_t at $t = 0$:

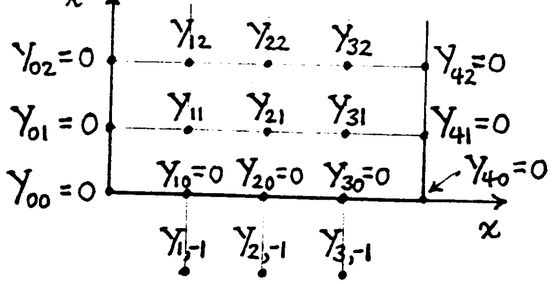
$$g(x) = y_t(x,0) \approx \frac{y(x,0) - y(x,-\Delta t)}{\Delta t} = \frac{Y_{i,0} - Y_{i,-1}}{\Delta t}$$

so $Y_{i,-1} \approx Y_{i0} - g_i \Delta t$. ☆

Now, $c = 10, L = 1, \Delta x = 0.25, \Delta t = 0.02$, $p(t) = q(t) = f(x) = 0$, $g(x) = \sin \pi x$,

so $r^2 = (c\Delta t/\Delta x)^2 = (.2/.25)^2 = 0.64$,

$g_1 = \sin .25\pi = 0.7071 = g_3$, $g_2 = \sin .5\pi = 1$.



Using ☆,

$Y_{1,-1} = 0 - (.7071)(.02) = -0.01414$

$Y_{2,-1} = 0 - (1)(.02) = -0.02$

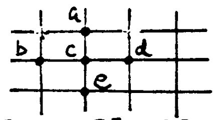
$Y_{3,-1} = 0 - (.7071)(.02) = -0.01414$

Actually, in this example there is symmetry about the midline $x=0.5$ so we need calculate only the Y_{1j}'s and the Y_{2j}'s since, by the symmetry, $Y_{3j}=Y_{1j}$ for each j. But let us calculate them nonetheless.

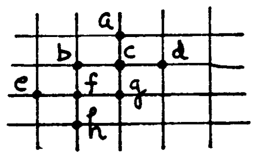
$$Y_{11}=r^2Y_{00}+2(1-r^2)Y_{10}+r^2Y_{20}-Y_{1,-1}=.64(0)+.72(0)+.64(0)+.01414=.01414$$
$$Y_{21}=r^2Y_{10}+2(1-r^2)Y_{20}+r^2Y_{30}-Y_{2,-1}=.64(0)+.72(0)+.64(0)+.02 = .02$$
$$Y_{31}=r^2Y_{20}+2(1-r^2)Y_{30}+r^2Y_{40}-Y_{3,-1}=.64(0)+.72(0)+.64(0)+.01414=.01414$$

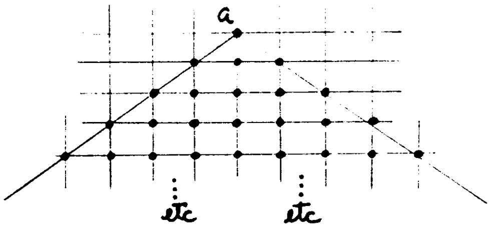
$$Y_{12}=r^2Y_{01}+2(1-r^2)Y_{11}+r^2Y_{21}-Y_{10}=.64(0)+.72(.01414)+.64(.02)-0=.02298$$
$$Y_{22}=r^2Y_{11}+2(1-r^2)Y_{21}+r^2Y_{31}-Y_{20}=.64(.01414)+.72(.02)+.64(.01414)-0=.03250$$
$$Y_{32}=r^2Y_{21}+2(1-r^2)Y_{31}+r^2Y_{41}-Y_{30}=.64(.02)+.72(.01414)+.64(0)-0=.02298$$

and so on.

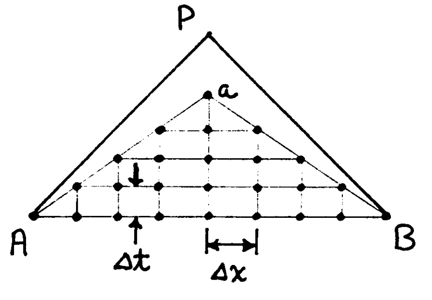


(f) From (16.1), we see that the computational scheme can be described by the diagram at the right: We can evaluate Y_a as a linear combination of Y_b, Y_c, Y_d, Y_e. Likewise Y_b was known as a linear combination of Y_e, Y_f, Y_g, Y_h. The upshot is that Y_a is determined as a linear combination of the Y's at all of the dots below it, within a triangle, as shown at the right →





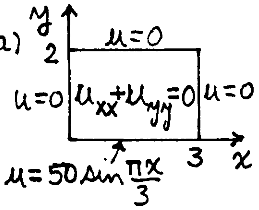
Now recall from (17) and Fig. 1 that initial data [$y(x,0)$ and $y_t(x,0)$] along the line AB serve to determine $Y(x,t)$ everywhere within the triangle APBA. Thus, for our finite difference calculation at a to be well founded, surely we need a to fall within the APBA triangle. Since the slope of AP is $1/c$ (and the slope of PB is $-1/c$), a will indeed fall within APBA if and only if $\Delta t/\Delta x < 1/c$, i.e., if and only if $r = c\Delta t/\Delta x < 1$. Otherwise we can expect "junk" and, sure enough, the calculation proves to be unstable (and, according to the Lax equivalence theorem, nonconvergent as well) if $r>1$.



CHAPTER 20

Section 20.2

1. (a)



Note: For the sake of space, we dont always include a picture like the one at the left, but we always urge the student to begin with a simple picture or sketch whenever there is one that is relevant. Also for brevity, our solutions often omit steps and details that the student should include, such as the separation process and derivation of the product solution forms.

$u(x,y) = (A+Bx)(C+Dy)+(E\sin kx+F\cos kx)(G\sinh ky+H\cosh ky)$

$u(0,y)=0= A(\ \text{"}\ )+ \qquad F(\ \text{"} \qquad \text{"}\ ) \Rightarrow A=F=0$, so

$u(x,y)= x(C'+D'y)+\sin kx\,(G'\sinh ky+H'\cosh ky)$

$u(3,y)=0=3(C'+D'y)+\sin 3k(\ \text{"} \qquad \text{"}\ )\Rightarrow C'=D'=0,\ 3k=n\pi\ (n=1,2,\dots)$

so $u(x,y)=\sum_1^\infty \sin\frac{n\pi x}{3}\left(G_n'\sinh\frac{n\pi y}{3}+H_n'\cosh\frac{n\pi y}{3}\right)$

$u(x,0)=50\sin\frac{\pi x}{3}=\sum_1^\infty H_n'\sin\frac{n\pi x}{3} \Longrightarrow H_1'=50$, others $=0$

so

$u(x,y)= 50\sin\frac{\pi x}{3}\cosh\frac{\pi y}{3}+\sum_1^\infty G_n'\sin\frac{n\pi x}{3}\sinh\frac{n\pi y}{3}$

$u(x,2)=0= 50\sin\frac{\pi x}{3}\cosh\frac{2\pi}{3}+\sum_1^\infty G_n'\sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}$

$-50\cosh\frac{2\pi}{3}\sin\frac{\pi x}{3}=\sum_1^\infty G_n'\sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}\Longrightarrow -50\cosh\frac{2\pi}{3}=G_1'\sinh\frac{2\pi}{3}$

so $G_1'=-50\coth\frac{2\pi}{3}$,
others $=0$

so $u(x,y)= 50\sin\frac{\pi x}{3}\left(\cosh\frac{\pi y}{3}-\coth\frac{2\pi}{3}\sinh\frac{\pi y}{3}\right)$

NOTE: The latter can be expressed more cogently by using the identity $\sinh(A-B)=\sinh A\cosh B-\cosh A\sinh B$.

$$u(x,y)= 50\sin\frac{\pi x}{3}\,\frac{\cosh\pi y/3\,\sinh 2\pi/3-\cosh 2\pi/3\,\sinh\pi y/3}{\sinh 2\pi/3}$$

$= 50\sin\frac{\pi x}{3}\,\frac{\sinh\frac{\pi}{3}(2-y)}{\sinh\frac{2\pi}{3}}$. Normally we will not carry out such rearrangement, but it can be important: see solution to Exercise 2d, below.

(b) $u(x,y)=(A+Bx)(C+Dy)+(E\sin kx+F\cos kx)(G\sinh ky+H\cosh ky)$

$u(0,y)= \quad A(\ \text{"}\ )+ \qquad F(\ \text{"} \qquad \text{"}\ )\rightarrow A=F=0$

$u(x,y)= x(C'+D'y)+\sin kx\,(G'\sinh ky+H'\cosh ky)$

$u(x,0)=0= C'x + \sin kx\,(H')\rightarrow C'=H'=0$

$u(x,y)= D'xy+G'\sin kx\sinh ky$

$$u(3,y)=0= 3D'y+G'\sin 3\kappa \sinh \kappa y \rightarrow D'=0,\ 3\kappa=n\pi \quad (n=1,2,\dots)$$
$$u(x,y)=\sum_1^\infty G_n' \sin\frac{n\pi x}{3}\sinh\frac{n\pi y}{3}$$
$$u(x,2) = 10\sin(\pi x/3)-4\sin\pi x = \sum_1^\infty G_n' \sinh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}$$
so $G_1'\sinh\frac{2\pi}{3}=10,\ G_3'\sinh 2\pi=-4$

$$u(x,y)=\frac{10}{\sinh\frac{2\pi}{3}}\sin\frac{\pi x}{3}\sinh\frac{\pi y}{3}-\frac{4}{\sinh 2\pi}\sin\pi x\sinh\pi y$$

(c) This time choose $X''/X=-Y''/Y=+\kappa^2$ since the expansion will be in y. Thus,
$$u(x,y)=(A+Bx)(C+Dy)+(E\sinh\kappa x+F\cosh\kappa x)(G\sin\kappa y+H\cos\kappa y)$$
$$u(x,0)=0=(A+Bx)C+(\ \text{''}\qquad\text{''}\ )H\rightarrow C=H=0$$
$$u(x,y)=(A'+B'x)y+(E'\sinh\kappa x+F'\cosh\kappa x)\sin\kappa y$$
$$u(x,2)=(\ \text{''}\ )2+(\ \text{''}\qquad\text{''}\ )\sin 2\kappa\rightarrow A'=B'=0,\ 2\kappa=n\pi,$$
$$u(x,y)=\sum_1^\infty\left(E_n'\sinh\frac{n\pi x}{2}+F_n'\cosh\frac{n\pi x}{2}\right)\sin\frac{n\pi y}{2}$$

$$u(0,y)=5\sin\pi y+4\sin 2\pi y-\sin 3\pi y=\sum_1^\infty F_n'\sin\frac{n\pi y}{2}$$
so $F_2'=5,\ F_4'=4,\ F_6'=-1$, others $=0$, so

$$u(3,y)=0=\sum_1^\infty\left(E_n'\sinh\frac{3n\pi}{2}+F_n'\cosh\frac{3n\pi}{2}\right)\sin\frac{n\pi y}{2}$$
gives $E_n'=-F_n'\coth(3n\pi/2)$ for all n,

so
$$u(x,y)=\sum_1^\infty F_n'\left(\cosh\frac{n\pi x}{2}-\coth\frac{3n\pi}{2}\sinh\frac{n\pi x}{2}\right)\sin\frac{n\pi y}{2}$$

$$\begin{aligned}&=5(\cosh\pi x-\coth 3\pi\sinh\pi x)\sin\pi y\\&\quad+4(\cosh 2\pi x-\coth 6\pi\sinh 2\pi x)\sin 2\pi y\\&\quad-(\cosh 3\pi x-\coth 9\pi\sinh 3\pi x)\sin 3\pi y\end{aligned}$$

(e) Expansion will be on x, so use $-\kappa^2$.
$$u(x,y)=(A+Bx)(C+Dy)+(E\sin\kappa x+F\cos\kappa x)(G\sinh\kappa y+H\cosh\kappa y)$$
$$u_x(0,y)=0=B(\ \text{''}\ )+\kappa E(\ \text{''}\qquad\text{''}\ )\rightarrow B=E=0$$
$$u(x,y)=C'+D'y+\cos\kappa x\,(G'\sinh\kappa y+H'\cosh\kappa y)$$
$$u(3,y)=0=C'+D'y+\cos 3\kappa(\ \text{''}\qquad\text{''}\ )\rightarrow C'=D'=0,\ 3\kappa=n\pi/2\ (n\text{ odd})$$
$$u(x,y)=\sum_{1,3,\dots}\cos\frac{n\pi x}{6}\left(G_n'\sinh\frac{n\pi y}{6}+H_n'\cosh\frac{n\pi y}{6}\right)\qquad ①$$
$$u(x,0)=50H(x-2)=\sum_{1,3,\dots}^\infty H_n'\cos\frac{n\pi x}{6}$$
$$\text{QRC:}\quad H_n'=\frac{2}{3}\int_0^3 50H(x-2)\cos\frac{n\pi x}{6}\,dx=\frac{200}{n\pi}\left(\sin\frac{n\pi}{2}-\sin\frac{n\pi}{3}\right)\qquad ②$$
$$u(x,2)=0=\sum_{1,3,\dots}^\infty\left(G_n'\sinh\frac{n\pi}{3}+H_n'\cosh\frac{n\pi}{3}\right)\cos\frac{n\pi x}{3}$$

so $G_n' \sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{3} = 0$ ③

The solution is given by ①, where H_n' is given by ② and then G_n' by ③.

(f) Expansion will be on x, so use $-k^2$.

$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$

$u(0,y) = 0 = A(\text{''}) + F(\text{''} \quad \text{''}) \rightarrow A=F=0$

$u(x,y) = x(C'+D'y) + \sin kx\,(G'\sinh ky + H'\cosh ky)$

$u_x(3,y) = 0 = C'+D'y + k\cos 3k(\text{''} \quad \text{''}) \rightarrow C'=D'=0,\ 3k=\frac{n\pi}{2}$ (n odd)

$u(x,y) = \sum_{1,3,\dots} \sin\frac{n\pi x}{6}\left(G_n'\sinh\frac{n\pi y}{6} + H_n'\cosh\frac{n\pi y}{6}\right)$ ①

$u(x,0) = 50H(x-2) = \sum_{1,3,\dots}^{\infty} H_n'\sin\frac{n\pi x}{6}$

QRS: $H_n' = \frac{2}{3}\int_0^3 50H(x-2)\sin\frac{n\pi x}{6}\,dx = \frac{200}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{n\pi}{2}\right)$ ②

$u(x,2) = 0 = \sum_{1,3,\dots}^{\infty}\left(G_n'\sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{2}\right)\sin\frac{n\pi x}{6}$

gives $G_n'\sinh\frac{n\pi}{3} + H_n'\cosh\frac{n\pi}{2} = 0$ ③

so the solution is given by ①, with H_n' and G_n' given by ② and ③.

(g) Expansion will be on x, so use $-k^2$.

$u(x,y) = (A+Bx)(C+Dy) + (E\sin kx + F\cos kx)(G\sinh ky + H\cosh ky)$

$u_x(0,y) = 0 = B(\text{''}) + kE(\text{''} \quad \text{''}) \rightarrow B=E=0$

$u(x,y) = C'+D'y + \cos kx\,(G'\sinh ky + H'\cosh ky)$

$u_x(3,y) = 0 = -k\sin 3k(\text{''} \quad \text{''}) \rightarrow 3k = n\pi \quad (n=1,2,\dots)$

$u(x,y) = C'+D'y + \sum_1^{\infty}\cos\frac{n\pi x}{3}\left(G_n\sinh\frac{n\pi y}{3} + H_n'\cosh\frac{n\pi y}{3}\right)$ ①

$u(x,2) = 0 = C'+D'2 + \sum_1^{\infty}\left(G_n'\sinh\frac{2n\pi}{3} + H_n'\cosh\frac{2n\pi}{3}\right)\cos\frac{n\pi x}{3}$

HRC: $C'+2D'=0$, ②

$G_n'\sinh\frac{2n\pi}{3} + H_n'\cosh\frac{2n\pi}{3} = 0$ ③

$u(x,0) = 50H(x-2) = C' + \sum_1^{\infty} H_n'\cos\frac{n\pi x}{3}$

HRC: $C' = \frac{1}{3}\int_0^3 50H(x-2)\,dx = 50/3$ ④

$H_n' = \frac{2}{3}\int_0^3 50H(x-2)\cos\frac{n\pi x}{3}\,dx = -\frac{100}{n\pi}\sin\frac{2n\pi}{3}$ ⑤

so $u(x,y)$ is given by ①, with C', D', G_n', H_n' given by ②-⑤.

2(d) In Exercise 1(e) we obtained this solution:

$$u(0,y)=\sum_{1,3,\dots}\left[\left(-\coth\frac{n\pi}{3}\right)\sinh\frac{n\pi y}{6}+\cosh\frac{n\pi y}{6}\right]\frac{200}{n\pi}\left(\sin\frac{n\pi}{2}-\sin\frac{n\pi}{3}\right)\overset{1}{\cancel{\cos}}\frac{n\pi x}{6}$$

If, for a particular value of y we sum 10 terms, 20 terms, 30, etc., we find that the results fail to settle down to a limit. Why?? Observe that as $n\to\infty$, $\coth\frac{n\pi}{3}\to 1$ so $[\;]\sim\left[-\sinh\frac{n\pi y}{6}+\cosh\frac{n\pi y}{6}\right]$

$$=\frac{1}{2}\left\{-\cancel{e^{n\pi y/6}}+e^{-n\pi y/6}+\cancel{e^{n\pi y/6}}+e^{-n\pi y/6}\right\}=e^{-n\pi y/6}\to 0.$$

However, the approach to zero is only by virtue of the cancellation of almost-equal oppositely signed large numbers (observe the $e^{n\pi y/6}$'s within the $-\sinh n\pi y/6$ and the $+\cosh n\pi y/6$). Carrying only a limited number of decimal places, Maple is unable to handle this calculation — as it stands.* BUT, as discussed above (See the NOTE in Exercise 1(a)), we can express the solution in the alternative form

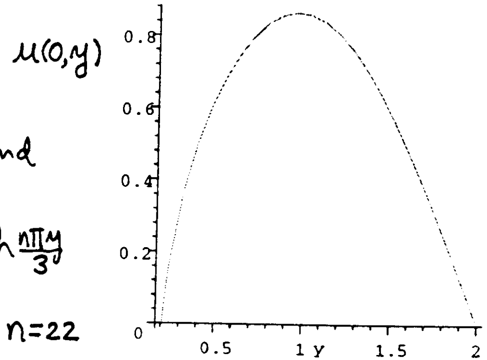
$$u(0,y)=\sum_{1,3,\dots}\frac{200}{n\pi}\,\frac{\sin\frac{n\pi}{2}-\sin\frac{n\pi}{3}}{\sinh\frac{n\pi}{3}}\sinh\frac{n\pi(2-y)}{6}, \qquad 웃$$

which contains a ratio of large numbers (the sinh's) rather than a difference of large numbers. Applying the Maple sum command to 웃 at $y=0.25$ gives

```
> sum((200/(2*i-1)/Pi)*(sin((2*i-1)*Pi/2)-sin((2*i-1)*Pi/3))*sinh((2
  *i-1)*Pi*(2-.25)/6)/sinh((2*i-1)*Pi/3),i=1..25);
                              .350771873
```

Similarly for $y=.5, .75, 1, 1.25, 1.5, 1.75, 2$ we obtain (settled down to three significant figures) .638, .816, .862, .778, .584, .312, 0. But it is easier to plot directly per

```
> with(plots):
> implicitplot(u=sum((200/(2*i-1)/Pi)*(sin((2*i-1)*Pi/2)-sin((2*i-1)
  *Pi/3))*sinh((2*i-1)*Pi*(2-y)/6)/sinh((2*i-1)*Pi/3),i=1..10),y=0..
  2,u=0..3);
```



---

* To clarify this point let us focus on the Maple calculation of the left- and right-hand sides of the identity

$$\sinh\frac{n\pi(2-y)}{6}=\sinh\frac{n\pi}{3}\cosh\frac{n\pi y}{6}-\sinh\frac{n\pi y}{6}\cosh\frac{n\pi y}{3}$$

with $y=1$, say. Increasing n we obtain

	n=1	n=10	n=20	n=21	n=22
LHS =	.5478534741	93.955	17655.95	29804.87	50313.36
RHS =	.5478534728	93.955	20000	30000	0

of which the LHS values are trustworthy and the RHS values are not; they are obtained as the difference of large numbers.

3.(a) From the bdy conditions there seems to be a bleak future for the $K=0$ terms so let us try omitting them. (Go ahead and do this if you can see your way clearly; if not, play it safe.)

$$u(x,y) = (A\sin Kx + B\cos Kx)(C\sinh Ky + D\cosh Ky)$$
$$u(0,y) = 0 = \qquad B\;(\qquad '' \qquad\qquad '' \qquad) \to B = 0$$
$$u(x,y) = \sin Kx\,(C'\sinh Ky + D'\cosh Ky)$$
$$u(2,y) = 0 = \sin 2K\;(\qquad '' \qquad\qquad '' \qquad) \to 2K = n\pi \quad (n=1,2,\dots)$$
$$u(x,y) = \sum_1^\infty \sin\frac{n\pi x}{2}\left(C_n'\sinh\frac{n\pi y}{2} + D_n'\cosh\frac{n\pi y}{2}\right)$$
$$u(x,0) = 100\sin\frac{\pi x}{2} = \sum_1^\infty D_n'\sin\frac{n\pi x}{2} \to D_1' = 100, \text{ all others} = 0$$

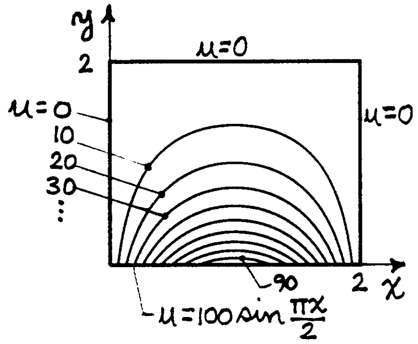
and $u(x,2)=0$ then gives the C_n''s, but it is more convenient to satisfy $u(x,2)=0$ first by expressing

$$u(x,y) = \sum_1^\infty E_n \sin\frac{n\pi x}{2}\sinh\frac{n\pi}{2}(2-y).$$

Then $\quad u(x,0) = 100\sin\frac{\pi x}{2} = \sum E_n \sin\frac{n\pi x}{2}\sinh n\pi$

so $\quad E_1\sinh\pi = 100$, others $=0$. Thus,

$$u(x,y) = 100\sin\frac{\pi x}{2}\,\frac{\sinh\frac{\pi}{2}(2-y)}{\sinh\pi}$$



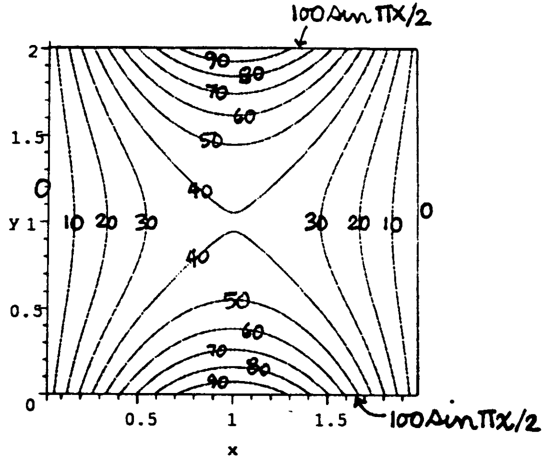
(b) Let's jump in with

$$u(x,y) = \sin\frac{\pi x}{2}\left(C'\sinh\frac{\pi y}{2} + D'\cosh\frac{\pi y}{2}\right)$$
$$u(x,0) = 100\sin\frac{\pi x}{2} = D'\sin\frac{\pi x}{2} \to D' = 100.$$
$$u(x,2) = 100\sin\frac{\pi x}{2} = \sin\frac{\pi x}{2}\left(C'\sinh\pi + 100\cosh\pi\right)$$

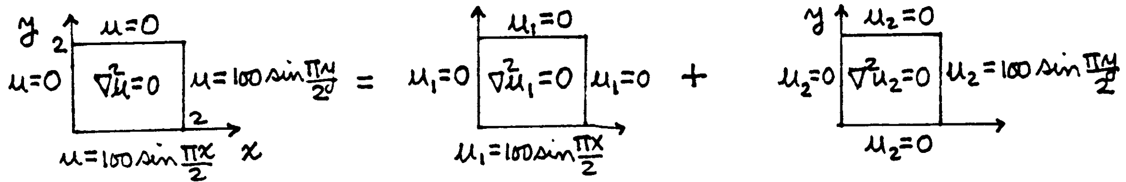
so $C' = 100(1-\cosh\pi)/\sinh\pi$ and

$$u(x,y) = 100\sin\frac{\pi x}{2}\left(\frac{1-\cosh\pi}{\sinh\pi}\sinh\frac{\pi y}{2} + \cosh\frac{\pi y}{2}\right)$$

```
> with(plots):
> u:=100*sin(Pi*x/2)*((1-cosh(Pi))*sinh(Pi*y/2)+sinh(Pi)*cosh(Pi*y/2
  ))/sinh(Pi):
> implicitplot({u=10,u=20,u=30,u=40,u=50,u=60,u=70,u=80,u=90},x=0..2
  ,y=0..2,grid=[100,100]);
```



(c)

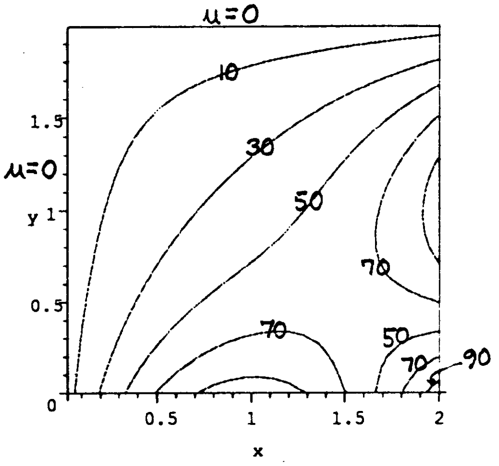


From (a), $u_1(x,y) = 100\sin\frac{\pi x}{2}\,\frac{\sinh\frac{\pi}{2}(2-y)}{\sinh\pi}$

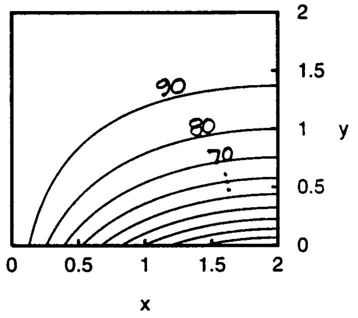
Similarly, $u_2(x,y) = 100\sin\frac{\pi y}{2}\,\frac{\sinh\frac{\pi}{2}x}{\sinh\pi}$

Adding these,

$$u(x,y) = \frac{100}{\sinh\pi}\left(\sin\frac{\pi x}{2}\sinh\frac{\pi}{2}(2-y) + \sin\frac{\pi y}{2}\sinh\frac{\pi x}{2}\right)$$

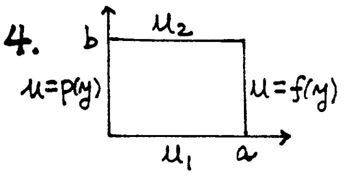


(d)



$$u(x,y) = 100\sin\frac{\pi x}{4}\left(\cosh\left(\frac{\pi y}{4}\right) - \frac{\sinh(\pi y/4)}{\tanh(\pi/2)}\right)$$

4.



$u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$

$u(x,0) = u_1 = (\text{"}\quad)C + (\quad\text{"}\qquad\text{"}\quad)G \to B=0,\ AC=u_1,\ G=0$

$u(x,y) = u_1 + D'y + (E'\cosh kx + F'\sinh kx)\sin ky$

$u(x,b) = u_2 = u_1 + D'b + (\quad\text{"}\qquad\text{"}\quad)\sin kb$

so $D' = (u_2-u_1)/b$, $kb = n\pi$ $(n=1,2,\dots)$

$$u(x,y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b} \qquad ①$$

$$u(0,y) = p(y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty E_n'\sin\frac{n\pi y}{b}$$

or,

$$p(y) - u_1 - (u_2-u_1)\frac{y}{b} = \sum_1^\infty E_n'\sin\frac{n\pi y}{b} \qquad (0<y<b)$$

HRS: $E_n' = \frac{2}{b}\int_0^b \left[p(y) - u_1 - (u_2-u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy$ ②

Finally,

$$u(a,y) = f(y) = u_1 + (u_2-u_1)\frac{y}{b} + \sum_1^\infty\left(E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b}$$

HRS:

$$E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b\left[f(y) - u_1 - (u_2-u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy \qquad ③$$

The solution is given by ①-③.

5. (a) $u(x,y) = (A+Bx)(C+Dy) + (E\cos kx + F\sin kx)(G\cosh ky + H\sinh ky)$

$u(0,y) = u_1 = A(C+Dy) + \qquad E(\quad '' \qquad '' \quad) \to AC = u_1,\ D=0,\ E=0$

$u(x,y) = u_1 + B'x + \sin kx(G'\cosh ky + H'\sinh ky)$

$u(a,y) = u_2 = u_1 + B'a + \sin ka(\quad '' \qquad '' \quad) \to B' = (u_2 - u_1)/a,\ ka = n\pi$

$u(x,y) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(G_n'\cosh\frac{n\pi y}{a} + H_n'\sinh\frac{n\pi y}{a}\right)$ ①

$u(x,0) = f(x) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty G_n'\sin\frac{n\pi x}{a}$

HRS:

$$G_n' = \frac{2}{a}\int_0^a \left[f(x) - u_1 - (u_2-u_1)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$ ②

Then

$u(x,b) = p(x) = u_1 + (u_2-u_1)\frac{x}{a} + \sum_1^\infty \left(G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$

HRS:

$$G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a \left[p(x) - u_1 - (u_2-u_1)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$ ③

Solution given by ①-③.

(b) This time we anticipate having to expand $p(y)$ and $f(y)$, so choose the sign of k^2 so as to obtain sines and cosines of y, not x.

$u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$

$u(x,0) = u_2 = (A+Bx)C + (\quad '' \qquad '' \quad)G \to AC = u_2,\ B=0,\ G=0$

$u(x,y) = u_2 + D'y + (E'\cosh kx + F'\sinh kx)\sin ky$

$u(x,b) = u_1 = u_2 + D'b + (\quad '' \qquad '' \quad)\sin kb \to D' = (u_1-u_2)/b,\ kb = n\pi$

$u(x,y) = u_2 + (u_1-u_2)\frac{y}{B} + \sum_1^\infty \left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$ ①

$u_x(0,y) = p(y) = 0 + \sum_1^\infty \frac{n\pi}{b}F_n'\sin\frac{n\pi y}{b} \quad (0<y<b)$

HRS:

$$\frac{n\pi}{b}F_n' = \frac{2}{b}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy,\quad F_n' = \frac{2}{n\pi}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy$$ ②

$u(a,y) = f(y) = u_2 + (u_1-u_2)\frac{y}{B} + \sum_1^\infty\left(E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b} \quad (0<y<b)$

HRS:

$$E_n'\cosh\frac{n\pi a}{b} + F_n'\sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b \left[f(y) - u_2 - (u_1-u_2)\frac{y}{B}\right]\sin\frac{n\pi y}{b}\,dy$$ ③

Solution is given by ①-③.

(c) As in (b), $u(x,y) = u_2 + (u_1-u_2)\frac{y}{B} + \sum_1^\infty\left(E_n'\cosh\frac{n\pi x}{b} + F_n'\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$ ①

$u_x(0,y) = p(y)$ gives, as in (b),

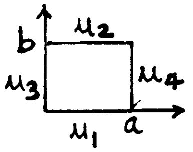
$$F_n' = \frac{2}{n\pi}\int_0^b p(y)\sin\frac{n\pi y}{b}\,dy$$ ②

Then $u_x(a,y) = f(y)$ gives

$$E_n'\sinh\frac{n\pi a}{b} + F_n'\cosh\frac{n\pi a}{b} = \frac{2}{n\pi}\int_0^b \left[f(y) - u_2 - (u_1-u_2)\frac{y}{B}\right]\sin\frac{n\pi y}{b}\,dy$$ ③

and the solution is given by ①-③.

6.



$\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$ leads to

$$u(x,y) = u_1 + (u_2 - u_1)\frac{y}{b} + \sum_1^\infty \left(A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}$$

$$A_n = \frac{2}{b}\int_0^b \left[u_3 - u_1 - (u_2 - u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy$$

$$A_n \cosh\frac{n\pi a}{b} + B_n \sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b \left[u_4 - u_1 - (u_2 - u_1)\frac{y}{b}\right]\sin\frac{n\pi y}{b}\,dy$$

alternatively,

$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$ leads to

$$u(x,y) = u_3 + (u_4 - u_3)\frac{x}{a} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}\right)$$

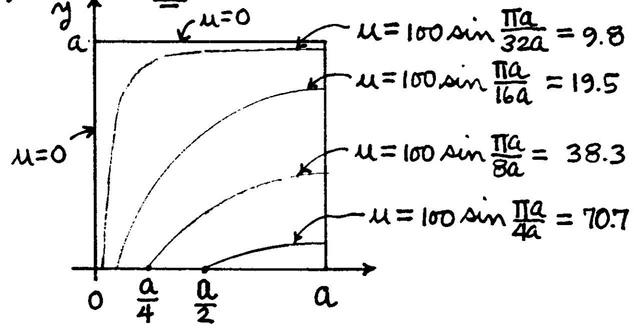
$$A_n = \frac{2}{a}\int_0^a \left[u_1 - u_3 - (u_4 - u_3)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$

$$A_n \cosh\frac{n\pi b}{a} + B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a \left[u_2 - u_3 - (u_4 - u_3)\frac{x}{a}\right]\sin\frac{n\pi x}{a}\,dx$$

7. If $b = a$ and $f(y) = g(x) = p(y) = q(x) = 100$, then (see Fig. 4) it is clear that $u_1 = u_2 = u_3 = u_4$ at the center, $(a/2, a/2)$. Also clear is that u at the center (in fact everywhere in the rectangle) is 100. Thus, $u_1 + u_2 + u_3 + u_4 = 100$ or, since $u_1 = u_2 = u_3 = u_4$ at the center, $4u_1 = 100$, $u_1 = 25$ (at the center).

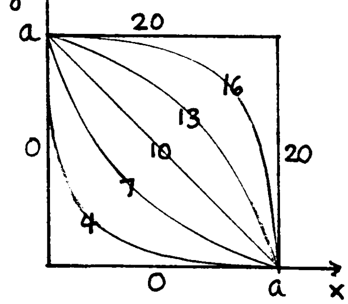
8. (a) Same as in Exercise 3(a), with the "2"'s changed to "a"'s.

(b) Sketch:

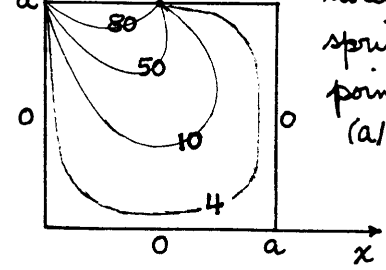


The sketch should be "topologically correct" – i.e., in its key features. In particular, each isotherm must be horizontal at the x=a edge because $u_x(a,y) = 0$.

(c)

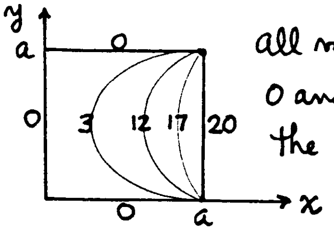


(d) Remember, these are rough sketches, not computer plots of actual solutions.



All values between 0 and 100 spring from the points (0,a) and (a/2, a).

(e)

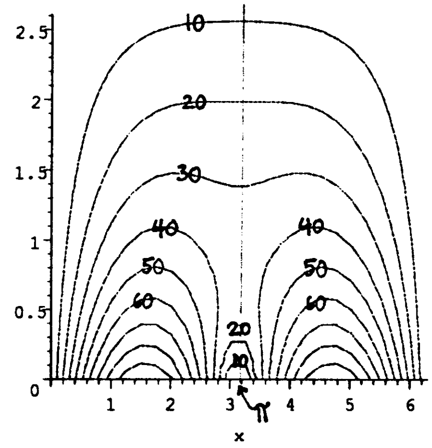


All values between 0 and 20 spring from the corners (a,0), (a,a).

NOTE: The preceding sketches of level curves have been fairly straightforward. In other cases the topography of these curves may be trickier. To illustrate, consider the problem $\nabla^2 u = 0$ in $0 < x < \pi, 0 < y < \pi$ with the boundary conditions $u(0,y) = u(x,\pi) = u_x(\pi,y) = 0, u(x,0) = 100 \sin x$. In particular, note that the "plate", say, is insulated at the right edge $x = \pi$, so the isotherms will have to be horizontal at that edge. Without deriving the solution, here is the Maple plotting of the isotherms, where we have plotted over the extended region $0 < x < 2\pi$ simply because that picture seems to make it easier to see the patterns.

```
> u:=sum(-(800/Pi)*(sin((2*i-1)*Pi/2)/(sinh((2*i-1)*Pi/2)*((2*i-1)^2
  -4)))*sin((2*i-1)*x/2)*sinh((2*i-1)*(Pi-y)/2),i=1..10):
> with(plots):
> implicitplot({u=90,u=80,u=70,u=60,u=50,u=40,u=30,u=20,u=10},x=0..2
  *Pi,y=0..Pi);
```

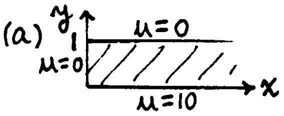
The tricky topological feature of the level curve pattern is the way the lower region, with "3 emanations" of curves gives way, above, to a region of single curves. Exploration of the details of that transition might make a nice computer project. To capture those details we'd surely need to include more terms in the sum than the 10 used above.



9. (11) and (13) give $u(x,y) = \sum_1^\infty \left(\frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b} dy \right) \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$

$$= \int_0^b \Big(\underbrace{\sum_1^\infty \frac{2}{b \sinh \frac{n\pi a}{b}} \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \sin \frac{n\pi \eta}{b}}_{K(\eta;\, x,y)} \Big) f(\eta) d\eta$$

10. (a)



Anticipating the Fourier series expansion to be on the finite edge $x=0$, write $\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$. Then

$u(x,y) = (A+Bx)(C+Dy) + (E e^{kx} + F e^{-kx})(G \cos ky + H \sin ky)$

u bdd as $x \to \infty \Rightarrow B = E = 0$, so

$u(x,y) = C' + D'y + e^{-kx}(G' \cos ky + H' \sin ky)$

$u(x,0) = 10 = C' + e^{-kx} G' \to C' = 10, G' = 0$ so

$u(x,y) = 10 + D'y + H' e^{-kx} \sin ky$

$u_y(x,1) = 0 = D' + kH' \cos k \, e^{-kx} \to D' = 0, \cos k = 0$ so $k = n\pi/2$ (n odd)

$u(x,y) = 10 + \sum_{1,3,..} H_n' e^{-n\pi x/2} \sin n\pi y/2,$

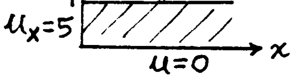
$u(0,y) = 0 = 10 + \sum_{1,3,..} H_n' \sin n\pi y/2, \quad -10 = \sum_{1,3,..} H_n' \sin \frac{n\pi y}{2} \quad (0<y<1)$

QRS: $H_n' = 2\int_0^1 (-10) \sin(n\pi y/2) dy = -40/n\pi$, so

$$u(x,y) = 10 - \frac{40}{\pi} \sum_{1,3,..} \frac{1}{n} \sin \frac{n\pi y}{2} e^{-n\pi x/2}$$

(b) Proceeding essentially as in (a) we will arrive at $u(x,y)=100$ (everywhere), which result could have been seen by inspection.

(c) $u(x,y)= (A+Bx)(C+Dy)+(Ee^{kx}+Fe^{-kx})(G\cos ky+H\sin ky)$



Sequence of application of the 4 boundary conditions: We must do the $y=0$ and $y=1$ b.c.'s before the $x=0$ one so as to get ready for the Fourier series expansion that will take place at the $x=0$ edge. But I advise applying any boundedness conditions (in this case at $x=\infty$) first since they knock terms out and simplify the solution form. Thus,

Boundedness as $x\to\infty \Rightarrow B=0$ and $E=0$, so

$u(x,y)=C'+D'y+e^{-kx}(G'\cos ky+H'\sin ky)$

$u(x,0)=0=C'+e^{-kx}G' \to G'=0$ and $C'=0$, so

$u(x,y)=D'y+H'e^{-kx}\sin ky$

$u(x,1)=0=D'+H'e^{-kx}\sin k \to D'=0, \sin k=0$ so $k=n\pi$ $(n=1,2,\dots)$

$u(x,y)=\sum_1^\infty H_n' e^{-n\pi x}\sin n\pi y$

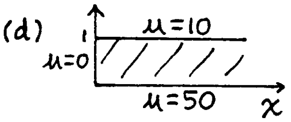
$u_x(0,y)=5=\sum_1^\infty -n\pi H_n'\sin n\pi y \quad (0<y<1)$

HRS:

$-n\pi H_n'=\frac{2}{1}\int_0^1 5\sin n\pi y\,dy$ so $H_n'=\frac{10}{n^2\pi^2}(\cos n\pi-1)=\begin{cases}-20/n^2\pi^2, & n \text{ odd}\\ 0 & , n \text{ even}\end{cases}$

so $u(x,y)=-\frac{20}{\pi^2}\sum_{1,3,\dots}^{\infty}\frac{1}{n^2}e^{-n\pi x}\sin n\pi y.$

(d)



$u(x,y)=(A+Bx)(C+Dy)+(Ee^{kx}+Fe^{-kx})(G\cos ky+H\sin ky)$

u bounded as $x\to\infty \Rightarrow B=E=0$ so

$u(x,y)= C'+D'y+e^{-kx}(G'\cos ky+H'\sin ky)$

$u(x,0)=50=C'+e^{-kx}G' \to C'=50, G'=0$ so

$u(x,y)= 50+D'y+H'e^{-kx}\sin ky$

$u(x,1)=10=50+D'+H'e^{-kx}\sin k \to D'=-40, k=n\pi$ $(n=1,2,\dots)$

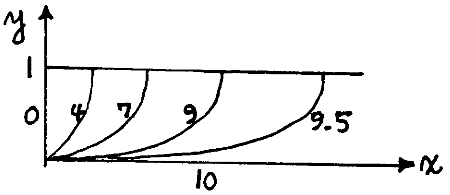
$u(x,y)=50-40y+\sum_1^\infty H_n' e^{-n\pi x}\sin n\pi y$

$u(0,y)=0=50-40y+\sum H_n'\sin n\pi y$

$40y-50=\sum_1 H_n'\sin n\pi y$

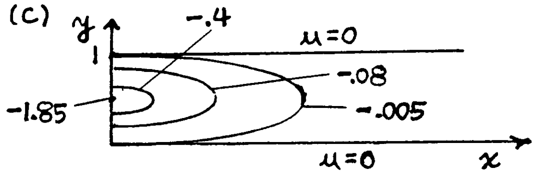
HRS: $H_n'=\frac{2}{1}\int_0^1(40y-50)\sin n\pi y\,dy=\frac{20}{n\pi}[(-1)^n-5]$

11. (a) The $u_y(x,1)=0$ condition implies that the isotherms are vertical at $y=1$. Also, the $y=0$ and $y=1$ b.c.'s show that $u(x,y)\sim 10$ as $x\to\infty$. Thus:



(b) $u=100$ everywhere.

(c)

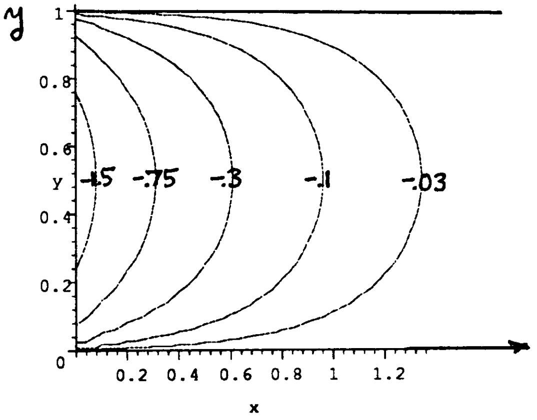


u will be negative at each (x,y) in the interior of the domain. Its largest negative value will be at (0,0.5), and the isotherm values will increase and approach zero as the isotherms penetrate more deeply into the strip. It's difficult to estimate the actual values so I used the solution

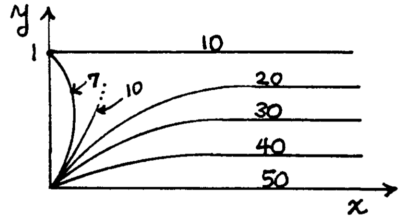
$$u(0,.5) = -\frac{20}{\pi^2}\sum_{1,3,\dots}\frac{1}{n^2}\sin\frac{n\pi}{2}$$

from Exercise 10(c) to determine that $u(0,.5) \approx -1.85$. Thus, at least qualitatively, the isothermal values will be somewhat as noted in the sketch, above. Let us check with a computer plot using Maple.

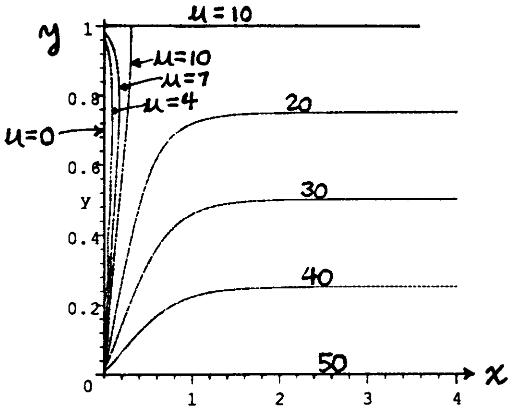
```
> u:=-(20/Pi^2)*sum(exp(-(2*i-1)*Pi*x)*sin((2*i-1)*Pi*y)/(2*i-1)^2,i
  =1..20):
> implicitplot({u=-1.5,u=-.75,u=-.3,u=-.1,u=-.03},x=0..2.5,y=0..1);
```



(d) This one is trickier. Isotherms between 0 and 10 extend from (0,0) to (0,1) and those between 10 and 50 asymptote (as $x\to\infty$) to a linear variation in y (see sketch), but as we move to the left along $y=0.8$, say, u falls from 20 to 0. Thus, it must pass through 10, so there must be a $u=10$ isotherm in the interior that starts at the origin and "heads north". Does that isotherm intersect the line $y=1$ or does it come in to the corner (0,1)? Attempting some sketches the former seems more reasonable, but to be sure let us do a Maple plot:



```
> with(plots):
> u:=50-40*y+sum((20/(i*Pi))*((-1)^i-5)*exp(-i*Pi*x)*sin(i*Pi*y),i=1
  ..20):
> implicitplot({u=4,u=7,u=10,u=20,u=30,u=40},x=0..4,y=0..1,grid=[400
  ,100]);
```



Thus, we see that the $u=10$ isotherm does intersect the line $y=1$ rather than coming in to the corner $(0,1)$. NOTE: Aside from the latter occurrence the isotherms begin/end at the corners $(0,0)$ and $(0,1)$. They don't quite do that in the figure but that is because our calculation sums only the first 20 terms of the series.

12. Observe that the $e^{-(x/10b)^2}$ factor is a slowly varying function of x. For instance, it diminishes from 1 to $e^{-1}=0.368$ only after x increases from 0 to 10b (i.e. ten widths, the width b being the natural length scale). Then, approximately, we can neglect the u_{xx} term in the PDE, which becomes

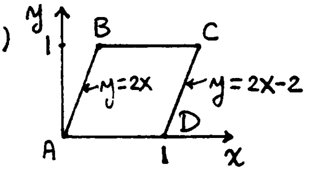
$$u_{yy} \approx 0 \rightarrow u = Ay+B,$$

where A, B can be slowly-varying functions of x.

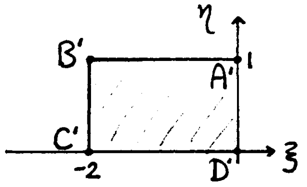
$$\left.\begin{array}{l} u(x,0)=0=0+B \\ u(x,b)=50e^{-(x/10b)^2}=Ab+B \end{array}\right\} \rightarrow \begin{array}{l} B=0, \\ A=50e^{-(x/10b)^2}/b, \end{array}$$

so $u(x,y) \approx 50\,e^{-(x/10b)^2}(y/b)$.

14. (a)



$$\left.\begin{array}{l} AB:\ y=2x,\ \text{so}\ \xi=0,\ 0<\eta<1 \\ BC:\ y=1,\ \text{so}\ \eta=1,\ -2<\xi<0 \\ CD:\ y=2x-2,\ \text{so}\ \xi=-2,\ 0<\eta<1 \\ AD:\ y=0,\ \text{so}\ \eta=0,\ -2<\xi<0 \end{array}\right\} \longrightarrow$$



(b) $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = -2\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

so $u_{xx}+u_{yy} = (-2\frac{\partial}{\partial \xi})(-2\frac{\partial}{\partial \xi})u + (\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta})(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta})u = 4u_{\xi\xi} + (\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta})(u_\xi+u_\eta)$

$$= 5u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}=0$$

Then, $u = X(\xi)Y(\eta)$ gives

$$5\frac{X''}{X}+2\frac{X'}{X}\frac{Y'}{Y}+\frac{Y''}{Y}=0$$

and because of this "mixed" term we are unable to complete the separation.

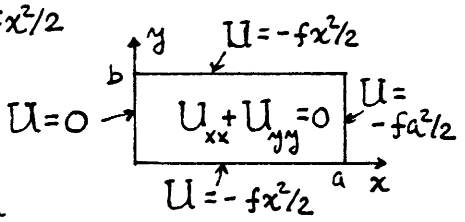
15. (a) With $u = \frac{f}{2}x^2 + U$, $u_{xx}+u_{yy} = f + U_{xx} + U_{yy} = f$ gives $U_{xx}+U_{yy}=0$

Then, $u(0,y)=0=0+U(0,y)$ gives $U(0,y)=0$,

$u(a,y)=0=fa^2/2+U(a,y)$ gives $U(a,y)=-fa^2/2$,

$u(x,0)=0=fx^2/2+U(x,0)$ gives $U(x,0)=-fx^2/2$,

and $u(x,b)=0=fx^2/2+U(x,b)$ gives $U(x,b)=-fx^2/2$



so the U problem is as summarized at the right. Of the b.c.'s the N (north) and S (south) are functions and the E and W are constants, so we will need Fourier series in $\underline{\underline{x}}$. Thus, with $U(x,y)=X(x)Y(y)$, set $\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$, so

$$U(x,y) = (A+Bx)(C+Dy) + (E\cos kx + F\sin kx)(G\cosh ky + H\sinh ky)$$
$$U(0,y)=0= A(\;''\;) + E(\;''\quad''\;) \rightarrow A=E=0$$
$$U(x,y) = x(C'+D'y) + \sin kx\,(G'\cosh ky + H'\sinh ky)$$
$$U(a,y) = -fa^2/2 = a(C'+D'y) + \sin ka(\;''\quad''\;) \rightarrow C'=-fa/2,\ D'=0,$$
$$k=n\pi/a\ (n=1,2,\ldots)$$

$$U(x,y) = -\frac{fax}{2} + \sum_1^\infty \sin\frac{n\pi x}{a}\left(G_n'\cosh\frac{n\pi y}{a} + H_n'\sinh\frac{n\pi y}{a}\right) \qquad ①$$

$$U(x,0) = -f\frac{x^2}{2} = -\frac{fa}{2}x + \sum_1^\infty G_n'\sin\frac{n\pi x}{a} \qquad ②$$

or, $$\frac{f}{2}x(a-x) = \sum_1 G_n'\sin\frac{n\pi x}{a} \qquad (0<x<a)$$

HRS: $$G_n' = \frac{2}{a}\int_0^a \frac{f}{2}x(a-x)\sin\frac{n\pi x}{a}\,dx = \frac{f}{a}\int_0^a (ax-x^2)\sin\frac{n\pi x}{a}\,dx \qquad ③$$

$$U(x,b) = -f\frac{x^2}{2} = -\frac{fa}{2}x + \sum_1^\infty \left(G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} \qquad ④$$

Comparing ② and ④ gives $G_n' = G_n'\cosh\frac{n\pi b}{a} + H_n'\sinh\frac{n\pi b}{a}$ so

$$H_n' = \frac{1-\cosh(n\pi b/a)}{\sinh(n\pi b/a)}\,G_n' \qquad ⑤$$

and so $U(x,y)$ is given by ①, ③, ⑤.

(b) Putting (15.4) and (15.6) into $u_{xx}+u_{yy}=f$ gives

$$\sum_1^\infty g_n''\sin\frac{n\pi y}{b} + \sum_1^\infty \left(-\frac{n\pi}{b}\right)^2 g_n\sin\frac{n\pi y}{b} = \sum_1^\infty f_n\sin\frac{n\pi y}{b}$$

so equating coefficients of sine terms gives

$$g_n'' - \left(\frac{n\pi}{b}\right)^2 g_n = f_n.$$

Then, $u(0,y)=0=\sum_1^\infty g_n(0)\sin\frac{n\pi y}{b} \rightarrow g_n(0)=0$

and $u(a,y)=0=\sum_1^\infty g_n(a)\sin\frac{n\pi y}{b} \rightarrow g_n(a)=0.$

(c) If $f(x,y)=xy$ then $f_n(x)=\frac{2}{b}\int_0^b xy\sin\frac{n\pi y}{b}\,dy = -\frac{2b(-1)^n}{n\pi}x$
so

$$g_n''-\left(\frac{n\pi}{b}\right)^2 g_n = -\frac{2b(-1)^n}{n\pi}x, \quad g_n(x)=\frac{2(-1)^n b^3}{n^3\pi^3}x + A\sinh\frac{n\pi x}{b}+B\cosh\frac{n\pi x}{b}$$

$$g_n(0)=0=B, \quad g_n(a)=0=\frac{2(-1)^n b^3}{n^3\pi^3}a + A\sinh\frac{n\pi a}{b} \text{ gives } A=\frac{-2(-1)^n ab^3}{n^3\pi^3\sinh\frac{n\pi a}{b}}$$

so

$$u(x,y)=\frac{2b^3}{\pi^3}\sum_1^\infty \frac{(-1)^n}{n^3}\left[x-a\frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)}\right]\sin\frac{n\pi y}{b}$$

16. (a) $u(x,y,z)=X(x)Y(y)Z(z)$ gives $\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$. Anticipating an expansion on x and y we seek sines and cosines in x and y. Thus, write $\frac{Y''}{Y}+\frac{Z''}{Z}=-\frac{X''}{X}=\alpha^2$,

and $\frac{Y''}{Y}=-\frac{Z''}{Z}+\alpha^2=-\beta^2$

so (omitting the special cases that give "ramp" terms, since the homogeneous b.c.'s on all faces except $z=c$ will, no doubt, knock out those terms)

$$X=A\cos\alpha x+B\sin\alpha x$$
$$Y=C\cos\beta y+D\sin\beta y$$
$$Z=E\cosh\sqrt{\alpha^2+\beta^2}\,z+F\sinh\sqrt{\alpha^2+\beta^2}\,z.$$

Now, $u(0,y,z)=0\to A=0$

$u(a,y,z)=0\to \alpha=m\pi/a \quad (m=1,2,\dots)$

$u(x,0,z)=0\to C=0$

$u(x,b,z)=0\to \beta=n\pi/b \quad (n=1,2,\dots)$

$u(x,y,0)=0\to E=0,$

so

$$u(x,y,z)=\sum_{m=1}^\infty\sum_{n=1}^\infty G_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sinh\omega_{mn}z$$

where

$$\omega_{mn}=\pi\sqrt{(m/a)^2+(n/b)^2}.$$

Finally,

$$u(x,y,c)=f(x,y)=\sum_{m=1}^\infty\sum_{n=1}^\infty (G_{mn}\sinh\omega_{mn}c)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

gives (by HR sine series on x and y) (16.4) for G_{mn}.

(b)

$$G_{mn}=\frac{400}{a^2\sinh(\pi\sqrt{m^2+n^2})}\int_0^a\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy$$

$$=\frac{1600}{\pi^2}\frac{1}{mn\sinh(\pi\sqrt{m^2+n^2})} \quad \text{if } m,n \text{ are both odd, 0 otherwise}$$

Using Maple on

$$u(a/2,a/2,a/2) = \frac{1600}{\pi^2}\sum_{1,3,..}^{\infty}\sum_{1,3,..}^{\infty} \frac{\sin m\pi/2 \,\sin n\pi/2}{mn}\;\frac{\sinh\left(\frac{\pi}{2}\sqrt{m^2+n^2}\right)}{\sinh\left(\pi\sqrt{m^2+n^2}\right)}$$

```
> u:=(1600/Pi^2)*sum(sum(sin((2*i-1)*Pi/2)*sin((2*j-1)*Pi/2)*sinh(Pi
  *sqrt((2*i-1)^2+(2*j-1)^2)/2)/((2*i-1)*(2*j-1)*sinh(Pi*sqrt((2*i-1
  )^2+(2*j-1)^2))),i=1..8),j=1..8):
> evalf(u);
                              16.66666666
```

The convergence was rapid; summing $i=1..2, j=1..2$ gave 16.6479, $i=1..4, j=1..4$ gave 16.666647, and $i=1..8, j=1..8$ gave

$$u(a/2,a/2,a/2) = 16.66666666$$

which does not change with the inclusion of more terms.

17. $\int_V \nabla^2 u\, dV = \int_V \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla} u)\, dV = \int_S \hat{\underset{\sim}{n}}\cdot \underset{\sim}{\nabla} u\, dA$ by divergence theorem

$= \int_S \frac{\partial u}{\partial n} dA$ by directional derivative formula,

so $\int_S \frac{\partial u}{\partial n} dA = \int_V f\, dV$

18. (a) Green's 1st identity: $\int_V (\underset{\sim}{\nabla} u\cdot \underset{\sim}{\nabla} v + u\nabla^2 v)\, dV = \int_S u \frac{\partial v}{\partial n} dA$ ①

$\nabla^2 u_1 = f$ in V

$\nabla^2 u_2 = f$ " "

$\nabla^2 u_1 - \nabla^2 u_2 = f - f$

or, $\nabla^2(u_1 - u_2) = 0$, or, $\nabla^2 w = 0$ in V ②

$u_1 = g$ on S

$u_2 = g$ on S

$u_1 - u_2 = g - g$ or $w = 0$ on S ③

Then, letting "u"="v"$=w$ in ①, and using ② and ③ gives

$$\int_V (\nabla w\cdot \nabla w + w \underbrace{\nabla^2 w}_{0})\, dV = \int_S \underbrace{w}_{0} \frac{\partial w}{\partial n} dA \quad ④$$

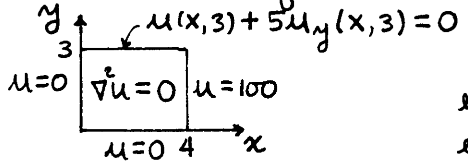
so $\int_V \underset{\sim}{\nabla} w\cdot \underset{\sim}{\nabla} w\, dV = \int_V (w_x^2 + w_y^2 + w_z^2)\, dV = 0$ ⑤

so $w_x = 0, w_y = 0, w_z = 0$ in V. Thus, $w =$ constant in V, and $w = 0$ on S implies that that constant is zero. Thus, $w(x,y,z) \equiv 0$ so $u_1(x,y,z) = u_2(x,y,z)$ and the solution is unique.

(b) Again we arrive at ④ and ⑤, so $w =$ constant in V, but this time we cannot argue that that constant is zero due to w being 0 on S, so all we can conclude is that $w =$ const. That is, two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$ differ by at most a constant.

(c) Again we arrive at ④: $\int_V (\underset{\sim}{\nabla} w\cdot \underset{\sim}{\nabla} w + w \underbrace{\nabla^2 w}) dV = \int_S w \frac{\partial w}{\partial n} dA = 0$ because w is zero on part of S and $\partial w/\partial n$ is zero on the rest of S. Thus, ⑤ holds again so $w =$ constant. But $w=0$ on part of S, so that constant is zero. Thus, $w \equiv 0$ so $u_1 - u_2 = 0$, $u_1 = u_2$, and the solution is unique.

19.



Anticipating the Fourier series (i.e., the eigenfunction series) expansion on the $x=4$ edge we write

$$\frac{X''}{X} = -\frac{Y''}{Y} = +k^2$$

so $u(x,y) = (A+Bx)(C+Dy) + (E\cosh kx + F\sinh kx)(G\cos ky + H\sin ky)$.

$u(0,y) = 0 = \quad A(\ " \quad) + \qquad\qquad E \ (\ " \qquad " \ ) \rightarrow A=E=0$

$u(x,y) = x(C'+D'y) + \sinh kx\,(G'\cos ky + H'\sin ky)$

$u(x,0) = 0 = C'x + G'\sinh kx \ \rightarrow \ C' = G' = 0,$

$u(x,y) = \ D'xy + H'\sinh kx \sin ky$

$u(x,3) + 5u_y(x,3) = 0 = \ 3D'x + H'\sin 3k \sinh kx$
$\qquad\qquad\qquad\qquad + 5D'x + 5kH'\cos 3k \sinh kx$

implies $D'=0$ and the characteristic equation

$$\sin 3k + 5k\cos 3k = 0.$$

Denoting the successive roots as k_n $(n=1,2,\dots)$, the Maple fsolve command gives $k_1 = 0.6266$, $k_2 = 1.6119$, $k_3 = 2.6432$, $k_4 = 3.6833$, $k_5 = 4.7265$.

H_n' remains arbitrary, so

$$u(x,y) = \sum_1^\infty H_n' \sinh k_n x \sin k_n y$$

Finally,

$$u(4,y) = 100 = \sum^\infty H_n' \sinh 4k_n \sin k_n y$$

The St.-Liou. problem is

$$Y'' + k^2 Y = 0 \quad (0<y<3)$$
$$Y(0)=0, \ Y(3) + 5Y'(3) = 0$$

with eigens $\lambda_n = k_n^2$ and $\phi_n(y) = \sin k_n y$. Thus,

$$H_n' \sinh 4k_n = \frac{\langle 100, \sin k_n y\rangle}{\langle \sin k_n y, \sin k_n y\rangle} = \frac{100\int_0^3 \sin k_n y\,dy}{\int_0^3 \sin^2 k_n y\,dy}$$

gives the H_n''s: $H_1' = 19.74$, $H_2' = 14.70$, $H_3' = 3.84$, $H_4' = 0.86$, $H_5' = 0.26, \dots$

so $u(2,1) \simeq 18.62 + 1.40 + 0.07 - 0.01 - \cdots \ \approx 20.08$.

NOTE: Since 3 of the 4 b.c.'s are homogeneous it is natural to anticipate the expansion to occur on the nonhomogeneous b.c., at $x=4$. However, the "ramp" term from $k=0$ is quite capable of handling both conditions $u(0,y)=0$ and $u(4,y)=0$ so we can expand on the $y=0$ and $y=3$ edges instead. The advantage of doing it this way is that the associated Sturm-Liouville problem will be simpler and, indeed, the expansion will merely be a half-range sine series. Let us go through it. The key point is that – anticipating the expansions to be on the $y=0$ and $y=3$ edges (hence on the x variable) – we write

$$\frac{X''}{X}=-\frac{Y''}{Y}=-k^2$$

and apply the $x=0$ and $x=4$ b.c.'s first. Thus,

$$u(x,y)=(A+Bx)(C+Dy)+(E\cos kx+F\sin kx)(G\cosh ky+H\sinh ky)$$
$$u(0,y)=0=A(\ \text{"}\ )+\qquad E(\ \text{"}\qquad\text{"}\ )\rightarrow A=E=0$$
$$u(x,y)=x(C'+D'y)+\sin kx(G'\cosh ky+H'\sinh ky)$$
$$u(4,y)=100=4C'+4D'y+\sin 4k(\ \text{"}\qquad\text{"}\ )\rightarrow C'=25,\ D'=0,\ k=n\pi/4$$
$$u(x,y)=25x+\sum_1^\infty \sin\frac{n\pi x}{4}\left(G_n'\cosh\frac{n\pi y}{4}+H_n'\sinh\frac{n\pi y}{4}\right)\qquad \text{①}$$

$$u(x,0)=0=25x+\sum_1^\infty G_n'\sin\frac{n\pi x}{4}$$

or,
$$-25x=\sum_1^\infty G_n'\sin\frac{n\pi x}{4},\quad (0<x<4)\qquad \text{②}$$

which is an eigenfunction expansion of $-25x$ in terms of the eigenfunctions $\sin(n\pi x/4)$ of the St.-Liou. problem

$$X''+k^2X=0\quad(0<x<4)$$

$X(0)=0,\ X(4)=0$ (not 100; the $25x$ "ramp term" in ① handles the 100)

but it is also simply a HR sine series, so

$$G_n'=\frac{2}{4}\int_0^4(-25x)\sin\frac{n\pi x}{4}\,dx\qquad \text{③}$$

Finally,

$$u(x,3)+5u_y(x,3)=0=25x+\sum_1^\infty \sin\frac{n\pi x}{4}\left(G_n'\cosh\frac{3n\pi}{4}+H_n'\sinh\frac{3n\pi}{4}\right)+5\sum \sin\frac{n\pi x}{4}\left(\frac{n\pi}{4}\right)\left(G_n'\sinh\frac{3n\pi}{4}+H_n'\cosh\frac{3n\pi}{4}\right)$$

or,
$$-25x=\sum_1^\infty \sin\frac{n\pi x}{4}\left[G_n'\left(\cosh\frac{3n\pi}{4}+\frac{5n\pi}{4}\sinh\frac{3n\pi}{4}\right)+H_n'\left(\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}\right)\right],\qquad \text{④}$$

which is of the same form as ②. In fact, comparing ② and ④ we see that

$$G_n'\left(\cosh\frac{3n\pi}{4}+5\frac{n\pi}{4}\sinh\frac{3n\pi}{4}\right)+H_n'\left(\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}\right)=G_n'$$

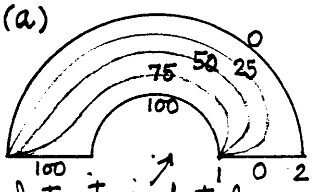
so

$$H_n'=\frac{1-\cosh\frac{3n\pi}{4}-\frac{5n\pi}{4}\sinh\frac{3n\pi}{4}}{\sinh\frac{3n\pi}{4}+\frac{5n\pi}{4}\cosh\frac{3n\pi}{4}}\qquad \text{⑤}$$

and $u(x,y)$ is given by ①-⑤.

Section 20.3

2. (a)



Intuitive sketch.

$$u(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$$
$$u(r,0)=(\;''\;\;''\;)C\;+\;(\;''\;\;\;\;''\;)G\to C=G=0$$
$$u(r,\theta)=(A'+B'\ln r)\theta+(E'r^k+F'r^{-k})\sin k\theta$$
$$u(r,\pi)=100=(A'+B'\ln r)\pi+(\;''\;\;\;''\;)\sin k\pi\to A'=100/\pi,B'=0,k=n$$
$$u(r,\theta)=100\theta/\pi+\sum_1^\infty(E_n'r^n+F_n'r^{-n})\sin n\theta \quad ①$$
$$u(1,\theta)=100=100\theta/\pi+\sum_1(E_n'+F_n')\sin n\theta$$
$$100(\pi-\theta)/\pi=\sum_1^\infty(E_n'+F_n')\sin n\theta$$

HRS: $E_n'+F_n'=\frac{2}{\pi}\int_0^\pi\frac{100}{\pi}(\pi-\theta)\sin n\theta\,d\theta$ ②

$$u(2,\theta)=0=100\theta/\pi+\sum_1^\infty(2^nE_n'+2^{-n}F_n')\sin n\theta$$

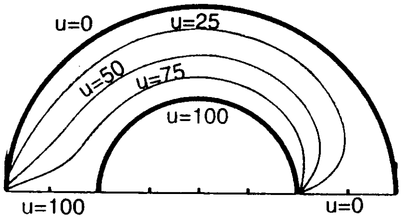
HRS: $2^nE_n'+2^{-n}F_n'=\frac{2}{\pi}\int_0^\pi(-100\theta/\pi)\sin n\theta\,d\theta$ ③

so $u(r,\theta)$ is given by ①, where E_n' and F_n' are found from ② and ③. These give $E_n'=\frac{200}{n\pi}\frac{(-1)^n-2^{-n}}{2^n-2^{-n}}$, $F_n'=\frac{200}{n\pi}\frac{-2^n+(-1)^n}{2^{-n}-2^n}$

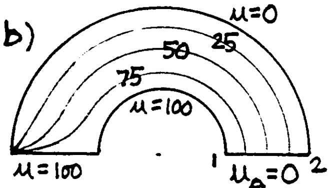
We used these maple commands:

```
f:= (200/Pi^2)*int((Pi-t)*sin(n*t), t=0..π)
g:= (-200/Pi^2)*int(t*sin(n*t), t=0..π);
with(linalg):
A:= array([[1,1],[2^n,2^(-n)]]);
B:= array([f, g]);
linsolve(A,B);
```

Though not asked for, here is a computer plot:



(b)



Sketch ↑

This time the $\theta=0$ edge is insulated: $u_\theta(r,0)=0$.

$$u(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$$
$$u_\theta(r,0)=0=(\;''\;\;)D\;+\;(\;''\;\;\;\;''\;)kH\to D=H=0$$
$$u(r,\theta)=A'+B'\ln r+(E'r^k+F'r^{-k})\cos k\theta$$
$$u(r,\pi)=100=A'+B'\ln r+(\;''\;\;\;''\;)\cos k\pi\to A'=100,B'=0,k=n/2$$

where $n=1,3,\dots$

$$u(r,\theta)=100+\sum_{1,3,\dots}^\infty(E_n'r^{n/2}+F_n'r^{-n/2})\cos(n\theta/2)$$

$$u(1,\theta)=100=100+\sum_{1,3,\dots}^\infty(E_n'+F_n')\cos(n\theta/2)$$

or, $0=\sum_{1,3,\dots}(E_n'+F_n')\cos(n\theta/2)\longrightarrow F_n'=-E_n'$

so

$$u(r,\theta)=100+\sum_{1,3,\dots}^\infty E_n'(r^{n/2}-r^{-n/2})\cos(n\theta/2)$$

NOTE: Since $u_\theta(r,0)=0$, the isotherms are perpendicular to the edge $\theta=0$.

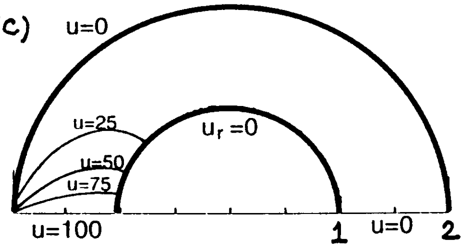
Finally,

$$u(2,\theta)=0=100+\sum_{1,3,\dots}^\infty E_n'(2^{n/2}-2^{-n/2})\cos(n\theta/2)\quad(0<\theta<\pi)$$

so, by QRC series, $E_n'(2^{n/2}-2^{-n/2}) = \frac{2}{\pi}\int_0^{\pi}(-100)\cos\frac{n\theta}{2}\,d\theta = -\frac{400}{n\pi}\sin\frac{n\pi}{2}$,

$$u(r,\theta) = 100 - \frac{400}{\pi}\sum_{1,3,..}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\;\frac{r^{n/2}-r^{-n/2}}{2^{n/2}-2^{-n/2}}\cos\frac{n\theta}{2}$$

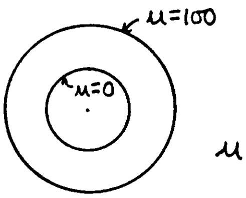
(c)



$$u(r,\theta) = 100\frac{\theta}{\pi} + \frac{200}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{n}\frac{r^n+r^{-n}}{2^n+2^{-n}}$$

NOTE: Since $u_r=0$ on $r=1$, the isotherms are perpendicular to that circle when they reach $r=1$.

(d)



u is 2π-periodic in θ so

$$u(r,\theta) = A+B\ln r + \sum_{1}^{\infty}\left[(C_nr^n+D_nr^{-n})\cos n\theta + (E_nr^n+F_nr^{-n})\sin n\theta\right]$$

$$u(1,\theta) = 0 = A+B\overset{0}{\cancel{\ln 1}} + \sum_{1}^{\infty}\left[(C_n+D_n)\cos n\theta + (E_n+F_n)\sin n\theta\right]$$

$$\Rightarrow A=0,\; C_n+D_n=0,\; E_n+F_n=0 \qquad ①$$

$$u(2,\theta) = 100 = \cancel{A}+B\ln 2 + \sum_{1}\left[(C_n2^n+D_n2^{-n})\cos n\theta + (E_n2^n+F_n2^{-n})\sin n\theta\right]$$

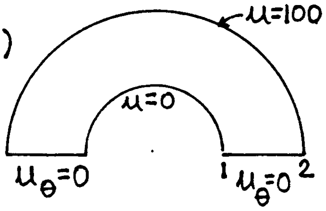
$$\Rightarrow B=100/\ln 2,\; 2^nC_n+2^{-n}D_n=0,\; 2^nE_n+2^{-n}F_n=0 \qquad ②$$

① and ② give $C_n=D_n=E_n=F_n=0$ so we simply have

$$u(r,\theta) = 100\frac{\ln r}{\ln 2}$$

and the $u=25,50,75$ isotherms are the circles $r=2^{1/4}, 2^{1/2}, 2^{3/4}$, respectively.
NOTE: We can see at the outset that u does not vary with θ so all we need is the $A+B\ln r$ part of the solution form.

(e)



$$u(r,\theta) = (A+B\ln r)(C+D\theta) + (Er^k+Fr^{-k})(G\cos k\theta + H\sin k\theta)$$

$$u_\theta(r,0) = 0 = (\;\text{"}\;)D + (\;\text{"}\quad\text{"}\;)Hk \to D=H=0$$

$$u(r,\theta) = A'+B'\ln r + (E'r^k+F'r^{-k})\cos k\theta$$

$$u_\theta(r,\pi) = 0 = (\;\text{"}\quad\text{"}\;)(-k)\sin k\pi \to k=n$$

$$u(r,\theta) = A'+B'\ln r + \sum_{1}^{\infty}(E_n'r^n+F_n'r^{-n})\cos n\theta$$

$$u(1,\theta) = 0 = A'+B'\overset{0}{\cancel{\ln 1}} + \sum_{1}^{\infty}(E_n'+F_n')\cos n\theta \to A'=0,\; E_n'+F_n'=0 \qquad ①$$

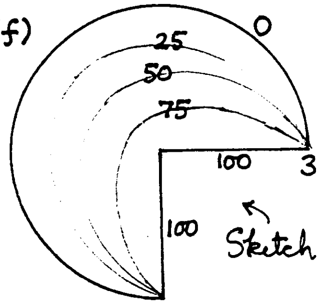
$$u(2,\theta) = 100 = 0+B'\ln 2 + \sum_{1}(E_n'2^n+F_n'2^{-n})\cos n\theta \to B'=100/\ln 2,\; E_n'2^n+F_n'2^{-n}=0 \qquad ②$$

① and ② $\to E_n'=F_n'=0$, so

$$u(r,\theta) = 100\frac{\ln r}{\ln 2},$$

as in (d). In fact, if you've studied the method of images you will see that the problem in (e) is, by that method, equivalent to the one in (d), which had the same simple solution.

(f)



$u(r,\theta) = (A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$

u bdd as $r\to 0 \Rightarrow B=F=0$, so

$u(r,\theta) = C'+D'\theta + r^k(G'\cos k\theta + H'\sin k\theta)$

$u(r,0)=100=C'+r^kG' \to C'=100,\ G'=0$

$u(r,\theta)=100+D'\theta+H'r^k\sin k\theta$

$u(r,3\pi/2)=100=100+\frac{3\pi}{2}D'+H'r^k\sin\frac{3\pi k}{2}$

$\to D'=0,\ 3\pi k/2=n\pi\ (n=1,2,\dots)$

$u(r,\theta)=100+\sum_1^\infty H_n' r^{2n/3}\sin\frac{2n\theta}{3}$

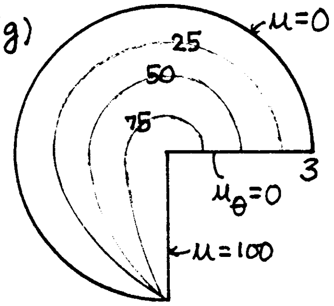
$u(3,\theta)=0=100+\sum_1^\infty H_n' 3^{2n/3}\sin\frac{2n\theta}{3}\quad (0<\theta<\frac{3\pi}{2})$

HRS: $H_n' 3^{2n/3} = \frac{2}{3\pi/2}\int_0^{3\pi/2}(-100)\sin\frac{2n\theta}{3}\,d\theta = \frac{200}{n\pi}(\cos n\pi -1)$

$H_n' = -400/(n\pi 3^{2n/3})$ for n odd, 0 for n even, so

$$u(r,\theta)=100-\frac{400}{\pi}\sum_{1,3,..}^\infty \frac{1}{n}\left(\frac{r}{3}\right)^{2n/3}\sin\frac{2n\theta}{3}$$

(g)



$u(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$

u bdd as $r\to 0 \Rightarrow B=F=0$ so

$u(r,\theta)=C'+D'\theta+r^k(G'\cos k\theta+H'\sin k\theta)$

$u_\theta(r,0)=0=D'+r^kkH' \to D'=H'=0$ so

$u(r,\theta)=C'+G'r^k\cos k\theta$

$u(r,3\pi/2)=100=C'+G'r^k\cos 3\pi k/2 \to C'=100$, and

$3\pi k/2=n\pi/2$ (n odd), or, $k=n/3$

$u(r,\theta)=100+\sum_{1,3,..}^\infty G_n' r^{n/3}\cos\frac{n\theta}{3}$

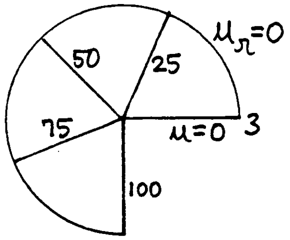
$u(3,\theta)=0=100+\sum_{1,3,..} G_n' 3^{n/3}\cos\frac{n\theta}{3}$

NOTE: Since $u_\theta(r,0)=0$, the isotherms are perpendicular to the edge $\theta=0$.

QRC: $G_n' 3^{n/3}=\frac{2}{3\pi/2}\int_0^{3\pi/2}(-100)\cos\frac{n\theta}{3}\,d\theta = -\frac{400}{n\pi}\sin\frac{n\pi}{2}$

so $u(r,\theta)=100-\frac{400}{\pi}\sum_{1,3,..}^\infty\frac{1}{n}\sin\frac{n\pi}{2}\left(\frac{r}{3}\right)^{n/3}\cos\frac{n\theta}{3}$

(h)



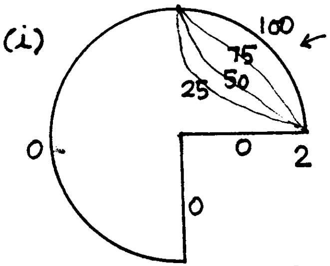
$u(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$

u bdd $\to B=F=0$

$\Downarrow$

$u(r,\theta)=\frac{200}{3\pi}\theta$ and the isotherms are radial lines.

(i)



← Rough sketch. Applying boundedness, we obtain

$u(r,\theta) = A + B\theta + r^k(C\cos k\theta + D\sin k\theta)$

$u(r,0) = 0 = A + r^k C \to A = C = 0$

$u(r,\theta) = B\theta + D r^k \sin k\theta$

$u(r,3\pi/2) = 0 = (3\pi/2)B + D r^k \sin(3\pi k/2)$

so $B = 0$ and $3\pi k/2 = n\pi$, so $k = 2n/3$ $(n = 1,2,\dots)$

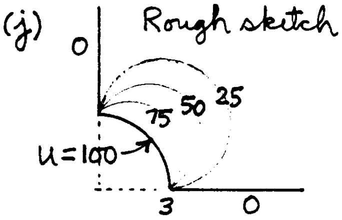
$u(r,\theta) = \sum_1^\infty D_n r^{2n/3} \sin(2n\theta/3)$.

$u(2,\theta) = \sum_1^\infty D_n 2^{2n/3} \sin(2n\theta/3)$ $(0<\theta<3\pi/2)$

HRS: $D_n 2^{2n/3} = \frac{2}{3\pi/2}\int_0^{3\pi/2} u(2,\theta)\sin\frac{2n\theta}{3}\,d\theta = \frac{400}{3\pi}\int_0^{\pi/2}\sin\frac{2n\theta}{3}\,d\theta = \frac{200}{n\pi}\left(1-\cos\frac{n\pi}{3}\right)$

so $u(r,\theta) = \frac{200}{\pi}\sum_1^\infty \frac{1-\cos\frac{n\pi}{3}}{n}\left(\frac{r}{2}\right)^{2n/3}\sin\frac{2n\theta}{3}$

(j)



Rough sketch.

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r\to\infty \Rightarrow B = E = 0$ so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

$u(r,0) = 0 = C' + r^k G' \to C' = G' = 0$

$u(r,\theta) = D'\theta + H' r^{-k}\sin k\theta$

$u(r,\pi/2) = 0 = \pi D'/2 + H' r^{-k}\sin k\pi/2 \to D' = 0,\ k = 2n$ $(n = 1,2,\dots)$

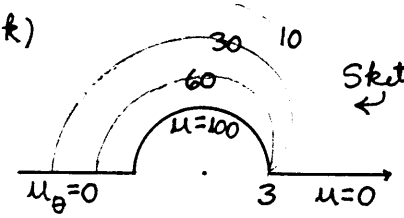
$u(r,\theta) = \sum_1^\infty H_n' r^{-2n}\sin 2n\theta$

$u(3,\theta) = 100 = \sum_1^\infty H_n' 3^{-2n}\sin 2n\theta$ $(0<\theta<\pi/2)$

HRS: $H_n' 3^{-2n} = \frac{2}{\pi/2}\int_0^{\pi/2} 100\sin 2n\theta\,d\theta = \frac{400}{2n\pi}(1-\cos n\pi)$

so $u(r,\theta) = \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\left(\frac{3}{r}\right)^{2n}\sin 2n\theta$

(k)



Sketch. ↙

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r\to\infty \Rightarrow B = E = 0$, so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

$u(r,0) = 0 = C' + G' r^{-k} \longrightarrow C' = G' = 0$

$u(r,\theta) = D'\theta + H' r^{-k}\sin k\theta$

$u_\theta(r,\pi) = 0 = D' + kH' r^{-k}\cos k\pi \to D' = 0,\ k\pi = n\pi/2$ (n odd), so

$u(r,\theta) = \sum_{1,3,\dots}^\infty H_n' r^{-n/2}\sin\frac{n\theta}{2}$

$u(3,\theta) = 100 = \sum_{1,3,\dots}^\infty H_n' 3^{-n/2}\sin\frac{n\theta}{2}$ $(0<\theta<\pi)$

QRS: $H_n' 3^{-n/2} = \frac{2}{\pi}\int_0^\pi 100\sin\frac{n\theta}{2}\,d\theta = 400/n\pi$

so

$u(r,\theta) = \frac{400}{\pi}\sum_{1,3,\dots}^\infty \frac{1}{n}\left(\frac{3}{r}\right)^{n/2}\sin n\theta$.

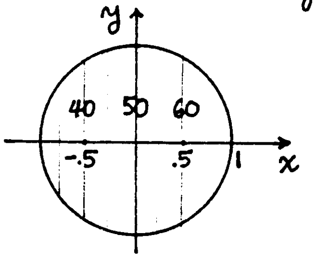
3. In all of these we can use (31) and (33), with $b=1$. Actually, the f's given are simple enough so that it is much easier to evaluate I, P_n, Q_n by matching terms in (32) rather than using (33).

(a) $f(\theta) = 50 + 20\cos\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\Rightarrow I = 50,\ P_1 = 20$, other P_n's and Q_n's $= 0$.

Thus, $u(r,\theta) = 50 + 20r\cos\theta$

$= 50 + 20x$.

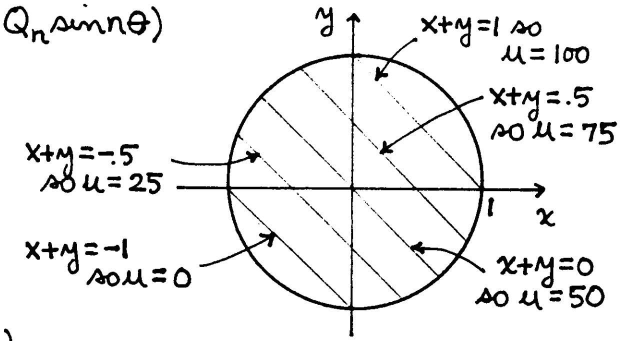


(b) $f(\theta) = 50 + 50(\cos\theta + \sin\theta) = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\Rightarrow I = 50,\ P_1 = Q_1 = 50$, others $= 0$.

$u(r,\theta) = 50 + 50(r\cos\theta + r\sin\theta)$

$= 50 + 50(x+y)$



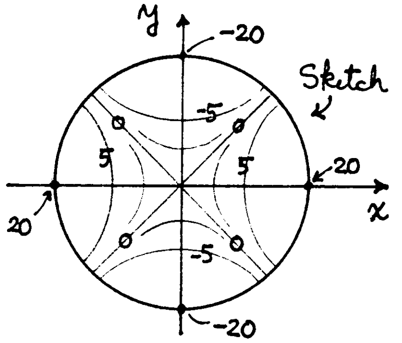
(c) $f(\theta) = 20\cos 2\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

$\rightarrow I = 0,\ P_2 = 20$, others $= 0$.

$u(r,\theta) = 20r^2\cos 2\theta = 20r^2(1 - 2\sin^2\theta)$

$= 20(x^2+y^2) - 40y^2 = 20(x^2 - y^2)$

so the isotherms are a family of hyperbolas, as sketched.



(e) $f(\theta) = 20\cos 3\theta = I + \sum_1^\infty (P_n \cos n\theta + Q_n \sin n\theta)$

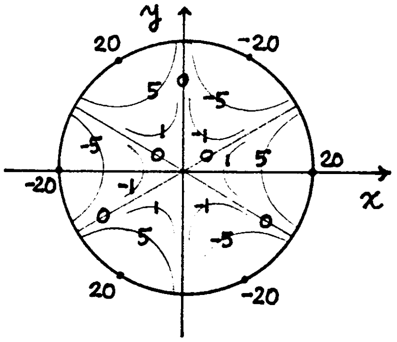
$\rightarrow I = 0,\ P_3 = 20$, others $= 0$.

$u(r,\theta) = 20r^3\cos 3\theta$

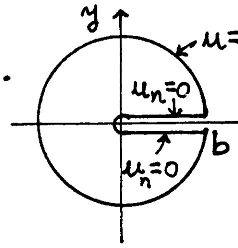
$= 20r^3(4\cos^3\theta - 3\cos\theta)$

$= 80x^3 - 60x(x^2+y^2)$

The $\cos 3\theta$ is 0 along the rays $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$, and the isotherms are as sketched at the right.



4.



On the $\theta=0$ edge $\frac{\partial u}{\partial n} = \underset{\sim}{\nabla u} \cdot \hat{n} = \left(\frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta\right)\cdot(-\hat{e}_\theta)$ $= -\frac{1}{r}\frac{\partial u}{\partial \theta}$, so $\partial u/\partial n = 0$ there implies that $\partial u/\partial \theta = 0$ there. Similarly, on the $\theta = 2\pi$ edge $\partial u/\partial n = \underset{\sim}{\nabla u}\cdot\underset{\sim}{\hat{n}} =$ $\left(\frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta\right)\cdot\hat{e}_\theta = \frac{1}{r}\frac{\partial u}{\partial \theta}$, so $\partial u/\partial n = 0$ there implies that $\partial u/\partial \theta = 0$ there.

$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to 0 \Rightarrow B = F = 0$, so

$u(r,\theta) = C' + D'\theta + r^k(G'\cos k\theta + H'\sin k\theta)$

$\frac{\partial u}{\partial \theta}(r,0) = 0 = D' + kr^k(0 + H') \to D' = H' = 0$ so

$u(r,\theta) = C' + G' r^k \cos k\theta$

$\frac{\partial u}{\partial \theta}(r,2\pi) = 0 = -kG'r'\sin 2\pi k \to 2\pi k = n\pi$ so $k = n/2$ $(n = 1,2,\dots)$

$u(r,\theta) = C' + \sum_1^\infty G'_n r^{n/2} \cos\frac{n\theta}{2}$

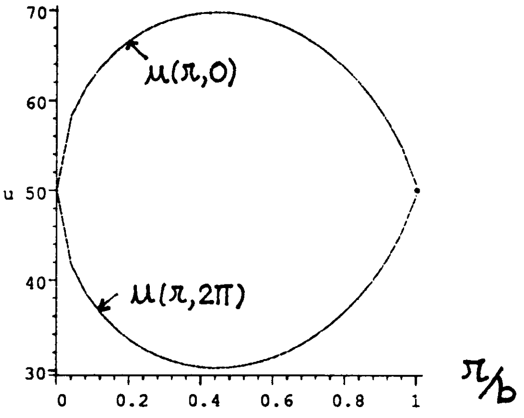
$u(b,\theta) = 50(1+\sin\theta) = C' + \sum_1^\infty G'_n b^{n/2} \cos\frac{n\theta}{2} \quad (0<\theta<2\pi)$

HRC: $\to C' = 50,\ G'_n = -\frac{400 b^{-n/2}}{\pi(n^2-4)}$ for n odd, 0 for n even, so

$u(r,\theta) = 50 - \frac{400}{\pi}\sum_{1,3,..}^{\infty} \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}\cos\frac{n\theta}{2}$

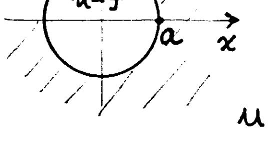
$u(r,0) = 50 - \frac{400}{\pi}\sum_{1,3,..}^{\infty} \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}$

$u(r,2\pi) = 50 + \frac{400}{\pi}\sum_{1,3,..}^{\infty} \frac{1}{n^2-4}\left(\frac{r}{b}\right)^{n/2}$



No, the field $u(r,\theta)$ is insensitive to the material (steel, brass, ...). The only place the nature of the specific material enters is in the diffusivity α^2 in the diffusion equation $\alpha^2\nabla^2 u = u_t$. The larger the diffusivity the faster u approaches steady state. Once at steady state, however, $u_t \to 0$ so $\alpha^2\nabla^2 u = 0$ and α^2 cancels out. Thus, the steady state temperature fields discussed in this chapter are completely insensitive to the specific material (i.e., to the diffusivity).

5.



$u(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

u bdd as $r \to \infty \Rightarrow B = 0, E = 0$ so

$u(r,\theta) = C' + D'\theta + r^{-k}(G'\cos k\theta + H'\sin k\theta)$

u 2π-periodic in $\theta \Rightarrow D' = 0, k = n$, so

$u(r,\theta) = C' + \sum_1^\infty r^{-n}(G'_n\cos n\theta + H'_n \sin n\theta)$ ①

$$u(a,\theta) = f(\theta) = C' + \sum_1^\infty a^{-n}(G_n' \cos n\theta + H_n' \sin n\theta)$$

so $C' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$, $a^{-n}G_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$ so $G_n' = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$,

$a^{-n}H_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$ so $H_n' = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$

As $r\to\infty$, ① gives $u(r,\theta) \sim C' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$, which is the average value of f.

6.(a) $\Phi(r,\theta) = (A + B\ln r)(C + D\theta) + (Er^k + Fr^{-k})(G\cos k\theta + H\sin k\theta)$

$\Phi_\theta(r,0) = 0 = (\quad '' \quad)D \quad + (\quad '' \quad '' \quad)Hk \to D = H = 0,$

$\Phi(r,\theta) = A' + B'\ln r + (E'r^k + F'r^{-k})\cos k\theta$

$\Phi_\theta(r,\pi) = 0 = \qquad (\quad '' \qquad '' \quad)(-k\sin k\pi) \to k = n \; (n = 1,2,\ldots)$

$\Phi(r,\theta) = A' + B'\ln r + \sum_1^\infty (E_n' r^n + F_n' r^{-n})\cos n\theta$

$\Phi_r(a,\theta) = 0 = \frac{B'}{a} + \sum_1^\infty (nE_n' a^{n-1} - nF_n' a^{-n-1})\cos n\theta \to B' = 0,\; F_n' = a^{2n}E_n'$

so

$$\Phi(r,\theta) = A' + \sum_1^\infty E_n'\left(r^n + \frac{a^{2n}}{r^n}\right)\cos n\theta$$

Finally, as $r\to\infty$

$$\Phi(r,\theta) = A' + E_1'\left(r + \frac{a^2}{r}\right)\cos\theta + E_2'\left(r^2 + \frac{a^4}{r^2}\right)\cos 2\theta + \cdots \sim Ur\cos\theta$$

implies A' = arbitrary, $E_1' = U$, $E_2' = E_3' = \cdots = 0$. The reasoning is as follows. Suppose $E_4' = E_5' = \cdots = 0$, say. Then the dominant term in Φ, as $r\to\infty$, is the $E_3'r^3$ term. Then Φ would be $\sim E_3'r^3$, which cannot (by any choice of E_3') be matched with $Ur\cos\theta$. Thus we need $E_3' = 0$. But then Φ would be $\sim E_2'r^2$, which is still too big as $r\to\infty$. Thus we need $E_2' = 0$. Then we have

$$\Phi(r,\theta) = A' + E_1'\left(r + \frac{a^2}{r}\right)\cos\theta \sim E'r\cos\theta \quad \text{as } r\to\infty,$$

for any value of A'. Finally, we can match $E'r\cos\theta$ with $Ur\cos\theta$ by choosing $E_1' = U$. Thus we obtain

$$\Phi(r,\theta) = A' + U\left(r + \frac{a^2}{r}\right)\cos\theta.$$

The arbitrary constant A' can be set $= 0$ without loss since it will drop out anyway when we take the gradient of Φ to obtain the velocity field. It is easily verified that (6.1) does indeed satisfy all the requirements in (38) of Section 16.10.

(b) $\Phi = U\left(r + \frac{a^2}{r}\right)\cos\theta = Ux + Ua^2x/(x^2+y^2)$

$\Psi_x = -\Phi_y = -Ua^2x\frac{(-1)2y}{(x^2+y^2)^2}$ so $\Psi = 2Ua^2\int\frac{xy\,\partial x}{(x^2+y^2)^2} = -\frac{Ua^2y}{x^2+y^2} + \overset{\overset{\text{arb.}}{\downarrow}}{A(y)}$

Then, $\Psi_y = \Phi_x$ gives

$$-\frac{Ua^2}{x^2+y^2}-\frac{Ua^2y(-1)(2y)}{(x^2+y^2)^2}+A'(y)=U+\frac{Ua^2}{x^2+y^2}+\frac{Ua^2x(-1)(2x)}{(x^2+y^2)^2}$$

or, $-Ua^2(x^2+y^2)+2Ua^2y^2+A'(y)(x^2+y^2)^2=U(x^2+y^2)^2+Ua^2(x^2+y^2)-2Ua^2x^2$
or, after cancellations, $A'(y)=U$, so $A(y)=Uy+\text{const}$. Thus,

$$\Psi(x,y)=-\frac{Ua^2y}{x^2+y^2}+Uy+\cancelto{0}{\text{const}}$$

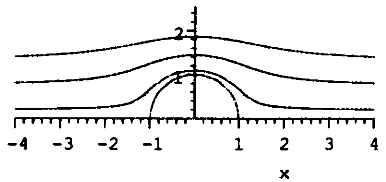
$\Psi(-.4,.2)=0.1875$, $\Psi(-4,.8)=0.7519$, $\Psi(-4,1.4)=1.3220$, $\Psi(-4,2)=1.9$.

Maple:
```
> with(plots):
> p:=y-y/(x^2+y^2);
```

$$p := y-\frac{y}{x^2+y^2}$$

```
> implicitplot({p=.1875,p=.7519,p=1.322,p=1.9,x^2+y^2=1},x=-4..4,y=0
  ..2,view=[-4..4,0..8]);
```

gives



NOTE: If we don't include the view option then the view will be $-4<x<4$ on x, but only $0<y\lesssim 2$ since the uppermost streamline reaches only $y\approx 2$. Since the print will be square then the x axis will appear compressed and the y axis elongated; e.g., the circle $r=1$ will be a tall and narrow ellipse. To keep the same x,y scales we need to force the printed y-interval to be $0<y<8$ (although I cut off the upper part by hand), which was accomplished by the view option.

7. (a) $\Phi(r,\theta)=(A+B\ln r)(C+D\theta)+(Er^k+Fr^{-k})(G\cos k\theta+H\sin k\theta)$

$\Phi_r(a,\theta)=0=\frac{B}{a}(C+D\theta)+k(Ea^{k-1}-Fa^{-k-1})(\quad ''\qquad ''\quad)\rightarrow B=0$ and $F=a^{2k}E$, so

$\Phi(r,\theta)=C'+D'\theta+(r^k+a^{2k}r^{-k})(G'\cos k\theta+H'\sin k\theta)$

$$\left.\begin{aligned}&\Phi(r,2\pi)-\Phi(r,0)=-\Gamma=2\pi D'+(r^k+a^{2k}r^{-k})(G'\cos 2\pi k+H'\sin 2\pi k-G')\\&\Phi_\theta(r,2\pi)-\Phi_\theta(r,0)=0=(r^k+a^{2k}r^{-k})(-kG'\sin 2\pi k+kH'\cos 2\pi k-kH')\end{aligned}\right\}\text{♀}$$

♀ gives $D'=-\Gamma/2\pi$ and $\begin{pmatrix}c-1 & s\\ -s & c-1\end{pmatrix}\begin{pmatrix}G'\\H'\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ where $c=\cos 2\pi k$, $s=\sin 2\pi k$. To avoid $G'=H'=0$, set

$$\begin{vmatrix}c-1 & s\\ -s & c-1\end{vmatrix}=(c-1)^2+s^2=0,\text{ so }\left.\begin{matrix}\cos 2\pi k=1\\ \text{and }\sin 2\pi k=0\end{matrix}\right\}\Rightarrow k=n\quad(n=1,2,\dots)$$

with G' and H' arbitrary. Thus far, then,

$$\Phi(r,\theta)=-\frac{\Gamma}{2\pi}\theta+\sum_1^\infty\left(r^n+\frac{a^{2n}}{r^n}\right)(G_n'\cos n\theta+H_n'\sin n\theta)+\cancelto{\text{can set}=0}{C'}$$

Finally, $\Phi(r,\theta)\sim Ur\cos\theta$ as $r\to\infty\Rightarrow G_1'=U$ and all other G_n''s and H_n''s are zero, so

$$\Phi(r,\theta)=U\left(r+\frac{a^2}{r}\right)\cos\theta-\frac{\Gamma}{2\pi}\theta.$$

(b) Set $\underset{\sim}{v} = \underset{\sim}{\nabla}\Phi = \Phi_r \hat{e}_r + \frac{1}{r}\Phi_\theta \hat{e}_\theta = \underbrace{U(1-\frac{a^2}{r^2})\cos\theta}_{v_r} \hat{e}_r + \underbrace{[-U(r+\frac{a^2}{r})\sin\theta - \frac{\Gamma}{2\pi}]\frac{1}{r}}_{v_\theta}\hat{e}_\theta = \underset{\sim}{0}$

$v_r = 0$ gives $r = a$ or $\theta = \pi/2$ or $\theta = 3\pi/2$. Consider these one at a time:

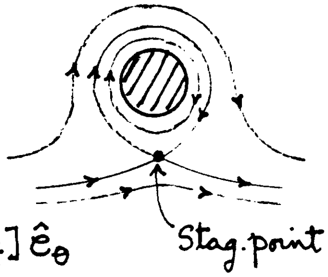
$r=a$: Then $v_\theta = 0$ gives $\theta = \sin^{-1}(-\Gamma/4\pi U a)$ provided that $\Gamma \leq 4\pi U a$ (let us consider Γ to be ≥ 0; if it is < 0 the story is essentially the same but with the swirl counterclockwise and the "lift" force downward instead of upward)

$\theta = \pi/2$: Then $v_\theta = 0$ gives $r^2 + (\Gamma/2\pi U)r + a^2 = 0$, but this has no positive roots.

$\theta = 3\pi/2$: Then $v_\theta = 0$ gives $r^2 - (\Gamma/2\pi U)r + a^2 = 0$ so $r = \frac{\Gamma}{4\pi U} + \sqrt{(\frac{\Gamma}{4\pi U})^2 - a^2}$, which $\to a$ as $\Gamma \to 4\pi U a$.

Thus, we see that when $\Gamma = 0$ there are stagnation points on the cylinder at $\theta = 0, \pi$. As Γ increases the stagnation points move downward on the cylinder (as in the last figure in Exercise 7) and are located at the two roots of $\theta = \sin^{-1}(-\Gamma/4\pi U a)$. When $\Gamma = 4\pi U a$ the two stagnation points merge at $\theta = 3\pi/2$. As Γ increases beyond $4\pi U a$ the stagnation point leaves the surface of the cylinder and moves "south" along $\theta = 3\pi/2$ to $r = (\Gamma/4\pi U) + \sqrt{(\Gamma/4\pi U)^2 - a^2}$.

NOTE: An interesting and challenging project consists of seeing what the flow pattern looks like for the "supercritical" case where $\Gamma > 4\pi U a$. Qualitatively, the student should find that the pattern is somewhat (topologically, at least) as we have sketched at the right.



(c) $\underset{\sim}{v} = \underset{\sim}{\nabla}\Phi = U(1-\frac{a^2}{r^2})\cos\theta\, \hat{e}_r + \frac{1}{r}[-U(r+\frac{a^2}{r})\sin\theta - \frac{\Gamma}{2\pi}]\hat{e}_\theta$

$= 0\,\hat{e}_r - (2U\sin\theta + \Gamma/2\pi)\hat{e}_\theta$ on $r = a$.

By Bernoulli, $p|_{r=a} = \text{const.} - (\sigma/2)(2U\sin\theta + \Gamma/2\pi)^2 = \text{const.} - 2U^2\sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta$,

$L = \sigma \int_0^{2\pi} (-2U^2\sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta)(a\sin\theta\, d\theta) = \sigma U \Gamma$

8. After applying boundedness we have

$u(r,\theta) = E' + F'\theta + r^k(G'\cos k\theta + H'\sin k\theta)$ ①

$u(r,0) - u(r,2\pi) = 0 = E' + G'r^k - [E' + 2\pi F' + r^k(G'\cos 2\pi k + H'\sin 2\pi k)]$

$u_\theta(r,0) - u_\theta(r,2\pi) = 0 = F' + kH'r^k - [F' + kr^k(-G'\sin 2\pi k + H'\cos 2\pi k)]$

or,

$-2\pi F' + r^k[(1-c)G' - sH'] = 0 \Rightarrow F' = 0$ and $(1-c)G' - sH' = 0$

and $r^k[sG' + (1-c)H'] = 0$ $sG' + (1-c)H' = 0$,

where $c = \cos 2\pi k$, $s = \sin 2\pi k$. To avoid $G' = H' = 0$ set $\begin{vmatrix} 1-c & -s \\ s & 1-c \end{vmatrix} = 0$ and obtain, as in Exercise 7(a), $k = n$ $(n = 1, 2, \ldots)$, where G' and H' are then arbitrary. Thus ① becomes

$$u(r,\theta) = E' + \sum_1^\infty r^n(G_n'\cos n\theta + H_n'\sin n\theta),$$

which is the same as (31). Then proceed as in Example 2.

9. $\frac{1}{2}+\sum_1^\infty \left(\frac{r}{b}\right)^n \cos n(\vartheta-\theta) = \frac{1}{2}+\mathrm{Re}\sum_1^\infty \left(\frac{r}{b}\right)^n e^{in(\vartheta-\theta)} = \frac{1}{2}+\mathrm{Re}\sum_1^\infty \left(\frac{r}{b}e^{i(\theta-\theta)}\right)^n$

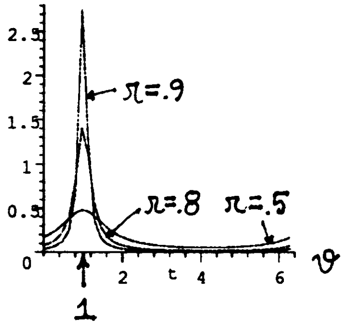
$= \frac{1}{2}-1+\mathrm{Re}\sum_0^\infty \left(\frac{r}{b}e^{i(\theta-\theta)}\right)^n = -\frac{1}{2}+\mathrm{Re}\,\frac{1}{1-\frac{r}{b}e^{i(\vartheta-\theta)}}$

$= -\frac{1}{2}+\mathrm{Re}\,\frac{1}{1-\frac{r}{b}e^{i(\vartheta-\theta)}}\,\frac{1-\frac{r}{b}e^{-i(\vartheta-\theta)}}{1-\frac{r}{b}e^{-i(\vartheta-\theta)}}$ (since we multiply $\frac{1}{a+ib}$ by $\frac{a-ib}{a-ib}$ to get it into the form $c+id$.)

$= -\frac{1}{2}+\frac{1-\frac{r}{b}\cos(\vartheta-\theta)}{1-2\frac{r}{b}\cos(\vartheta-\theta)+\frac{r^2}{b^2}} = \frac{1}{2}\,\frac{b^2-r^2}{b^2-2br\cos(\theta-\theta)+r^2}$. ✓

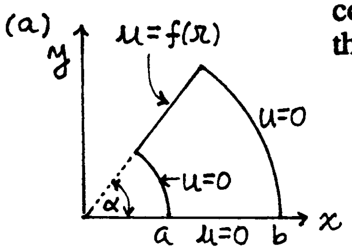
10. $d^2u/dx^2=0$ on $a<x<b$ with $u(a)=u_1$ and $u(b)=u_2$ given. Solving gives $u(x)=u_1+\frac{u_2-u_1}{b-a}(x-a)$. That is, the solution is linear. Thus, surely u at the midpoint $(a+b)/2$ is the average value $(u_1+u_2)/2$, as is easily shown.

12.



(I forgot to include $r=0$, which gives the constant value $P(0,\theta-1)=1/2\pi$.)

13. NOTE: This is a nice problem pedagogically since it falls between the case where the Sturm-Liouville expansion is equivalent to (and could therefore been handled as) a half- or quarter-range cosine or sine series and the more sophisticated cases where the expansion is a series of special functions such as Bessel functions or Legendre polynomials. In this case the eigenfunctions are still elementary functions, but not merely cosines and sines. Also, this problem forms a natural companion for the problem where u is prescribed on one of the circular-arc edges, discussed as Example 1 in this section.



$$\frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = -k^2$$

$$\to u(r,\theta) = (A+B\ln r)(E+F\theta) + [C\cos(k\ln r)+D\sin(k\ln r)][G\cosh k\theta+H\sinh k\theta]$$

Since the St.-Liou. expansion will be in r, we need to be sure to do the $r=a$ and $r=b$ boundary conditions before any expansions in r can be attempted. Actually, it looks like we can also do the $u(r,0)=0$ condition early.

$u(r,0)=0=(A+B\ln r)E+[C\cos(k\ln r)+D\sin(k\ln x)]G \to E=G=0$, so

$u(r,\theta)=(A'+B'\ln r)\theta+[C'\cos(k\ln r)+D'\sin(k\ln r)]\sinh k\theta$

$$u(a,\theta)=0=(A'+B'\ln a)\theta+[C'\cos(\kappa\ln a)+D'\sin(\kappa\ln a)]\sinh\kappa\theta$$
$$u(b,\theta)=0=(A'+B'\ln b)\theta+[C'\cos(\kappa\ln b)+D'\sin(\kappa\ln b)]\sinh\kappa\theta$$

so $\left.\begin{array}{l}A'+B'\ln a=0\\ A'+B'\ln b=0\end{array}\right\}\to A'=B'=0.$

and
$$C'\cos(\kappa\ln a)+D'\sin(\kappa\ln a)=0 \qquad ①$$
$$C'\cos(\kappa\ln b)+D'\sin(\kappa\ln b)=0 \qquad ②$$

To avoid $C'=D'=0$, set
$$\begin{vmatrix}\cos(\kappa\ln a) & \sin(\kappa\ln a)\\ \cos(\kappa\ln b) & \sin(\kappa\ln b)\end{vmatrix}=\cos(\kappa\ln a)\sin(\kappa\ln b)-\cos(\kappa\ln b)\sin(\kappa\ln b)$$
$$=\sin(\kappa\ln a-\kappa\ln b)$$
$$=\sin(\kappa\ln\tfrac{a}{b})=0$$

so $\kappa\ln(a/b)=n\pi$ $(n=1,2,\dots)$ or, $\kappa_n=n\pi/\ln(\frac{a}{b})$.

With that choice of κ we now need to solve ① and ② for the resulting nontrivial values of C', D'. With $\kappa_n=n\pi/\ln(\frac{a}{b})$, ② and ① will be redundant, so let us discard ②, say, and solve ① for D' in terms of C':
$$D'=-\cot(\kappa_n\ln a)\,C'.$$

Thus far,
$$u(r,\theta)=\sum_1^\infty C_n'\left[\cos(\kappa_n\ln r)-\cot(\kappa_n\ln a)\sin(\kappa_n\ln r)\right]\sinh\kappa_n\theta$$
$$=\sum_1^\infty C_n'\,\frac{\cos(\kappa_n\ln r)\sin(\kappa_n\ln a)-\cos(\kappa_n\ln a)\sin(\kappa_n\ln r)}{\sin(\kappa_n\ln a)}\sinh\kappa_n\theta$$
$$=\sum_1^\infty I_n\sin(\kappa_n\ln r-\kappa_n\ln a)\sinh\kappa_n\theta$$
$$=\sum_1^\infty I_n\sin\left(\kappa_n\ln\frac{r}{a}\right)\sinh\kappa_n\theta \qquad ③$$

where we've combined $-C_n'/\sin(\kappa_n\ln a)$ as "I_n" for simplicity.

(c) $$u(r,\alpha)=f(r)=\sum_1^\infty(I_n\sinh\kappa_n\alpha)\,\phi_n(r) \qquad (a<r<b) \qquad ④$$

where $\phi_n(r)=\sin(\kappa_n\ln\frac{r}{a})$ are the eigenfunctions of the Sturm-Liouville problem
$$r^2R''+rR'+\kappa^2R=0 \quad (a<r<b) \qquad ⑤$$
$$R(a)=0,\ R(b)=0.$$

To identify the weight function for the inner product multiply ⑤ by $1/r$ so $(rR')'+\kappa^2\frac{1}{r}R=0$. Thus, the weight function is $1/r$, so ③ gives
$$I_n\sinh\kappa_n\alpha=\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}=\frac{\int_a^b f(r)\sin(\kappa_n\ln\frac{r}{a})\frac{1}{r}\,dr}{\int_a^b\sin^2(\kappa_n\ln\frac{r}{a})\frac{1}{r}\,dr}. \qquad ⑥$$

Then the solution is given by ③ and ⑥.

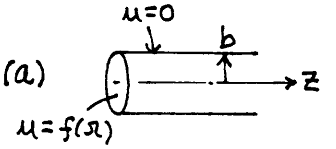
14. $R'' + \frac{1}{r}R' - k^2 R = 0$. To identify a, b, c in (50) (page 239), write out (46) (pg 238): $x^a y'' + ax^{a-1}y' + bx^c y = 0$

or, $y'' + ax^{-1}y' + bx^{c-a}y = 0$

so $a = 1, b = -k^2, c - a = 0$ so $c = a = 1$. Then $\alpha = 2/2 = 1$, $\nu = 0/2 = 0$, so (50) gives
$$R(r) = r^0 Z_0(\sqrt{1 \cdot k^2}\, r) = Z_0(kr).$$
Since $b < 0$, $Z_0 \to I_0, K_0$ so
$$R(r) = CI_0(kr) + DK_0(kr).$$

15. Same as in Exercise 14 but with $-k^2 \to +k^2$. Thus,
$$R(r) = r^0 Z_0(\sqrt{k^2}\, r) = Z_0(kr)$$
$$= CJ_0(kr) + DY_0(kr)$$

16. (a)



Surely the eventual St.-Liou expansion will be on r rather than on z, so write $u(r,z) = R(r)Z(z)$ in $u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$ and obtain

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -k^2.$$ Then $$R = \begin{cases} A + B\ln r, & k = 0 \\ CJ_0(kr) + DY_0(kr), & k \neq 0 \end{cases}$$
$$Z = \begin{cases} E + Fz, & k = 0 \\ Ge^{kz} + He^{-kz}, & k \neq 0 \end{cases}$$

so $u(r,z) = (A + B\ln r)(E + Fz) + \left[CJ_0(kr) + DY_0(kr)\right](Ge^{kz} + He^{-kz})$

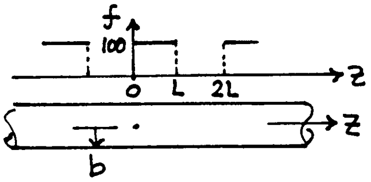
Boundedness as $r \to 0 \Rightarrow B = D = 0$, and bddness as $z \to \infty \Rightarrow G = 0$, so
$$u(r,z) = E' + F'z + C'J_0(kr)e^{-kz}.$$

Then
$$u(b,z) = 0 = E' + F'z + C'J_0(kb)\,e^{-kz} \to E' = F' = 0,\ J_0(kb) = 0$$ with $kb = z_n$ where the z_n's are the known positive roots of $J_0(z) = 0$. Thus,
$$u(r,z) = \sum_1^\infty C_n' J_0\!\left(z_n \frac{r}{b}\right) e^{-z_n z/b} \qquad ①$$

Finally,
$$u(r,0) = f(r) = \sum_1^\infty C_n' J_0\!\left(z_n \frac{r}{b}\right) \qquad (0 \leq r < b)$$

where the $J_0(z_n \frac{r}{b})$'s are the eigenfunctions of the S-L problem
$$rR'' + R' + k^2 rR = 0, \qquad (0 < r < b)$$
$$R(0) \text{ bdd},\ R(b) = 0$$
with weight function r. Thus
$$C_n' = \frac{\langle f, J_0 \rangle}{\langle J_0, J_0 \rangle} = \frac{\int_0^b f(r) J_0(z_n \frac{r}{b})\, r\,dr}{\int_0^b J_0^2(z_n \frac{r}{b})\, r\,dr} = \frac{2}{b^2 J_1^2(z_n)} \int_0^b f(r) J_0(k_n r)\, r\,dr \qquad ②$$
The solution is given by ① and ②.

(b)



This time we anticipate the expansion to be on the z variable so write

$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

so $u(r,z) = (A+B\ln r)(E+Fz) + [CI_0(kr)+DK_0(kr)](G\cos kz + H\sin kz)$

Boundedness as $r \to 0 \Rightarrow B=0$ and $D=0$ and boundedness in z implies that $F=0$, so

$$u(r,z) = A' + I_0(kr)(G'\cos kz + H'\sin kz).$$

Understand that this means

$$u(r,z) = A' + I_0(k_1 r)(G_1'\cos k_1 z + H_1'\sin k_1 z) + \cdots + I_0(k_N r)(G_N'\cos k_N z + H_N'\sin k_N z)$$

for any set of k_j's. Looking ahead to the expansion of $f(z)$ in a classical Fourier series

$$f(z) = a_0 + \sum_1^\infty \left(a_n \cos\frac{n\pi z}{L} + b_n \sin\frac{n\pi z}{L}\right)$$

$$a_0 = \text{ave. value} = 50, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(z)\cos\frac{n\pi z}{L}\,dz = \frac{100}{L}\int_0^L \cos\frac{n\pi z}{L}\,dz = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(z)\sin\frac{n\pi z}{L}\,dz = \frac{100}{L}\int_0^L \sin\frac{n\pi z}{L}\,dz = \begin{cases} 200/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

we can see that we should (and can) choose the k's as $n\pi/L$. Thus, write

$$u(r,z) = A' + \sum_1^\infty I_0(n\pi r/L)\left(G_n'\cos\frac{n\pi z}{L} + H_n'\sin\frac{n\pi z}{L}\right)$$

Finally,

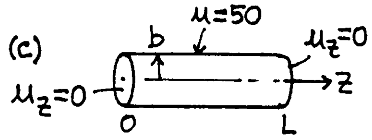
$$u(b,z) = f(z) = 50 + \sum_{1,3,..}^{\infty} \frac{200}{n\pi}\sin\frac{n\pi z}{L} = A' + \sum_1^\infty I_0\left(\frac{n\pi b}{L}\right)\left(G_n'\cos\frac{n\pi z}{L} + H_n'\sin\frac{n\pi z}{L}\right)$$

so $A'=50$, G_n''s all $=0$, $I_0\left(\frac{n\pi b}{L}\right)H_n' = \frac{200}{n\pi}$ for n odd and 0 for n even. Thus

$$u(r,z) = 50 + \frac{200}{\pi}\sum_{1,3,..}^{\infty} \frac{1}{n}\frac{I_0(n\pi r/L)}{I_0(n\pi b/L)}\sin\frac{n\pi z}{L}$$

NOTE: Alternatively, we could have started with the form $u(r,z) = A + \sum_1^\infty \left[B_n(r)\cos\frac{n\pi z}{L} + C_n(r)\sin\frac{n\pi z}{L}\right]$.

(c)

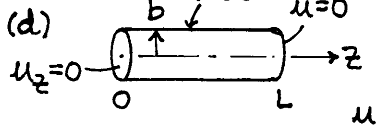


$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

$$\Downarrow$$

$u(r,z) = 50$, which can be seen, by inspection, at the outset.

(d)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

$$u(r,z) = (A+B\ln r)(C+Dz) + [EI_0(kr)+FK_0(kr)](G\cos kz + H\sin kz)$$

u bdd as $r \to 0 \Rightarrow B=F=0$, so

$$u(r,z) = C' + D'z + I_0(kr)(G'\cos kz + H'\sin kz)$$

$$u_z(r,0) = 0 = D' + I_0(kr)\,kH' \to D' = H' = 0, \text{ so}$$

$$u(r,z) = C' + G'I_0(kr)\cos kz$$

$u(r,L)=0=C'+G'I_0(kr)\cos kL \rightarrow C'=0$ and $kL=n\pi/2$ (n odd)

$$u(r,z)=\sum_{1,3,..} G_n' I_0\left(\frac{n\pi r}{2L}\right)\cos\frac{n\pi z}{2L}$$

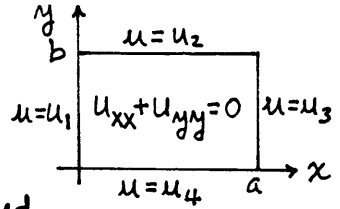
$$u(b,z)=50=\sum_{1,3,..} G_n' I_0\left(\frac{n\pi b}{2L}\right)\cos\frac{n\pi z}{2L} \qquad (0<z<L)$$

and we can use either a quarter-range cosine formula or the St.-Liou. theory. By QRC:

$$G_n' I_0\left(\frac{n\pi b}{2L}\right)=\frac{2}{L}\int_0^L 50\cos\frac{n\pi z}{2L}\,dz=\frac{200}{n\pi}\sin\frac{n\pi}{2}$$

so

$$u(r,z)=\frac{200}{\pi}\sum_{1,3,..}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\,\frac{I_0(n\pi r/2L)}{I_0(n\pi b/2L)}\cos\frac{n\pi z}{2L} \qquad ①$$



NOTE: Recall from Exercise 6 of Sec. 20.2 that for the problem shown at the right we can either write $\frac{X''}{X}=-\frac{Y''}{Y}=+k^2$ (giving cosines and sines on y), apply the u_4 and u_2 bc's first and then do the Fourier series expansions that will be needed on the western and eastern edges OR we can write $\frac{X''}{X}=-\frac{Y''}{Y}=-k^2$ (giving cosines and sines on x), apply the u_1 and u_3 bc's first and then do the Fourier series expansions that will be needed on the southern and northern edges. The resulting series solutions will look different but will sum to the same result. Likewise in this problem we could instead proceed as follows: $\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-k^2$

$$\rightsquigarrow \quad u(r,z)=(A+B\ln r)(C+Dz)+[EJ_0(kr)+FY_0(kr)](G\cosh kz+H\sinh kz)$$

u bdd $\rightarrow$ $u(r,z)=C'+D'z+J_0(kr)(G'\cosh kz+H'\sinh kz)$

Next we must do

$$u(b,z)=50=C'+D'z+J_0(kb)(\;\;''\qquad\;\;''\;\;)$$

so $C'=50$, $D'=0$ and $k_n=z_n/b$ where z_n's are the positive roots of $J_0(z)=0$.

Thus, $u(r,z)=50+\sum_1^\infty J_0(k_n r)(G_n'\cosh k_n z+H_n'\sinh k_n z)$

Then,

$$u_z(r,0)=0=\sum_1^\infty k_n H_n' J_0(k_n r) \rightarrow H_n'\text{'s}=0$$

so

$$u(r,z)=50+\sum_1^\infty G_n' J_0(k_n r)\cosh k_n z \qquad ②$$

Finally,

$$u(r,L)=0=50+\sum_1^\infty G_n' J_0(k_n r)\cosh k_n L$$

or,

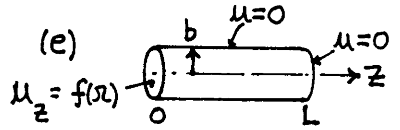
$$-50=\sum_1^\infty (G_n'\cosh k_n L)J_0(k_n r)$$

gives

$$G_n'\cosh k_n L=\frac{\langle -50, J_0(k_n r)\rangle}{\langle J_0(k_n r), J_0(k_n r)\rangle}=\text{etc.} \qquad ③$$

The solution given by ② and ③ is equivalent to that given by ① but in a different form. My choice would be ① since it is simpler to work with, if only because we need the k_n's in ② and ③ — i.e., the z_n roots of $J_0(z)=0$.

(e)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -k^2.$$ (This time $f(r)$ is not a constant so we have no choice but to do our expansion in r. Hence, choose $-k^2$.)

$$u(r,z) = (A+B\ln r)(C+Dz)+\left[EJ_0(kr)+FY_0(kr)\right](G\cosh kz+H\sinh kz)$$

u bdd as $r\to 0 \Rightarrow$ $u(r,z) = C'+D'z + J_0(kr)(G'\cosh kz+H'\sinh kz)$

$$u(b,z)=0 = C'+D'z + J_0(kb)(G'\cosh kz+H'\sinh kz)$$

so $C'=D'=0$ and $k = z_n/b \equiv k_n$ where $J_0(z_n)=0$.

$$u(r,z) = \sum_1^\infty J_0(k_n r)(G_n'\cosh k_n z + H_n'\sinh k_n z) \qquad ①$$

Then

$$u_z(r,0) = f(r) = \sum_1^\infty k_n H_n' J_0(k_n r) \qquad (0<r<b)$$

so $$k_n H_n' = \frac{\langle f, J_0\rangle}{\langle J_0, J_0\rangle}, \quad H_n' = \frac{1}{k_n}\frac{\int_0^b f(r)J_0(k_n r)r\,dr}{\int_0^b J_0^2(k_n r)r\,dr} \qquad ②$$ ← denom. can be evaluated

Finally,

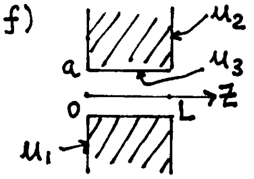
$$u(r,L)=0=\sum_1^\infty (G_n'\cosh z_n L + H_n'\sinh z_n L)J_0(k_n r)$$

gives

$$G_n'\cosh z_n L + H_n'\sinh z_n L = 0 \quad \text{or} \quad G_n' = -(\tanh z_n L)H_n' \qquad ③$$

and the solution is given by ①-③.

(f)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = k^2$$

so $u(r,z) = (A+B\ln r)(C+Dz)+\left[EI_0(kr)+FK_0(kr)\right](G\cos kz+H\sin kz)$

u bdd as $r\to\infty \Rightarrow B=E=0$ so

$$u(r,z) = C'+D'z + K_0(kr)(G'\cos kz+H'\sin kz)$$

$u(r,0) = u_1 = C' + K_0(kr)G' \rightarrow C'=u_1$ and $G'=0$ so

$$u(r,z) = u_1 + D'z + H'K_0(kr)\sin kz$$

$u(r,L) = u_2 = u_1 + D'L + H'K_0(kr)\sin kL$ so $D' = \frac{u_2-u_1}{L}$, $k = n\pi/L$

$$u(r,z) = u_1 + (u_2-u_1)\frac{z}{L} + \sum_1^\infty H_n' K_0(n\pi r/L)\sin(n\pi z/L) \qquad ①$$

Finally,

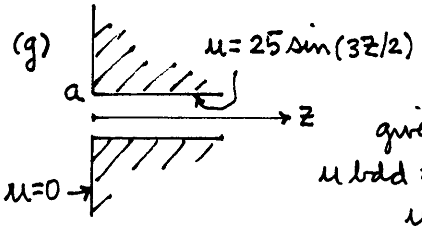
$$u(a,z) = u_3 = u_1 + (u_2-u_1)\frac{z}{L} + \sum_1^\infty H_n'K_0(n\pi a/L)\sin(n\pi z/L)$$

or

$$u_3 - u_1 - (u_2-u_1)\frac{z}{L} = \sum_1^\infty H_n'K_0(n\pi a/L)\sin(n\pi z/L) \qquad (0<z<L)$$

HRS: $H_n'K_0(n\pi a/L) = \frac{2}{L}\int_0^L \left[u_3-u_1-(u_2-u_1)\frac{z}{L}\right]\sin\frac{n\pi z}{L}\,dz = $ etc. ②
and the solution is given by ① and ②.

(g)



$$\frac{R''+\frac{1}{r}R'}{R} = -\frac{Z''}{Z} = +k^2$$

gives $u(r,z) = (A+B\ln r)(C+Dz) + [EI_0(kr)+FK_0(kr)](G\cos kz + H\sin kz)$

u bdd $\Rightarrow B = E = 0$ and $D=0$, so

$$u(r,z) = A' + K_0(kr)(G'\cos kz + H'\sin kz)$$

and $u(a,z) = 25\sin(3z/2) = A' + K_0(ka)(G'\cos kz + H'\sin kz)$

gives $A'=0$, $G'=0$, $K_0(ka)H' = 25$ and $k = 3/2$, so

$$u(r,z) = 25\,\frac{K_0(3r/2)}{K_0(3a/2)}\sin\frac{3z}{2}$$

17. Students often have trouble with this one.

$$\Phi'' + \cot\phi\,\Phi' + k^2\Phi = 0$$

With $\mu = \cos\phi$, $\frac{d}{d\phi} = \frac{d}{d\mu}\frac{d\mu}{d\phi} = -\sin\phi\frac{d}{d\mu} = -\sqrt{1-\mu^2}\frac{d}{d\mu}$, so we have

$$\left(-\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\left(-\sqrt{1-\mu^2}\frac{d\Phi}{d\mu}\right) + \frac{\mu}{\sqrt{1-\mu^2}}\left(-\sqrt{1-\mu^2}\frac{d\Phi}{d\mu}\right) + k^2\Phi = 0$$

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} + \sqrt{1-\mu^2}\,\frac{(\frac{1}{2})(-2\mu)}{\sqrt{1-\mu^2}}\,\frac{d\Phi}{d\mu} - \mu\frac{d\Phi}{d\mu} + k^2\Phi = 0$$

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + k^2\Phi = 0$$

18. $A_n = \frac{2n+1}{2c^n}\int_{-1}^{1} f P_n\,d\mu = \frac{2n+1}{2c^n}\int_0^1 100 P_n\,d\mu = 50\,\frac{2n+1}{c^n}\int_0^1 P_n(\mu)\,d\mu.$

The maple commands with(orthopoly):
int(P(j,x), x=0..1);

gives, for $j = 0,1,2,\dots,8$, the values $1, 1/2, 0, -1/8, 0, 1/16, 0, -5/128, 0, \dots$

Thus,

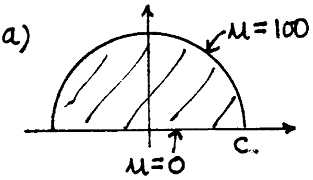
$A_0 = 50$, $A_1 = \frac{150}{c}(\frac{1}{2}) = \frac{75}{c}$, $A_2 = 0$, $A_3 = \frac{350}{c^3}(-\frac{1}{8}) = -\frac{175}{4c^3}$, $A_4 = 0$, $A_5 = \frac{550}{c^5}(\frac{1}{16}) = \frac{275}{8c^5}$,

$A_6 = 0$, $A_7 = \frac{750}{c^7}(-\frac{5}{128}) = -\frac{1875}{64c^7}$, $A_8 = 0, \dots$

so (81) gives

$$\begin{aligned} u(\rho,\phi) &= \sum_0^\infty A_n\rho^n P_n(\cos\phi) \\ &= 50P_0 + \frac{75}{c}\rho P_1 - \frac{175}{4c^3}\rho^3 P_3 + \frac{275}{8c^5}\rho^5 P_5 - \frac{1875}{64c^7}\rho^7 P_7 + \cdots \\ &= 50\Big[P_0(\cos\phi) + \frac{3}{2}\left(\frac{\rho}{c}\right)P_1(\cos\phi) - \frac{7}{8}\left(\frac{\rho}{c}\right)^3 P_3(\cos\phi) \\ &\quad + \frac{11}{16}\left(\frac{\rho}{c}\right)^5 P_5(\cos\phi) - \frac{75}{128}\left(\frac{\rho}{c}\right)^7 P_7(\cos\phi) + \cdots\Big] \end{aligned}$$

19. (a)



By (81), $u(\rho,\phi)=\sum_0^\infty A_n \rho^n P_n(\cos\phi)$

$u(\rho,\pi/2)=0=\sum_0^\infty A_n P_n(0)\rho^n$.

Now, the $P_n(0)$'s are 0 if $n=$ odd, so $A_0=A_2=A_4=\cdots=0$ and $u(\rho,\phi)=\sum_{1,3,..} A_n\rho^n P_n(\cos\phi)$

Then,

$$u(c,\phi)=100=\sum_{1,3,..} A_n c^n P_n(\cos\phi) \qquad (0<\phi<\pi/2 \text{ or } 0<\mu<1)$$

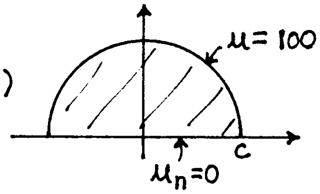
$$\text{so } A_n c^n = \frac{\langle 100, P_n\rangle}{\langle P_n,P_n\rangle}=\frac{\int_0^1 100 P_n(\mu)\,d\mu}{\int_0^1 P_n^2(\mu)\,d\mu} = \overset{n=1,}{150}, \overset{3,}{-\frac{175}{2}}, \overset{5,}{\frac{275}{4}}, \overset{7,}{-\frac{1875}{32}}, \overset{9,}{\frac{3325}{64}}, \overset{\cdots}{\cdots}$$

so

$$u(\rho,\phi)=150\left(\frac{\rho}{c}\right)P_1(\cos\phi)-\frac{175}{2}\left(\frac{\rho}{c}\right)^3P_3(\cos\phi)+\frac{275}{4}\left(\frac{\rho}{c}\right)^5P_5(\cos\phi)$$

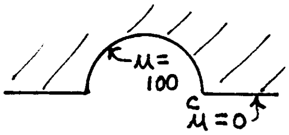
$$-\frac{1875}{32}\left(\frac{\rho}{c}\right)^7P_7(\cos\phi)+\frac{3325}{64}\left(\frac{\rho}{c}\right)^9P_9(\cos\phi)-\cdots$$

(b)



We can see by inspection that $u(\rho,\phi)=$ constant $=100$. Of course, a detailed solution will lead to this result; see that solution outlined in the Answers to Selected Exercises.

(c)



In Example 5 we set $B=0$ in (80) to give boundedness at $\rho=0$, but in this case $\rho=0$ is not relevant since $c<\rho<\infty$. Rather, boundedness as $\rho\to\infty \Rightarrow A_n=0$ for $n\geq 1$ so $u(\rho,\phi)=A_0+\sum_1^\infty \frac{B_n}{\rho^{n+1}}P_n(\cos\phi)$

Then,

$$u(\rho,\pi/2)=0=A_0+\sum_1^\infty \frac{B_n}{\rho^{n+1}}P_n(0)$$

Now, $P_n(0)=0$ if n is odd, so $A_0=B_2=B_4=\cdots=0$ and

$$u(\rho,\phi)=\sum_{1,3,\ldots}\frac{B_n}{\rho^{n+1}}P_n(\cos\phi).$$

Then,

$$u(c,\phi)=100=\sum_{1,3,..}^\infty \frac{B_n}{c^{n+1}}P_n(\cos\phi) \qquad (0<\phi<\pi/2 \text{ or } 0<\mu<1)$$

so $\qquad B_n/c^{n+1}=\dfrac{\langle 100,P_n\rangle}{\langle P_n,P_n\rangle}=$ same values as in part (a) above.

Thus, $\quad u(\rho,\phi)=\dfrac{B_1}{\rho^2}P_1(\cos\phi)+\dfrac{B_3}{\rho^4}P_3(\cos\phi)+\dfrac{B_5}{\rho^6}P_5(\cos\phi)+\cdots$

$$=150\left(\frac{c}{\rho}\right)^2P_1(\cos\phi)-\frac{175}{2}\left(\frac{c}{\rho}\right)^4P_3(\cos\phi)+\frac{275}{4}\left(\frac{c}{\rho}\right)^6P_5(\cos\phi)$$

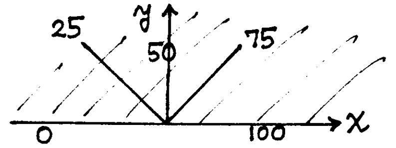
$$-\frac{1875}{32}\left(\frac{c}{\rho}\right)^8P_7(\cos\phi)+\frac{3325}{64}\left(\frac{c}{\rho}\right)^{10}P_9(\cos\phi)-\cdots$$

For ex., at $\rho=2c$ and $\phi=0$, this gives $u(2c,0)=37.5-5.47+1.07-0.23+0.05-\cdots=32.92$, and $u(40,0) = 9.38 - 0.34 + 0.02 - \cdots = 9.06$, and these results seem quite reasonable.

Section 20.4

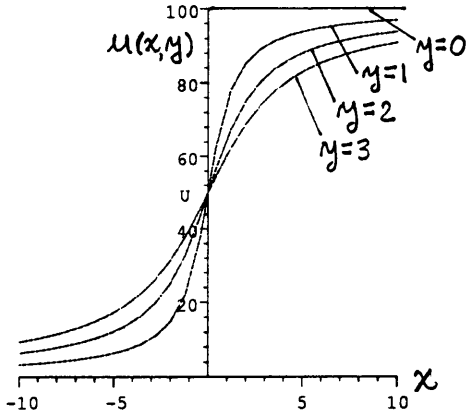
1. (a) $u(x,y) = \frac{y}{\pi}\int_0^\infty 100 \frac{d\xi}{(\xi-x)^2+y^2} = 50 + \frac{100}{\pi}\tan^{-1}\frac{x}{y}$, where $-\frac{\pi}{2} < \tan^{-1} < \frac{\pi}{2}$

(b) We see from (a) that the isotherms are $x/y =$ constant lines, i.e., rays out of the origin:



That is, $u=25$ on $y=-x$, $u=50$ on $x=0$, and $u=75$ on $y=x$.

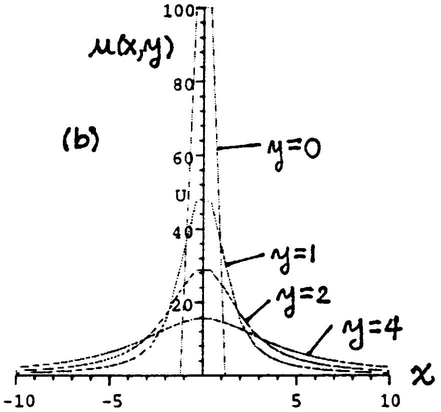
(c)



2. (a) $u(x,y) = \frac{y}{\pi}\int_{-1}^{1} \frac{100\, d\xi}{(\xi-x)^2+y^2}$

$$= \frac{100}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) - \tan^{-1}\left(\frac{-1-x}{y}\right)\right]$$

(b)



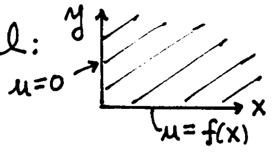
(c) $$\sim \frac{100}{\pi}\left[\frac{1-x}{y} - \frac{-1-x}{y}\right] = \frac{200}{\pi y}$$

3. (a) $u(-x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(-x-\xi)^2+y^2}\,d\xi \qquad (\xi = -\eta)$

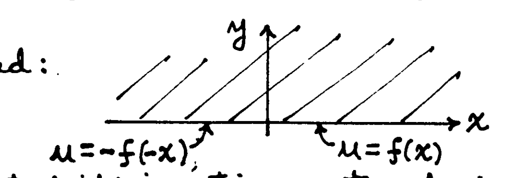
$$= \frac{y}{\pi}\int_{\infty}^{-\infty} \frac{f(-\eta)}{(-x+\eta)^2+y^2}(-d\eta) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\eta)}{(x-\eta)^2+y^2}\,d\eta = u(x,y)$$, so $u(x,y)$ is an even function of x.

(b) Same as in (a), except that $f(-\eta) = -f(\eta)$ so we obtain $u(-x,y) = -u(x,y)$.

4. (a) Original:



Extended:



to build in antisymmetry about $x=0$

so $u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f_{ext}(\xi)}{(x-\xi)^2+y^2}\,d\xi$ where "f_{ext}" is the ex<u>tended</u> f, on $-\infty < x < \infty$

$$= \frac{y}{\pi}\int_{-\infty}^{0} \frac{-f(-\xi)}{(x-\xi)^2+y^2}\,d\xi + \frac{y}{\pi}\int_{0}^{\infty} \frac{f(\xi)}{(x-\xi)^2+y^2}\,d\xi \qquad (\text{Let } \xi=-\eta \text{ in first integral})$$

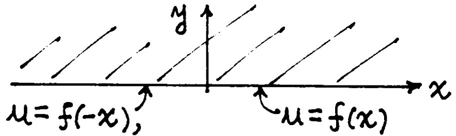
$$= \frac{y}{\pi}\int_{\infty}^{0} \frac{-f(\eta)}{(x+\eta)^2+y^2}(-d\eta) + \quad '' \qquad (\text{now let } \eta=\xi \text{ in first integral.})$$

$$= -\frac{y}{\pi}\int_{0}^{\infty} \frac{f(\xi)}{(x+\xi)^2+y^2}\,d\xi + \quad '' \quad = \int_0^{\infty} Q(\xi;x,y)\,f(\xi)\,d\xi,$$

where the new kernel is

$$Q(\xi;x,y) = \frac{y}{\pi}\left[\frac{1}{(x-\xi)^2+y^2} - \frac{1}{(x+\xi)^2+y^2}\right].$$

(b) Extended:



to build in symmetry about $x=0$

so $u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f_{ext}(\xi)}{(x-\xi)^2+y^2}\,d\xi$

$$= \frac{y}{\pi}\int_{-\infty}^{0} \frac{f(-\xi)}{(x-\xi)^2+y^2}\,d\xi + \frac{y}{\pi}\int_{0}^{\infty} \frac{f(\xi)}{(x-\xi)^2+y^2}\,d\xi$$

$$= \frac{y}{\pi}\int_{\infty}^{0} \frac{f(\eta)}{(x+\eta)^2+y^2}(-d\eta) + \quad ''$$

$$= \frac{y}{\pi}\int_{0}^{\infty} \frac{f(\eta)}{(x+\eta)^2+y^2}\,d\eta + \quad '' \quad = \int_0^{\infty} Q(\xi;x,y)\,f(\xi)\,d\xi,$$

where the new kernel is $Q(\xi;x,y) = \frac{y}{\pi}\left[\frac{1}{(x-\xi)^2+y^2} + \frac{1}{(x+\xi)^2+y^2}\right].$

5. (a) Take a Fourier transform on x: $(i\omega)^2\hat{u} + \hat{u}_{yy} = 0$, $\hat{u}_{yy} - \omega^2\hat{u} = 0$

$$\hat{u}(\omega,y) = A\cosh\omega y + B\sinh\omega y$$

$$\left.\begin{aligned}\hat{u}(\omega,0) &= \hat{f}(\omega) = A\\ \hat{u}(\omega,a) &= \hat{g}(\omega) = A\cosh\omega a + B\sinh\omega a\end{aligned}\right\} \text{so } A = \hat{f}(\omega) \text{ and } B = (\hat{g}(\omega) - \hat{f}(\omega)\cosh\omega a)/\sinh\omega a$$

so

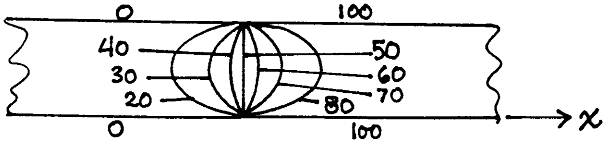
$$\hat{u}(\omega,y) = \hat{f}(\omega)\cosh\omega y + [\hat{g}(\omega) - \hat{f}(\omega)\cosh\omega a]\sinh\omega y/\sinh\omega a$$

$$= [\frac{\hat{f}(\omega)\cosh\omega y\sinh\omega a - \hat{f}(\omega)\cosh\omega a\sinh\omega y}{\sinh\omega a} + \hat{g}(\omega)\frac{\sinh\omega y}{\sinh\omega a}$$

$$= \hat{f}(\omega)\frac{\sinh\omega(a-y)}{\sinh\omega a} + \hat{g}(\omega)\frac{\sinh\omega y}{\sinh\omega a}$$

$$u(x,y) = F^{-1}\{ \quad '' \quad + \quad '' \quad \}$$

(b)



6. If $f(x)=\delta(x-x_0)$ then (12) gives $u(x,y)=\int_{-\infty}^{\infty}P(\xi-x,y)\delta(\xi-x_0)\,d\xi$ $=P(x_0-x,y)$, or, $P(x-x_0,y)$ since P is an even function of its first argument.

Section 20.5

1\. (a)

letter	α	β	γ	δ	T_{east}	T_{north}	T_{west}	T_{south}	f_p
a	1	1	1	1	b	20	10(3/4)	d	−20(3/4)(1/4)
b	1	1	1	1	c	20	a	e	-20(3/4)(1/2)
c	$4\sqrt{(3/4)}-3$	1	1	3/4	20	20	b	20	-20(3/4)(3/4)
d	1	1	1	1	e	a	5	g	-20(1/4)(1/2)
e	$4(\sqrt{1/2}-1/2)$	1	1	1	20	b	d	20	-20(1/2)(1/2)
g	1	1	1	3/4	20	d	2.5	20	-20(1/4)(1/4)

$h=.25$

$$\begin{array}{llllll}
7.5 & +\ d & +\ b & +\ 20 & -4a & = -.234375 \\
a & +\ e & +\ c & +\ 20 & -4b & = -.46875 \\
1.366b & +\ 30.476 & +\ 58.8677 & +\ 22.8571 & -6.9760c & = -.703125 \\
5 & +\ g & +\ e & +\ a & -4d & = -.15625 \\
1.0938d & +\ 20 & +26.4088 & +\ b & -4.4142e & = -.3125 \\
2.5 & +\ 30.4761 & +20.0 & +1.1428d & -4.666g & = -.078125
\end{array}$$

NOTE: Let us denote u_a as a, u_b as b, etc, for brevity

or,
$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1.366 & -6.976 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1.0938 & -4.414 & 0 \\ 0 & 0 & 0 & 1.143 & 0 & -4.666 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ g \end{bmatrix}=\begin{bmatrix} -27.73 \\ -20.47 \\ -112.90 \\ -5.16 \\ -46.72 \\ -53.05 \end{bmatrix}\rightarrow$$

$a=14.77$
$b=18.24$
$c=19.76$
$d=13.12$
$e=17.97$
$g=14.58$

(b) Same $\alpha,\beta,\gamma,\delta$'s as in (a).

a: $0+d+b+0-4a=0$

b: $a+e+c+0-4b=0$

c: $\frac{2}{1.4641}b+\frac{2}{1.3125}50+\frac{2}{.6795}50+0-2\frac{1.2141}{.3481}c=0$

d: $0+g+e+a-4d=0$

e: $\frac{2}{1.8284}d+50+\frac{2}{1.5146}50+b-2\frac{1.8284}{.8284}e=0$

g: $0+\frac{2}{1.3125}50+50+\frac{2}{1.75}d-2\frac{1.75}{.75}g=0$

or
$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1.3660 & -6.9756 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1.0939 & -4.4143 & 0 \\ 0 & 0 & 0 & 1.1429 & 0 & -4.6667 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ g \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ -223.36 \\ 0 \\ -116.02 \\ -126.19 \end{bmatrix}\rightarrow$$

$a=9.94$
$b=20.42$
$c=36.02$
$d=19.36$
$e=35.71$
$g=31.78$

(c) Same $\alpha, \beta, \gamma, \delta$'s as in (a).

$$\left.\begin{aligned}
&a:\ 0+d+b+0-4a=-6.25\\
&b:\ a+e+c+0-4b=-6.25\\
&c:\ \frac{2}{1.4641}b+0+0+0-2\frac{1.2141}{.3481}c=-6.25\\
&d:\ 0+g+e+a-4d=-6.25\\
&e:\ \frac{2}{1.8284}d+0+0+b-2\frac{1.8284}{.8284}e=-6.25\\
&g:\ 0+0+0+\frac{2}{1.75}d-2\frac{1.75}{.75}g=-6.25
\end{aligned}\right\}$$

so the 6x6 matrix is the same as in (b), but the vector on the right-hand-side is different, namely, $[-6.25, -6.25, -6.25, -6.25, -6.25, -6.25]^T$. The result is
$a = 3.41$, $b = 3.61$, $c = 1.60$, $d = 3.77$,
$e = 3.17$, $g = 2.26$

(d)

	α	β	γ	δ
a	1	1	.25	1
b	.75	1	1	1
c	.5	1	1	1
d	.25	1	1	1
e	1	1	.75	1
g	1	1	.5	1

$h=2$

$$\left.\begin{aligned}
&a:\ 0+g+1.6b+0-10a=-200\\
&b:\ 1.1429a+c+0+0-4.6667b=-200\\
&c:\ 1.3333g+d+0+b-6c=-200\\
&d:\ 1.6e+0+0+c-10d=-200\\
&e:\ 0+0+1.1429d+g-4.6667e=-200\\
&g:\ 0+e+1.3333c+a-6g=-200
\end{aligned}\right\}\rightarrow
\begin{aligned}
a&=37.02\\
b&=65.83\\
c&=64.90\\
d&=37.02\\
e&=65.83\\
g&=64.90
\end{aligned}$$

NOTE: The symmetry present implies that $a=d$ and $b=e$, and our results bear that out.

(e) Same $\alpha, \beta, \gamma, \delta$'s as in (d). Take $u(0,8)=(0+100)/2=50$

$$\left.\begin{aligned}
&a:\ 0+g+1.6b+50-10a=0\\
&b:\ 1.1429a+c+0+100-4.6667b=0\\
&c:\ 1.3333g+d+0+b-6c=0\\
&d:\ 1.6e+0+0+c-10d=0\\
&e:\ 0+0+1.1429d+g-4.6667e=0\\
&g:\ 0+e+1.3333c+a-6g=0
\end{aligned}\right\}\rightarrow
\begin{aligned}
a&=9.23\\
b&=24.72\\
c&=4.83\\
d&=0.60\\
e&=0.73\\
g&=2.73
\end{aligned}$$

(f) Same $\alpha, \beta, \gamma, \delta$'s as in (d). Take $u(2,0)=(200+0)/2=100$.

$$\left.\begin{aligned}
&a:\ 0+g+1.6b+0-10a=144\\
&b:\ 1.1429a+c+0+0-4.6667b=160\\
&c:\ 1.3333g+d+0+b-6c=80\\
&d:\ 1.6e+100+0+c-10d=32\\
&e:\ 0+200+1.1429d+g-4.6667e=16\\
&g:\ 0+e+1.3333c+a-6g=64
\end{aligned}\right\}\rightarrow
\begin{aligned}
a&=-22.80\\
b&=-44.53\\
c&=-21.77\\
d&=10.92\\
e&=39.37\\
g&=-12.74
\end{aligned}$$

(g)

	α	β	γ	δ
a	1	1	1	.2680
b	1	1	1	1
c	1	1	1	1
d	1	1	1	1
e	1	1	.2680	1

a: $25+294.270+b+0-9.463a=-80$
b: $a+e+c+0-4b=-80$
c: $b+d+0+0-4c=-80$
d: $e+0+0+c-4d=-80$
e: $294.270+25+1.577d+b-9.463e=-80$
gives $a=48.26, b=57.45, c=45.71, d=45.38, e=55.83$

$h=2$

(h) Same $\alpha,\beta,\gamma,\delta$'s as in (g).

a: $100+588.540+b+157.729-9.463a=120$
b: $a+e+c+100-4b=120$
c: $b+d+100+100-4c=120$
d: $e+100+100+c-4d=120$
e: $588.540+100+1.577d+b-9.463e=120$

gives

$a=81.45$
$b=44.45$
$c=43.38$
$d=49.08$
$e=72.96$

(i) Same $\alpha,\beta,\gamma,\delta$'s as in (g). $u_1=u_3=u_4=u_5=0$, $u_2(x)=10x$.

a: $0+0+b+31.546-9.463a=-2400$
b: $a+e+c+40-4b=0$
c: $b+d+0+60-4c=4000$
d: $e+0+0+c-4d=6400$
e: $0+0+1.577d+b-9.463e=2400$

gives

$a=201.7$
$b=-523.2$
$c=-1661.7$
$d=-2183.6$
$e=-672.8$

(j) By the noted symmetry, $h=a$, $g=b$, $e=c$, so we need only do points a,b,c,d, and use $e=c$ when we do point d. "$h=2$.

a: $0+100+100+100-4a=0$
b: $100+100+c+100-4b=0$
c: $b+d+0+100-4c=0$
d: $100+\underline{\underline{c}}+0+c-4d=0$

gives

$a=75 \quad =h$
$b=90.38=g$
$c=61.54=e$
$d=55.77$

(k) By the noted antisymmetry, $h=-a$, $g=-b$, $e=-c$, $d=0$, so we need only do points a,b,c, and use the fact that $d=0$ when we do point c. "$h=2$.

a: $0+0+0+50-4a=0$
b: $0+0+c+50-4b=0$
c: $b+\underline{\underline{0}}+0+50-4c=0$

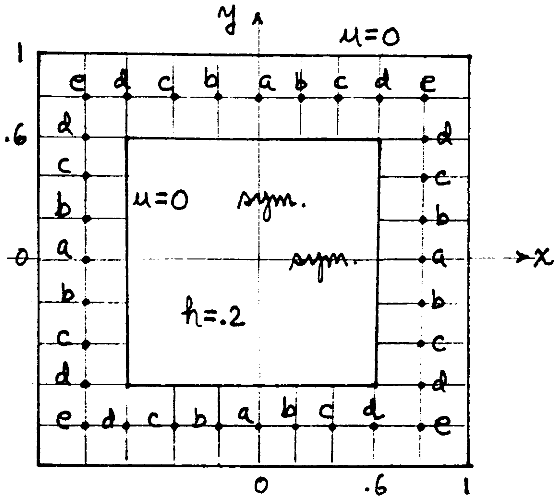
gives

$a=12.5$ so $h=-12.5$
$b=16.67$
$c=16.67$ so $e=-16.67$, and $d=0$.

(l) This time there is neither symmetry nor antisymmetry, due to the $f(x,y) = 10(x^2+y^2)$ term. "h" = 2.

$$\left.\begin{array}{ll} a: & 0+0+0+50-4a = 1600 \\ b: & 0+0+c+50-4b = 2880 \\ c: & b+d+0+50-4c = 4000 \\ d: & 0+e+0+c-4d = 3200 \\ e: & g+50+0+d-4e = 2720 \\ g: & 0+50+e+0-4g = 1600 \\ h: & 0+50+0+0-4h = 320 \end{array}\right\} \text{gives} \quad \begin{array}{l} a = -3875 \\ b = -1136.4 \\ c = -1665.6 \\ d = -1526.2 \\ e = -1239.0 \\ g = -709.7 \\ h = -67.5 \end{array}$$

2.(a)



By the symmetries there are only 5 unknowns rather than 32: a, b, c, d, e.

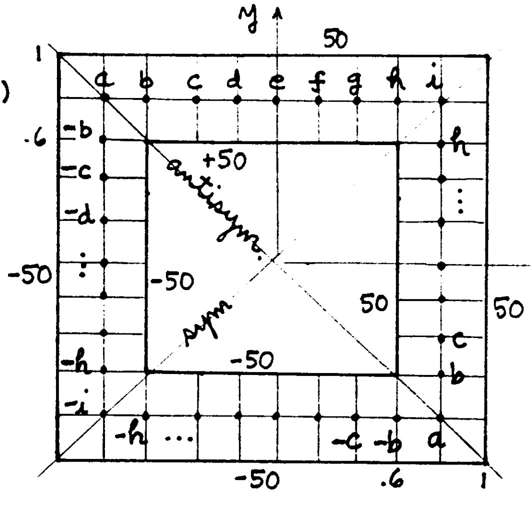
$$\left.\begin{array}{ll} a: & b+0+b+0-4a = -4 \\ b: & a+0+c+0-4b = -4 \\ c: & b+0+d+0-4c = -4 \\ d: & c+0+e+0-4d = -4 \\ e: & d+d+0+0-4e = -4 \end{array}\right\} \text{gives} \quad \begin{array}{l} a = 2 \\ b = 2 \\ c = 2 \\ d = 2 \\ e = 2 \end{array}$$

(b) The figure in (a) still applies, with $u=100$ on outer boundary.

$$\left.\begin{array}{ll} a: & b+0+b+100-4a = 0 \\ b: & a+0+c+100-4b = 0 \\ c: & b+0+d+100-4c = 0 \\ d: & c+0+e+100-4d = 0 \\ e: & d+d+100+100-4e = 0 \end{array}\right\} \text{gives} \quad \begin{array}{l} a = 50.30 \\ b = 50.60 \\ c = 52.08 \\ d = 57.74 \\ e = 78.87 \end{array}$$

These results seem reasonable since e is the most exposed to the $u=100$ boundary condition, d next, and so on. Also, a being just slightly more than 50 looks good too.

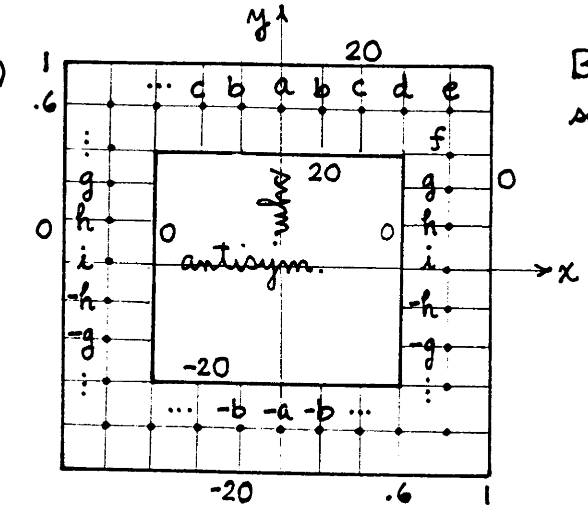
(c)



By the antisymmetry $a=0$, so we need to do b, c, …, i.

$$\left.\begin{array}{ll} b: & 0+0+c+50-4b = 0 \\ c: & b+50+d+50-4c = 0 \\ d: & c+50+e+50-4d = 0 \\ e: & d+50+f+50-4e = 0 \\ f: & e+50+g+50-4f = 0 \\ g: & f+50+h+50-4g = 0 \\ h: & g+50+i+50-4h = 0 \\ i: & h+h+50+50-4i = 0 \end{array}\right\} \text{gives} \quad \begin{array}{l} (a=0) \\ b = 23.21 \\ c = 42.82 \\ d = 48.08 \\ e = 49.48 \\ f = 49.86 \\ g = 49.96 \\ h = 49.99 \\ i = 49.99 \end{array}$$

(d)

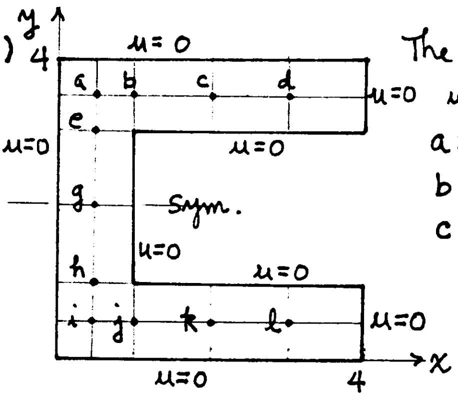


By the antisymmetry about x-axis, $i=0$, so we need only do $a,b,\dots,h$.

$$\left.\begin{array}{ll} a: & b+20+b+20-4a=0\\ b: & a+20+c+20-4b=0\\ c: & b+20+d+20-4c=0\\ d: & c+10+e+20-4d=0\\ e: & d+f+0+20-4e=0\\ f: & 10+g+0+e-4f=0\\ g: & 0+h+0+f-4g=0\\ h: & 0+0+0+g-4h=0\end{array}\right\}\text{gives}\quad\begin{array}{r} a=19.79\\ b=19.59\\ c=18.56\\ d=14.64\\ e=10.00\\ f=5.36\\ g=1.43\\ h=0.36\\ (i=0)\end{array}$$

Observe that these results look correct. For instance, the exact solution for e is 10, by inspection, and (to 4 significant figures) we do obtain that value.

3.(a)



The symmetry about $y=2$ implies that $h=e,\dots,l=d$, so we need only do a,b,c,d,e,g.

a: $h=.5,\ \alpha=\beta=\gamma=\delta=1$

b: $h=1,\ \alpha=1,\ \beta=\gamma=\delta=.5$

c: $h=1,\ \alpha=1,\ \beta=.5,\ \gamma=1,\ \delta=.5$

d: $h=1,\ \alpha=1,\ \beta=.5,\ \gamma=1,\ \delta=.5$

e: $h=1,\ \alpha=\beta=\gamma=.5,\ \delta=1$

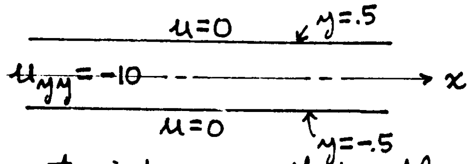
g: $h=1,\ \alpha=.5,\ \beta=1,\ \gamma=.5,\ \delta=1$

so (20) gives

$$\left.\begin{array}{ll} a: & 0+e+b+0-4a=.5^2(-10)\\ b: & 2.667a+0+1.333c+0-12b=1^2(-10)\\ c: & b+0+d+0-10c=1^2(-10)\\ d: & c+0+0+0-10d=1^2(-10)\\ e: & 0+1.333g+0+2.667a-12e=1^2(-10)\\ g: & 0+e+0+e-10g=1^2(-10)\end{array}\right\}\text{gives}\quad\begin{array}{r} a=1.250\\ b=1.248\\ c=1.237\\ d=1.124\\ e=1.250\\ g=1.250\end{array}$$

$(h=e,\ i=a,\ j=b,\ k=c,\ l=d)$

NOTE: (1) In doing computations it is important to proceed thoughtfully. For instance, in the present problem we expect the temperatures along the centerline (namely, d,c,b,a,e,g,h,i,j,k,l) to be very close to those along the centerline of a straight infinite strip (shown at the right), which we mention because that problem is only one-dimensional ($\overset{0}{\cancel{u_{xx}}}+u_{yy}=-10$) and easily solved: $u=-5(y^2-\frac{1}{4})$.



Then on the centerline ($y=0$), $u=1.25$, which result makes our results above look good. Note that we expect the largest deviation from 1.25 to be at d and l, where the end-effects are greatest. Sure enough, the temperatures there are the lowest. (2) At a, for example, we chose $h=.5$. We do expect that, as a rather "natural" choice, to be convenient, but it is to be emphasized that the choice of h at any given computation point is nominal. To illustrate, let us re-obtain the governing equation at a, which was

$$0+e+b+0-4a = .5^2(-10), \qquad ♀$$

using other h values:

$h=1$: $\alpha=\beta=\gamma=\delta=.5$. Then (20) gives

$$(4)(0)+(4)(e)+(4)(b)+(4)(0)-2\frac{.5}{(.5)^4}a = 1^2(-10)$$

or, $$0+4e+4b+0-16a=-10,$$

which does agree with ♀.

$h=2$: $\alpha=\beta=\gamma=\delta=.25$. Then (20) gives

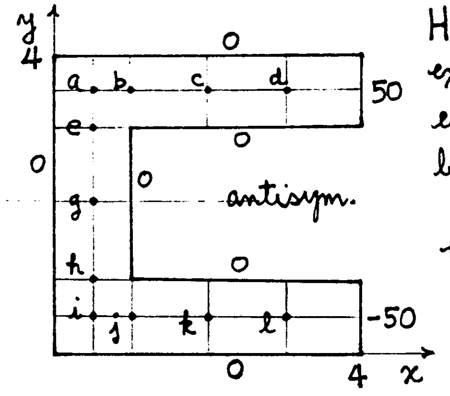
$$0+16e+16b+0-64a = 2^2(-10)$$

which, again, agrees with ♀.

We can even let h be small, such as $h=.1$, say. Then $\alpha=\beta=\gamma=\delta=5$ and (20) would once again give ♀. In the text we noted that $\alpha,\beta,\gamma,\delta$ are smaller than 1, but that was just for definiteness.

(b) See Answers to Selected Exercises.

(c)



Having developed some experience in the preceding exercises, we urge you to use your "intuition" to estimate the temperatures at the 11 nodal points before proceeding.

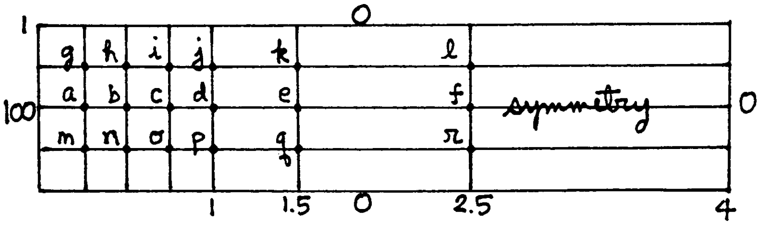
By the antisymmetry about $y=2$ we have $g=0$, $h=-e$, $i=-a$, $j=-b$, $k=-c$, $l=-d$.

$$\left.\begin{aligned} &a\!: 0+e+b+0-4a=0\\ &b\!: 2.667a+0+1.333c+0-12b=0\\ &c\!: b+0+d+0-10c=0\\ &d\!: c+0+50+0-10d=0\\ &e\!: 0+1.333\underset{0}{\cancel{g}}+0+2.667a-12e=0\end{aligned}\right\}\text{gives}\quad \begin{aligned} a&=0.016\\ b&=0.060\\ c&=0.511\\ d&=5.051\\ e&=0.004\end{aligned}$$

($g=0$, $h=-e$, $i=-a$, $j=-b$, $k=-c$, $l=-d$)

where we've used the same $h,\alpha,\beta,\gamma,\delta$'s as in part (a).

4. (a) Due to the symmetry noted, $m=g$, $n=h$, $o=i$, $p=j$, $q=k$ and $r=l$. We only need to do g,h,i,j,k,l,a,b,c,d,e,f.



$g,h,i,a,b,c:\quad h=.25,\ \alpha=\beta=\gamma=\delta=1$

$j,d:\quad h=.5,\ \alpha=1,\ \beta=\gamma=\delta=.5$

$k,e:\quad h=1,\ \alpha=1,\ \beta=\delta=.25,\ \gamma=.5$

$l,f:\quad h=1.5,\ \alpha=1,\ \beta=\delta=.1667,\ \gamma=.6667$

Thus,

$$\left.\begin{array}{ll} g: & 100+a+h+0-4g=0 \\ h: & g+b+i+0-4h=0 \\ i: & h+c+j+0-4i=0 \\ j: & 2.667i+4d+1.333k+0-12j=0 \\ k: & 2.667j+16e+1.333l+0-36k=0 \\ l: & 1.8k+36f+0+0-75l=0 \\ a: & 100+g+b+g-4a=0 \\ b: & a+h+c+h-4b=0 \\ c: & b+i+d+i-4c=0 \\ d: & 2.667c+4j+1.333e+4j-12d=0 \\ e: & 2.667d+16k+1.333f+16k-36e=0 \\ f: & 1.8e+36l+0+36l-75f=0 \end{array}\right\} \text{gives} \quad \begin{array}{l} g=43.18 \\ h=19.58 \\ i=8.97 \\ j=3.91 \\ k=0.78 \\ l=0.06 \\ a=53.13 \\ b=26.17 \\ c=12.40 \\ d=5.48 \\ e=1.11 \\ f=0.08 \end{array}$$

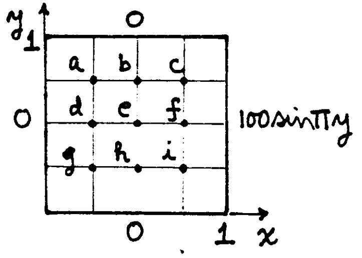
(b) By comparison, the exact (analytical) solution, obtained by separation of variables [equations (11) and (13) in Section 20.2] is

$a=54.47,\ b=26.10,\ c=12.03,\ d=5.50,\ e=1.14,\ f=0.05$

6. (a) See the Answers to Selected Exercises.

(b) $$\left.\begin{array}{l} 19.9-25\approx C\,.5^p \\ 19.2-21.3\approx C\,.25^p \end{array}\right\} \text{dividing gives } 2^p\approx 3.64,\ p\approx 1.87.$$

(c) Let us show the finite-difference calculation for $h=1/4$ first:



Symmetry about $y=0.5$ implies $g=a,\ h=b,\ i=c$, so we only need to do a,b,c,d,e,f.

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 & 1 & 0 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -70.7107 \\ 0 \\ 0 \\ -100 \end{bmatrix} \quad \text{gives } e=21.3388$$

Doing the same for $h=1/6$ we have (after using the symmetry) 15 equations on 15 unknowns. This time we obtain, for u at the center point, 20.5695. Also, the exact solution there is 19.927. Thus,

$$\left.\begin{array}{l} 19.927-21.339\approx C(1/4)^p \\ 19.927-20.570\approx C(1/6)^p \end{array}\right\} \text{dividing gives } 1.5^p=2.196,\ p\approx 1.94.$$

7. (a) a: $0+e+b+0-4a=0$

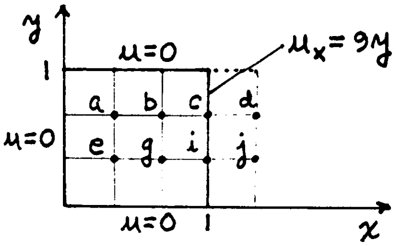
b: $a+g+c+0-4b=0$

bc@c: $(c-b)/(1/3) = 9(2/3)$

e: $0+0+g+a-4e=0$

g: $e+0+i+b-4g=0$

bc@i: $(i-g)/(1/3) = 9(1/3)$,



which is the same as (7.4) and which gives

$a=0.326,\ b=1.032,\ c=3.032,\ e=0.274,\ g=0.768,\ i=1.768$

(b) a: $0+e+b+0-4a=0$

b: $a+g+c+0-4b=0$

c: $b+i+d+0-4c=0$

e: $0+0+g+a-4e=0$

g: $e+0+i+b-4g=0$

i: $g+0+j+c-4i=0$

bc@c: $(d-b)/(2/3)=9(2/3)$

bc@i: $(j-g)/(2/3)=9(1/3)$

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & -3/2 & 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3/2 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ g \\ i \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 3 \end{bmatrix},$$

which gives

$a=0.176,\ b=0.545,\ c=1.552,\ d=4.545,\ e=0.158,\ g=0.455,\ i=1.115,\ j=2.455$

(c) Exact solution is

$$u(x,y) = -\frac{18}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\frac{\sinh n\pi x \sin n\pi y}{\cosh n\pi}$$

so

$u(1/3,1/3) = u_e = 0.164$

$u(2/3,1/3) = u_g = 0.497$

$u(1/3,2/3) = u_a = 0.176$

$u(2/3,2/3) = u_b = 0.592$

$u(1,1/3) = u_i = 1.228$

$u(1,2/3) = u_c = 1.843$

Considering that the grid is extremely coarse, the results in part (b) are not too bad, while those in part (a) are off by a great deal.

(d) In (a), change the bc@c from $(c-b)/(1/3) = 9(2/3)$, or, $c-b=2$, to

$(c-b)/(1/3) + 3c = 9(2/3)$, or, $2c-b=2$.

Also, change the bc@i to

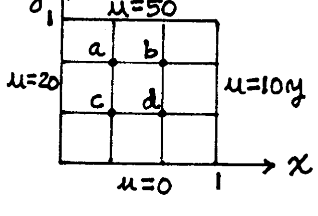
$(i-g)/(1/3) + 3i = 9(1/3)$, or, $2i-g=1$.

Thus, the point is that – at least for a rectangular region – the handling of Neumann or Robin boundary conditions is not much harder than for Dirichlet conditions.

8. (a) $A=1+x^2, B=0, C=1$ so $B^2-AC=-1-x^2<0$, hence elliptic.

$$(1+x_j^2)\frac{U_{j-1,k}-2U_{jk}+U_{j+1,k}}{(\Delta x)^2} + \frac{U_{j,k+1}-2U_{jk}+U_{j,k-1}}{(\Delta y)^2} = 0$$

or, with $\Delta x = \Delta y = h$,



$$(1+x_j^2)U_{j-1,k}+U_{j,k-1}+(1+x_j^2)U_{j+1,k}+U_{j,k+1}-(4+2x_j^2)U_{jk}=0$$

applying this to the present case (and using our abbreviated notation $U_a \equiv$ "a", and so on),

$$\begin{aligned}
a&:\ (1+0)20+c+(1+\tfrac{4}{9})b+50-(4+\tfrac{2}{9})a=0\\
b&:\ (1+\tfrac{1}{9})a+d+(1+1)\tfrac{20}{3}-50-(4+\tfrac{8}{9})b=0\\
c&:\ (1+0)20+0+(1+\tfrac{4}{9})d+a-(4+\tfrac{2}{9})c=0\\
d&:\ (1+\tfrac{1}{9})c+0+(1+1)\tfrac{10}{3}+b-(4+\tfrac{8}{9})d=0
\end{aligned}$$

or,

$$\begin{bmatrix} -13/9 & 13/9 & 1 & 0 \\ 10/9 & -44/9 & 0 & 1 \\ 1 & 0 & -38/9 & 13/9 \\ 0 & 1 & 10/9 & -44/9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} -70 \\ 110/3 \\ -20 \\ -20/3 \end{bmatrix}$$

(b) $A=1$, $B=0$, $C=2$ so $B^2-AC=-2<0$, hence elliptic:

$$\frac{U_{j-1,k}-2U_{jk}+U_{j+1,k}}{(\Delta x)^2}+2\,\frac{U_{j,k-1}-2U_{jk}+U_{j,k+1}}{(\Delta y)^2}-\frac{U_{j+1,k}-U_{j-1,k}}{\Delta x}=4$$

or, with $\Delta x=\Delta y=h$,

$$(1+h)U_{j-1,k}+2U_{j,k-1}+(1-h)U_{j+1,k}+2U_{j,k+1}-6U_{jk}=4h^2$$

applying this,

$$\begin{aligned}
a&:\ (1+\tfrac{1}{3})20+2c+(1-\tfrac{1}{3})b+2(50)-6a=4/9\\
b&:\ (1+\tfrac{1}{3})a+2d+(1-\tfrac{1}{3})\tfrac{20}{3}+2(50)-6b=4/9\\
c&:\ (1+\tfrac{1}{3})20+0+(1-\tfrac{1}{3})d+2a-6c=4/9\\
d&:\ (1+\tfrac{1}{3})c+0+(1-\tfrac{1}{3})\tfrac{10}{3}+2b-6d=4/9
\end{aligned}$$

or,

$$\begin{bmatrix} -6 & \frac{2}{3} & 2 & 0 \\ 4/3 & -6 & 0 & 2 \\ 2 & 0 & -6 & 2/3 \\ 0 & 2 & 4/3 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} -1136/9 \\ -936/9 \\ -236/9 \\ -16/9 \end{bmatrix}$$

CHAPTER 21

Section 21.2

2. $|z_1z_2| = |(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)| = \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$
$= \sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2}$

$|z_1||z_2| = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} = \sqrt{x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2} = |z_1z_2|$ ✓

4. (a) Quadratic formula gives $z = (2\pm\sqrt{4-8})/2 = 1\pm i$. Consider $z = 1+i$, say. Then
$z^2 = (1+i)(1+i) = 0+2i$ per (4)
$-2z = (-2+0i)(1+i) = -2-2i$ per (4)
$z^2-2z = (z^2)+(-2z) = (0+2i)+(-2-2i) = -2$ per (3)
$z^2-2z+2 = (z^2-2z)+(2+0i)$ per (6)
$= (-2+0i)+(2+0i) = 0$ per (3). ✓

5. (b) Using induction, we first observe that the equality holds for $n=1$. Next, suppose it holds for $n=k$. Then
$|z^{k+1}| = |z^kz| = |z^k||z|$ per (9)
$= |z|^k|z|$ per assumption
$= |z|^{k+1}$, which completes the proof by induction.

(c) $|z_1z_2z_3| = |(z_1z_2)(z_3)| = |z_1z_2||z_3|$ per (9)
$= |z_1||z_2||z_3|$ per (9) again.

6. (e) Using induction, first observe that the equality holds for $n=1$. Next, suppose it holds for $n=k$. Then
$\overline{z^{k+1}} = \overline{z^kz} = \overline{z^k}\,\overline{z}$ per (14b)
$= \bar{z}^k\bar{z}$ per assumption
$= \bar{z}^{k+1}$, which completes the proof.

8. $z_1z_2 = (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1) = 0$ gives $\left.\begin{array}{l} x_1x_2-y_1y_2=0 \\ x_1y_2+x_2y_1=0. \end{array}\right\}$ ♀

Regarding ♀ as a linear system on x_1, y_1, if x_1, y_1 are not both 0 then we must have the determinant $= x_2^2+y_2^2=0$. Thus, if $z_1\neq 0$ then we must have $z_2=0$. Similarly, if $z_2\neq 0$ then we need $z_1=0$. Thus, z_1 and z_2 cannot both be nonzero.

9. (a) $(2-i)^3 = (4-4i-1)(2-i) = (3-4i)(2-i) = 6-11i-4 = 2-11i$

(e) $\left(\frac{1+i}{2-i}\right)^3 = \left(\frac{1+i}{2-i}\right)^2\left(\frac{1+i}{2-i}\right) = \frac{2i}{3-4i}\cdot\frac{1+i}{2-i} = \frac{-2+2i}{+2-11i}\;\frac{+2+11i}{+2+11i} = \frac{-26-18i}{125} = -\frac{26}{125}-\frac{18}{125}i$

(g) $\operatorname{Im}(1+i)^3 = \operatorname{Im} 2i(1+i) = \operatorname{Im}(-2+2i) = 2$

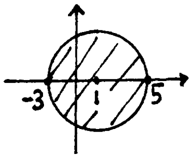
(h) $\left(\operatorname{Re}\frac{1}{1+i}\right)^3 = \left(\operatorname{Re}\frac{1}{1+i}\,\frac{1-i}{1-i}\right)^3 = \left(\operatorname{Re}\frac{1-i}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = 1/8$

10. (a) $\left|\frac{1-i}{1+i}\right| = \left|\frac{1-i}{1+i}\,\frac{1-i}{1-i}\right| = \left|\frac{-2i}{2}\right| = 1$

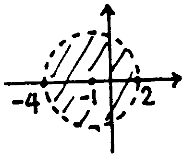
11.(a) $|z_1+z_2| = |(2+3i)+(4-i)| = |6+2i| = \sqrt{36+4} = \sqrt{40} = 6.324$
$|z_1|+|z_2| = |2+3i|+|4-i| = \sqrt{13}+\sqrt{17} = 7.729$ $\underline{\text{is}}$ ≥ 6.324. ✓

Section 21.3

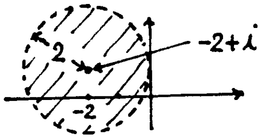
1. (a)



(b)

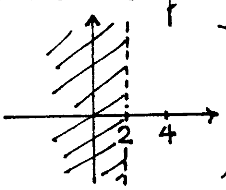


(c) $|z-(-2+i)|<2$



(d) The circle of radius 2, centered at $-2+i$.

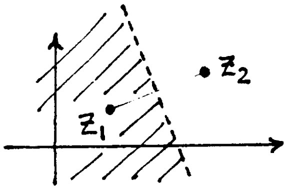
(e)



The open half plane $\text{Re}\,z<2$, i.e., $x<2$.

(f) $(x+1)^2+y^2 \leq x^2+y^2$ gives $2x+1 \leq 0$, hence, the half-plane $x \leq -1/2$. Could also have seen this by noting that $|z+1|=|z|$ is the locus of points equidistant from $z=-1$ and $z=0$, i.e., the line $x=-1/2$. Then $|z+1|\leq|z|$ is the half-plane $x \leq -1/2$.

(g)



(h) $\text{Re}(z-i)=\text{Re}(x+iy-i)=x$, so it is the half-plane $x>3$.

(i) $\sqrt{(x+1)^2+y^2} = \sqrt{x^2+y^2}+1$

$(x+1)^2+y^2 = x^2+y^2+2\sqrt{x^2+y^2}+1$

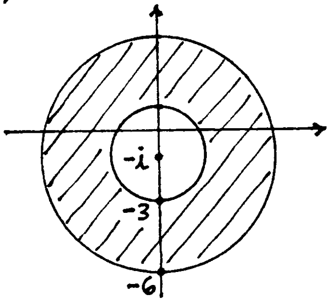
$\cancel{x^2}+2x+\cancel{1}+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}+2\sqrt{x^2+y^2}+\cancel{1}$

$x=\sqrt{x^2+y^2}$ ♀

Squaring gives $y=0$. However, we see from ♀ that we need $x\geq 0$. Thus, the set is comprised of the nonnegative x-axis.

(j) The half-plane $x<2$.

(k) $2 \leq |z-(-i)| \leq 5$



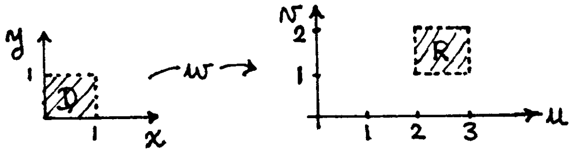
(l) $\text{Im}(z-i)>1$

$\text{Im}(x+i(y-1))>1$

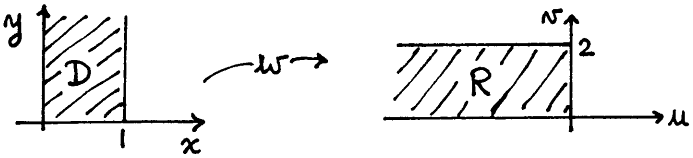
$y-1>1$, $y>2$.

The half-plane $y>2$.

2. (a) $w = z+2+i$



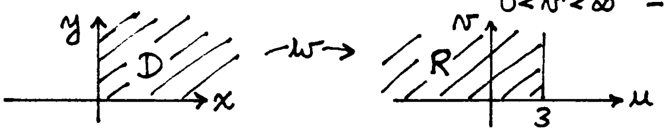
(b) $w=2iz=2i(x+iy)=\underbrace{-2y}_{u}+i\underbrace{2x}_{v}$. $0<x<1 \Rightarrow 0<v<2$
$0<y<\infty \Rightarrow -\infty<u<0$, so D and R are as shown:



(open regions)

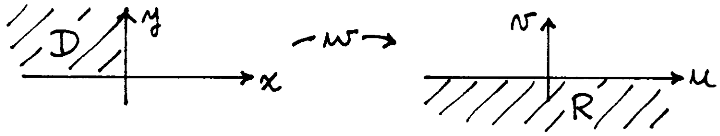
(c) $w = iz+3 = i(x+iy)+3 = \underbrace{3-y}_{u} + i\underbrace{x}_{v}$. $\quad 0<x<\infty,\ 0<y<\infty$

$\qquad\qquad\qquad\qquad 0<v<\infty \quad -\infty<u<3$



(open regions)

(d) $w = z^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{2xy}_{v}$. $\quad -\infty<x<0,\ 0<y<\infty \Rightarrow -\infty<u<\infty,\ -\infty<v<0$



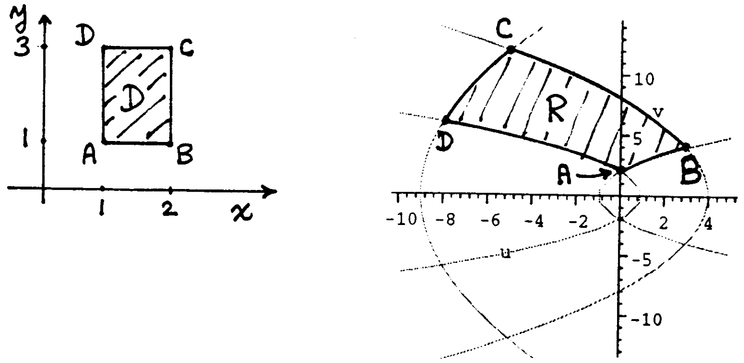
(e) $w = z^2 = \underbrace{(x^2-y^2)}_{u} + i\underbrace{2xy}_{v}$. Image of $x=1$: $\left.\begin{matrix} u=1-y^2 \\ v=2y \end{matrix}\right\} u = 1-\frac{v^2}{4}$

Image of $x=2$: $\left.\begin{matrix} u=4-y^2 \\ v=4y \end{matrix}\right\} u = 4-\frac{v^2}{16}$

Image of $y=1$: $\left.\begin{matrix} u=x^2-1 \\ v=2x \end{matrix}\right\} u = \frac{v^2}{4}-1$

Image of $y=3$: $\left.\begin{matrix} u=x^2-9 \\ v=6x \end{matrix}\right\} u = \frac{v^2}{36}-9$

maple:
```
> with(plots):
> implicitplot({u=1-v^2/4,u=4-v^2/16,u=
  -1+v^2/4,u=-9+v^2/36},u=-10..5,v=-14..14);
```

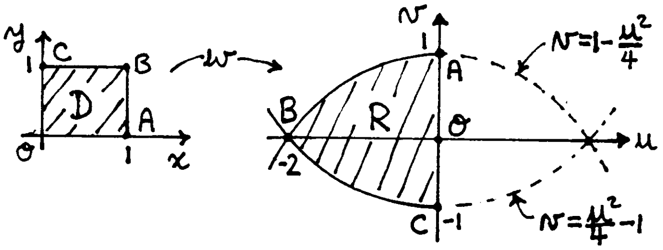


(f) $w = iz^2 = i[(x^2-y^2)+i2xy] = \underbrace{-2xy}_{u} + i\underbrace{(x^2-y^2)}_{v}$

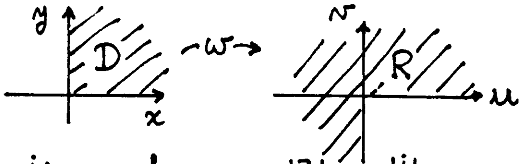
$x=0$: $u=0,\ v=-y^2$

$x=1$: $\left.\begin{matrix} u=-2y \\ v=1-y^2 \end{matrix}\right\} \to v = 1-\frac{u^2}{4}$

$y=0$: $u=0,\ v=x^2$
$y=1$: $u=-2x,\ v=x^2-1$ $\Big\} \to v = \frac{u^2}{4}-1$



(g) $w = z^3 = (x+iy)^3 = \underbrace{(x^3-3xy^2)}_{u} + i(\underbrace{3x^2y-y^3}_{v})$



3. $|e^z| \neq e^{|z|}$. For example, if $z=i$ then $|e^z|=|e^i|=1$ whereas $e^{|z|}=e^{|i|}=e$. It follows from this single counterexample that, in general, $|w(z)| \neq w(|z|)$.

4. $\overline{e^z} = e^x(\cos y - i\sin y)$ from (8)
$e^{\bar z} = e^{x-iy} = e^x(\cos y - i\sin y) = \overline{e^z}$. However, in general $\overline{w(z)} \neq w(\bar z)$. For example, if $w(z) = i$ then $\overline{w(z)} = -i$ but $w(\bar z) = i$.

5. (20a)-(20d) are important. Their derivation is simple, following immediately from the definitions of $\cos z, \sin z, \cosh z, \sinh z$. For ex.,

$$\cos iz = (e^{i(iz)}+e^{-i(iz)})/2 = (e^{-z}+e^{z})/2 = \cosh z.$$

6.(a)
$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}(\cos y_1\cos y_2 - \sin y_1\sin y_2 + i(\sin y_1\cos y_2 + \cos y_1\sin y_2)) \\ &= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)] \\ &= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}. \end{aligned}$$

(b) Let us use induction and the result established in part (a). Surely the proposition holds for $n=1$. Assume it holds for $n=k$. Then

$$\begin{aligned} (e^z)^{k+1} = (e^z)^k e^z &= e^{kz}e^z \text{ by assumption} \\ &= e^{kz+z} \text{ by (a)} \\ &= e^{(k+1)z}, \text{ so we have proof by induction.} \end{aligned}$$

7.(b)
$$\cos z_1\cos z_2 - \sin z_1\sin z_2 = \frac{e^{iz_1}+e^{-iz_1}}{2}\,\frac{e^{iz_2}+e^{-iz_2}}{2} - \frac{e^{iz_1}-e^{-iz_1}}{2i}\,\frac{e^{iz_2}-e^{-iz_2}}{2i}$$
$$= \frac{1}{4}\Big(e^{i(z_1+z_2)} + \cancel{e^{i(z_1-z_2)}} + \cancel{e^{-i(z_1-z_2)}} + e^{-i(z_1+z_2)}$$
$$+ e^{i(z_1+z_2)} - \cancel{e^{i(z_1-z_2)}} - \cancel{e^{-i(z_1-z_2)}} + e^{-i(z_1+z_2)}\Big)$$
$$= \frac{1}{4}(2e^{i(z_1+z_2)} + 2e^{-i(z_1+z_2)})$$

(e)
$$\cos x\cosh y - i\sin x\sinh y = \frac{e^{ix}+e^{-ix}}{2}\,\frac{e^{y}+e^{-y}}{2} - i\,\frac{e^{ix}-e^{-ix}}{2i}\,\frac{e^{y}-e^{-y}}{2}$$
$$= \frac{1}{4}(\cancel{e^{y+ix}} + e^{-y+ix} + e^{y-ix} + \cancel{e^{-y-ix}} - \cancel{e^{y+ix}} + e^{-y+ix} + e^{y-ix} - \cancel{e^{-y-ix}})$$
$$= \frac{1}{4}(2e^{-y+ix} + 2e^{y-ix}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \cos(x+iy).$$

8.(b)
$$\cosh z_1\cosh z_2 + \sinh z_1\sinh z_2 = \frac{e^{z_1}+e^{-z_1}}{2}\,\frac{e^{z_2}+e^{-z_2}}{2} + \frac{e^{z_1}-e^{-z_1}}{2}\,\frac{e^{z_2}-e^{-z_2}}{2}$$
$$= \frac{1}{4}(e^{z_1+z_2} + \cancel{e^{z_1-z_2}} + \cancel{e^{-z_1+z_2}} + e^{-(z_1+z_2)} + e^{z_1+z_2} - \cancel{e^{z_1-z_2}} - \cancel{e^{-z_1+z_2}} + e^{-(z_1+z_2)})$$
$$= \frac{1}{4}(2e^{(z_1+z_2)} + 2e^{-(z_1+z_2)}) = \cosh(z_1+z_2).$$

(e) $\cosh x\cos y + i\sinh x\sin y = \frac{e^x+e^{-x}}{2}\,\frac{e^{iy}+e^{-iy}}{2} + i\,\frac{e^x-e^{-x}}{2}\,\frac{e^{iy}-e^{-iy}}{2i}$

$$= \frac{1}{4}\left(e^{x+iy} + \cancel{e^{x-iy}} + \cancel{e^{-x+iy}} + e^{-(x+iy)} + e^{x+iy} - \cancel{e^{x-iy}} - \cancel{e^{-x+iy}} + e^{-(x+iy)}\right)$$

$$= \frac{1}{4}\left(2e^{(x+iy)} + 2e^{-(x+iy)}\right) = \cos(x+iy)$$

9. (a) $e^{2+\pi i} = e^2(\cos\pi + i\sin\pi) = -e^2$ (b) $e^{1-i} = e[\cos 1 - i\sin 1]$

(c) $e^{-\pi i/4} = \cos\pi/4 - i\sin\frac{\pi}{4} = \frac{1}{\sqrt2} - i\frac{1}{\sqrt2}$ (d) $\sin(3+\pi i) = \sin 3\cos\pi i + \sin\pi i\cos 3$
$= \sin 3\cosh\pi + i\sinh\pi\cos 3$

(e) $\cos(-2+3\pi i) = \cos 2\cos 3\pi i + \sin 2\sin 3\pi i = \cos 2\cosh 3\pi + i\sin 2\sinh 3\pi$

(f) $\sec(1+i) = 1/\cos(1+i) = 1/(\cos 1\cos i - \sin 1\sin i) = 1/(\cos 1\cosh 1 - i\sin 1\sinh 1)$

$$= \frac{1}{\cos 1\cosh 1 - i\sin 1\sinh 1}\,\frac{\cos 1\cosh 1 + i\sin 1\sinh 1}{\cos 1\cosh 1 + i\sin 1\sinh 1}$$

$$= \frac{\cos 1\cosh 1}{(\cos 1\cosh 1)^2 + (\sin 1\sinh 1)^2} + i\,\frac{\sin 1\sinh 1}{(\cos 1\cosh 1)^2 + (\sin 1\sinh 1)^2}$$ [See (g), below.]

(g) $\csc(1-i) = 1/\sin(1-i) = \frac{2i}{e^{i(1-i)} - e^{-i(1-i)}} = \frac{2i}{e^{1+i} - e^{-1-i}}$ (Now multiply top and bottom by complex conj. of denominator)

$$= \frac{2i}{e^{1+i} - e^{-1-i}}\,\frac{e^{1-i} - e^{-1+i}}{e^{1-i} - e^{-1+i}} = \frac{2i[e(\cos 1 - i\sin 1) - e^{-1}(\cos 1 + i\sin 1)]}{e^2 - e^{2i} - e^{-2i} + e^{-2}}$$

$$= \frac{2i[(e\cos 1 - e^{-1}\cos 1) - i(e\sin 1 + e^{-1}\sin 1)]}{(e^2+e^{-2}) - (e^{2i}+e^{-2i})} = \frac{i\cos 1\,(e-e^{-1}) + \sin 1\,(e+e^{-1})}{\cosh 2 - \cos 2}$$

$$= \frac{2\sin 1\cosh 1}{\cosh 2 - \cos 2} + i\,\frac{2\cos 1\sinh 1}{\cosh 2 - \cos 2}$$

NOTE: Our procedure has been different in (f) and (g); we could have used the method of (g) in (f) or visa versa. Note also that the Maple symbol for i is I. The command

evalf(csc(1-I));

gives .6215180172+.3039310016I

(h) $\tan\left(-\frac{3\pi i}{4}\right) = \frac{\sin(-3\pi i/4)}{\cos(-3\pi i/4)} = -i\,\frac{\sinh 3\pi/4}{\cosh 3\pi/4} = -i\tanh\frac{3\pi}{4}$

(i) $\cot\left(\frac{\pi i}{4}\right) = \frac{\cos(\pi i/4)}{\sin(\pi i/4)} = \frac{\cosh\pi/4}{i\sinh\pi/4} = -i\coth\frac{\pi}{4}$

(j) $\sinh(3+\pi i) = \sinh 3\cosh\pi i + \sinh\pi i\cosh 3$ (Here I've used Exercise 8(c), but we could simply use the definition of sinh.)
$= \sinh 3\cos\pi + i\cancelto{0}{\sin\pi}\cosh 3$
$= -\sinh 3$

(k) $\cosh(1-\pi i) = \cosh 1\cosh(-\pi i) + \sinh(1)\sinh(-\pi i)$ (by Exercise 8(b))
$= \cosh 1\cos\pi + (\sinh 1)(-i)\sin\pi = -\cosh 1.$

(l) $\tanh(2+4\pi i) = \frac{\sinh(2+4\pi i)}{\cosh(2+4\pi i)} = \frac{\sinh 2\cosh 4\pi i + \sinh 4\pi i\cosh 2}{\cosh 2\cosh 4\pi i + \sinh 2\sinh 4\pi i}$

$$= \frac{\sinh 2\cos 4\pi + i\sin 4\pi\cosh 2}{\cosh 2\cos 4\pi + i\sinh 2\sin 4\pi} = \frac{\sinh 2}{\cosh 2} = \tanh 2$$

10. The step $|\cos z + i\sin z| = \sqrt{\cos^2 z + \sin^2 z}$ is incorrect. It holds if $\cos z$ and $\sin z$ are both real, but they are not.

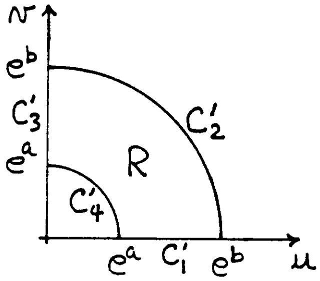
11. (a) $e^z = e^x(\cos y + i\sin y) = 1 = 1 + 0i \Rightarrow$ $e^x\cos y = 1,$ ①
$e^x\sin y = 0.$ ②

Now, $e^x \neq 0$ for all x so ② $\Rightarrow \sin y = 0$ so $y = 0, \pm\pi, \pm 2\pi, \dots$. For $y = 0, \pm 2\pi, \pm 4\pi, \dots$ ① becomes $e^x = 1$ so $x = 0$; for $y = \pm\pi, \pm 3\pi, \dots$ ① becomes $-e^x = 1$ which has no real roots for x. Thus, $e^z = 1$ has only the roots $z = 0 + 2n\pi i$ where $n = 0, \pm 1, \pm 2, \dots$.

(b) $e^{z_1} = e^{z_2} \rightarrow e^{z_1 - z_2} = 1$, and (a) $\rightarrow z_1 - z_2 = 2n\pi i$ or $z_1 = z_2 + 2n\pi i$.

12. $w = e^z = \underbrace{e^x\cos y}_{u} + i\,\underbrace{e^x\sin y}_{v}$

$$\left.\begin{array}{l}
C_1:\ u = e^x,\ v = 0,\ a < x < b \rightarrow e^a < u < e^b \\
C_2:\ u = e^b\cos y,\ v = e^b\sin y,\ 0 < y < \pi/2 \\
\qquad \text{or, } u^2 + v^2 = (e^b)^2 \\
C_3:\ u = 0,\ v = e^x,\ a < x < b \rightarrow e^a < v < e^b \\
C_4:\ u = e^a\cos y,\ v = e^a\sin y,\ 0 < y < \pi/2 \\
\qquad \text{or, } u^2 + v^2 = (e^a)^2.
\end{array}\right\} \Rightarrow$$

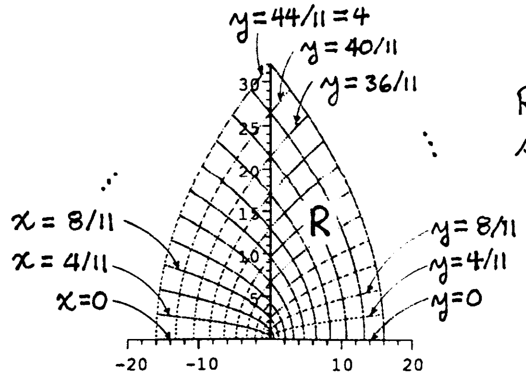


13. $\sin z = \sin(x + iy) = \sin x \cos iy + \sin iy \cos x = \underbrace{\sin x \cosh y}_{u} + i\,\underbrace{\sinh y \cos x}_{v}$,

$y = 0, -\pi/2 < x < \pi/2 \rightarrow u = \sin x, v = 0$ gives the segment $-1 < u < 1$ of the u axis
$x = \pi/2, 0 < y < \infty \rightarrow u = \cosh y, v = 0$ gives the segment $1 < u < \infty$ of the u axis
$x = -\pi/2, 0 < y < \infty \rightarrow u = -\cosh y, v = 0$ gives the segment $-\infty < u < -1$ of the u axis
Further, $z = i \rightarrow w = i\sinh 1$, so R is evidently the <u>upper</u> half plane, not the lower half plane.

14. (a) $(4 + 4i)^2 = 16 + 32i - 16 = 32i$

```
> with(plots):
> conformal(z^2,z=0..4+4*I,w=-20..20+32*I,grid=[12,12],numxy=[40,40]
  );
```

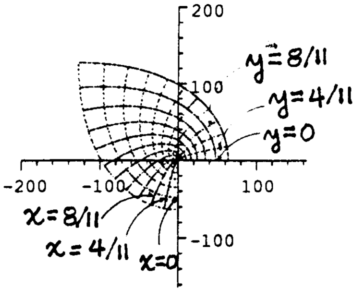


R is the image of the square $0 < x < 4$, $0 < y < 4$.

15.(a)

```
> conformal(z^3,z=0..4+4*I,w=-200-164*I..164+200*I,grid=[12,12],numx
  y=[40,40]);
```

NOTE: The image of the region $0<x<\infty, 0<y<\infty$ would be the 1st, 2nd, and 3rd quadrants.



16.(a) $I = \operatorname{Im}\int_0^\infty e^{-x}e^{i\omega x}\,dx = \operatorname{Im}\int_0^\infty e^{-(1-i\omega)x}\,dx$

$$= \operatorname{Im}\left.\frac{e^{-(1-i\omega)x}}{-1+i\omega}\right|_0^\infty = \operatorname{Im}\left(0-\frac{1}{-1+i\omega}\right) = \operatorname{Im}\left(\frac{1}{1-i\omega}\,\frac{1+i\omega}{1+i\omega}\right)$$

$= \operatorname{Im}\frac{1+i\omega}{1+\omega^2} = \omega/(1+\omega^2)$. NOTE: In more detail, $e^{-(1-i\omega)x}=0$ at $x=\infty$ because $\lim_{x\to\infty}|e^{-(1-i\omega)x}| = \lim_{x\to\infty}|e^{-x}e^{i\omega x}| = \lim_{x\to\infty}e^{-x}|e^{i\omega x}|$ $= \lim_{x\to\infty}e^{-x} = 0$. Now, if $|e^{-(1-i\omega)x}|\to 0$ as $x\to\infty$, then $e^{-(1-i\omega)x}\to 0$ as $x\to\infty$.

(b) $I = \int_0^\infty e^{-st}\cos\omega t\,dt$. Be careful; if s is complex (with nonzero imaginary part) then $e^{-st}\cos\omega t = \operatorname{Re}(e^{-st}e^{i\omega t}) = \operatorname{Re}(e^{-(s-i\omega)t})$ is <u>not true</u>.* To use the method let us assume that s is <u>real</u>, with $s>0$. Then

$$I = \operatorname{Re}\int_0^\infty e^{-(s-i\omega)t}\,dt = \operatorname{Re}\left.\frac{e^{-(s-i\omega)t}}{-(s-i\omega)}\right|_0^\infty = \operatorname{Re}\left(0-\frac{1}{-s+i\omega}\right) = \operatorname{Re}\left(\frac{1}{s-i\omega}\,\frac{s+i\omega}{s+i\omega}\right)$$

$$= \operatorname{Re}\left(\frac{s+i\omega}{s^2+\omega^2}\right) = s/(s^2+\omega^2).$$

17. (a) The problem is that $\cos mx\cos nx = (\operatorname{Re}e^{imx})(\operatorname{Re}e^{inx}) \neq \operatorname{Re}(e^{i(m+n)x})$. Let us show that:

$$\operatorname{Re}e^{i(m+n)x} = \cos(m+n)x,$$

but
$$\cos mx\cos nx = \tfrac{1}{2}(e^{imx}+e^{-imx})\tfrac{1}{2}(e^{inx}+e^{-inx})$$
$$= \tfrac{1}{4}(e^{i(m+n)x}+e^{-i(m+n)x}+e^{i(m-n)x}+e^{-i(m-n)x})$$
$$= \tfrac{1}{2}\cos(m+n)x + \tfrac{1}{2}\cos(m-n)x \neq \cos(m+n)x.$$

18. (b) $v'+2v = 10e^{i3t}$. Seek $v_p = Ae^{i3t}$. $(3i+2)Ae^{i3t} = 10e^{i3t}$ so $A = 10/(2+3i)$.

$$x_p(t) = \operatorname{Im}v_p(t) = \operatorname{Im}\frac{10e^{i3t}}{2+3i}\,\frac{2-3i}{2-3i} = \frac{10}{13}\operatorname{Im}[(\cos 3t+i\sin 3t)(2-3i)]$$
$$= \frac{10}{13}(2\sin 3t-3\cos 3t)$$

---

* Nevertheless, it is fortuitous that the result $I = s/(s^2+\omega^2)$ is correct even if s is complex (provided that $\operatorname{Re}s>0$).

(c) $x_p(t) = \frac{10}{17}(3\cos 5t + 5\sin 3t)$

(d) $x'' + x' = 100e^{i5t}$, $x_p = Ae^{i5t}$, $(-25+5i)A\cancel{e^{i5t}} = 100\cancel{e^{i5t}}$ so $A = 100/(-25+5i)$.
$x_p(t) = \operatorname{Im}\left(\frac{20}{-5+i}\,\frac{-5-i}{-5-i}\,e^{i5t}\right) = -\frac{20}{26}\operatorname{Im}[(5+i)(\cos 5t + i\sin 5t)]$
$= -\frac{10}{13}(\cos 5t + 5\sin 5t)$.

(e) $x'''' + 2x' + x = 10e^{it}$, $x_p = Ae^{it}$, $(1+2i+1)A\cancel{e^{it}} = 10\cancel{e^{it}}$ so $A = 10/(2+2i)$.
$x_p(t) = \operatorname{Im}\left(\frac{10}{2+2i}e^{it}\right) = 5\operatorname{Im}\left[\frac{(\cos t + i\sin t)}{1+i}\,\frac{1-i}{1-i}\right] = \frac{5}{2}(\sin t - \cos t)$.

(f) $x'''' - x' + 5x = 20e^{i2t}$, $x_p = Ae^{i2t}$, $(16-2i+5)A\cancel{e^{i2t}} = 20\cancel{e^{i2t}}$, $A = 20/(21-2i)$
$x_p(t) = \operatorname{Re}\left(\frac{20}{21-2i}\,\frac{21+2i}{21+2i}(\cos 2t + i\sin 2t)\right) = \frac{20}{445}(21\cos 2t - 2\sin 2t)$

(g) $x'''' - 2x' - 3x = 60e^{i3t}$, $x_p = Ae^{i3t}$, $(81-6i-3)A\cancel{e^{i3t}} = 60\cancel{e^{i3t}}$, $A = 60/(78-6i)$
$x_p(t) = \operatorname{Im}\frac{30}{39-3i}e^{i3t} = \operatorname{Im}\left(\frac{10}{13-i}\,\frac{13+i}{13+i}(\cos 3t + i\sin 3t)\right)$

$= \frac{10}{170}\operatorname{Im}[(13+i)(\cos 3t + i\sin 3t)] = \frac{1}{17}(13\sin 3t + \cos 3t)$.

Section 21.4

1. (a) $z = -3i$, $r = 3$, $\theta = -\pi/2$ rad $= -90°$
   (b) $z = 8i$, $r = 8$, $\theta = \pi/2$ rad $= 90°$
   (c) $z = -6$, $r = 6$, $\theta = \pi$ rad $= 180°$ (Recall that $-\pi < \theta \le \pi$ for principal arg.)
   (d) $z = 1+5i$, $r = \sqrt{26}$, $\theta = \tan^{-1}5 = 1.373$ rad $= 78.69°$
   (e) $z = -4-3i$, $r = 5$, $\theta = \tan^{-1}\frac{3}{4} = -2.498$ rad $= -143.13°$
   (f) $z = 2-12i$, $r = \sqrt{148} = 2\sqrt{37}$, $\theta = \tan^{-1}\left(\frac{-12}{2}\right) = -1.406$ rad $= -80.538°$
   (g) $z = -1+i$, $r = \sqrt{2}$, $\theta = 3\pi/4$ rad $= 135°$
   (h) $z = -1-i$, $r = \sqrt{2}$, $\theta = -3\pi/4$ rad $= -135°$
   (i) $z = 0.2+i$, $r = \sqrt{1.04}$, $\theta = \tan^{-1}\left(\frac{1}{.2}\right) = 1.373$ rad $= 87.433°$

2. $\operatorname{Re}(re^{i\theta}) = \operatorname{Re}(r\cos\theta + ir\sin\theta) = r\cos\theta$
   $\operatorname{Im}(\;''\;) = \operatorname{Im}(\quad''\quad) = r\sin\theta$

3. Product: $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2e^{i(\theta_{10}+2m\pi+\theta_{20}+2n\pi)}$
   $= r_1r_2e^{i(\theta_{10}+\theta_{20})}\underbrace{e^{i2(m+n)\pi}}_{=1 \text{ for all integers } m \text{ and } n} = r_1r_2e^{i(\theta_{10}+\theta_{20})}$

Similarly for z_1/z_2.

4. (a) $(-1+i)^{10} = (2^{1/2} e^{(3\pi/4)i})^{10} = 2^5 e^{15\pi i/2} = 2^5 e^{6\pi i} e^{3\pi i/2} = \underbrace{32 e^{3\pi i/2}}_{\text{polar}} = \underbrace{-32i}_{\text{cartesian}}$

Of course the polar form $32e^{15\pi i/2}$ was OK too but usually we prefer the arg to be in $0 \leq \theta < 2\pi$ or in $-\pi < \theta \leq \pi$ (the latter for the principal arg)

$(-1+i)^{20} = 2^{10} e^{15\pi i} = \underbrace{1024 e^{\pi i}}_{\text{Polar}} = \underbrace{-1024}_{\text{Cartesian}}$

(b) $(1+i)^{10} = (2^{1/2} e^{\pi i/4})^{10} = 2^5 e^{5\pi i/2} = \underbrace{32 e^{\pi i/2}}_{\text{Polar}} = \underbrace{32i}_{\text{Cartesian}}$

$(1+i)^{20} = (2^{1/2} e^{\pi i/4})^{20} = 2^{10} e^{5\pi i} = \underbrace{1024 e^{\pi i}}_{\text{Polar}} = \underbrace{-1024}_{\text{Cartesian}}$

(c) $(1+2i)^{10} = (5^{1/2} e^{1.107i})^{10} = 5^5 e^{11.07i} = \underbrace{3125 e^{4.787i}}_{\text{Polar}} = \underbrace{237-3116i}_{\text{Cartesian}}$

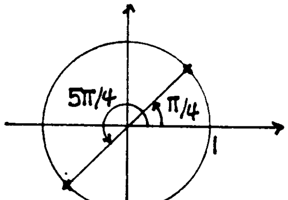
$(1+2i)^{20} = (5^{\frac{1}{2}} e^{1.107i})^{20}$ Before continuing, note that since we are multiplying the 1.107 by 20 we will not obtain very accurate results (i.e., say to 4 significant figures), so let us include more places to begin with: $\tan^{-1} 2 = 1.1071487$.

$= (5^{1/2} e^{1.1071487i})^{20} = 5^{10} e^{22.14297i} = 5^{10} e^{15.85979i} = 5^{10} e^{9.57660i}$

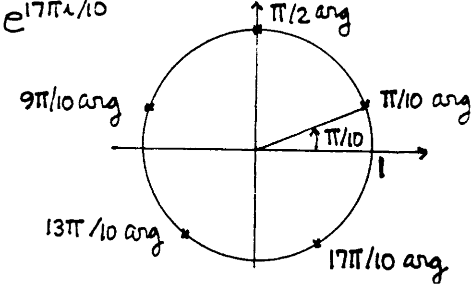
$= \underbrace{5^{10} e^{3.2934i}}_{\text{polar}} = \underbrace{5^{10}(-0.9885 - .1512i) = -9653320.3 - 1476562.5i}_{\text{Cartesian}}$

The maple command (1+2*I)^20 gives
$-9653287 - 1476984i$

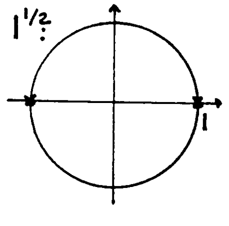
5. (a) $i^{1/2} = (1e^{\pi i/2})^{1/2}, (1e^{5\pi i/2})^{1/2}$
$= e^{\pi i/4}, e^{5\pi i/4}$

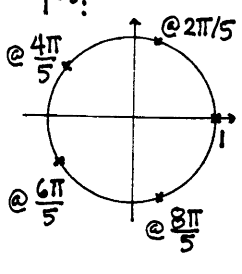


$i^{1/5} = (1e^{\pi i/2})^{1/5}, (1e^{5\pi i/2})^{1/5}, (1e^{9\pi i/2})^{1/5}, (1e^{13\pi i/2})^{1/5}, (1e^{17\pi i/2})^{1/5}$
$= e^{\pi i/10}, e^{\pi i/2}, e^{9\pi i/10}, e^{13\pi i/10}, e^{17\pi i/10}$

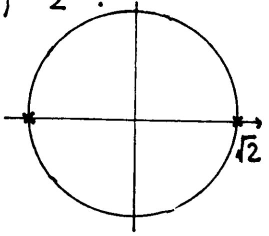


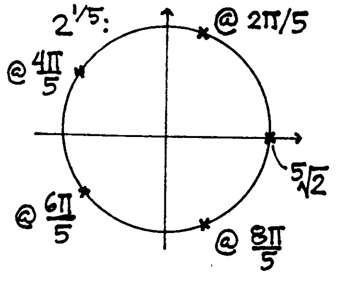
(b)





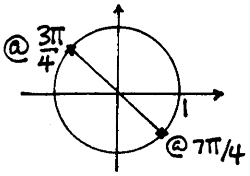
(c)





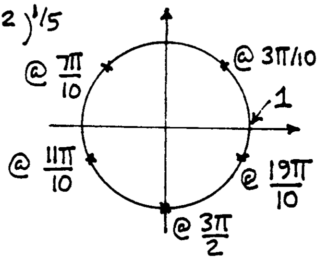
(d) $(-i)^{1/2} = (1e^{3\pi i/2})^{1/2}, (1e^{7\pi i/2})^{1/2}$

$= 1e^{3\pi i/4}, 1e^{7\pi i/4}$



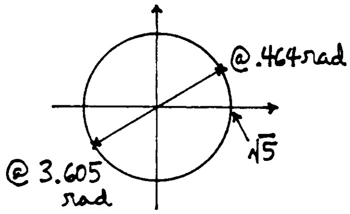
$(-i)^{1/5} = (1e^{3\pi i/2})^{1/5}, (1e^{7\pi i/2})^{1/5}, (1e^{11\pi i/2})^{1/5}, (1e^{15\pi i/2})^{1/5}, (1e^{19\pi i/2})^{1/5}$

$= 1e^{3\pi i/10}, 1e^{7\pi i/10}, 1e^{11\pi i/10}, 1e^{3\pi i/2}, 1e^{19\pi i/10}$



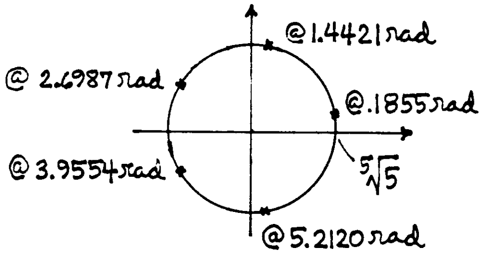
(g) $(3+4i)^{1/2} = (5e^{0.92730i})^{1/2}, (5e^{7.21048i})^{1/2}$

$= \sqrt{5}\, e^{.4637i}, \sqrt{5}\, e^{3.6052i}$



$(3+4i)^{1/5} = (5e^{i.92730})^{1/5}, (5e^{7.21048i})^{1/5}, (5e^{13.49367i})^{1/5}, (5e^{19.77686i})^{1/5}, (5e^{26.06004i})^{1/5}$

$= \sqrt[5]{5}\, e^{.1855i}, \sqrt[5]{5}\, e^{1.4421i}, \sqrt[5]{5}\, e^{2.6987i}, \sqrt[5]{5}\, e^{3.9554i}, \sqrt[5]{5}\, e^{5.2120i}$



6. (a) $\log(-2) = \log(2e^{(\pi+2n\pi)i}) = \ln 2 + (2n+1)\pi i \quad (n=0,\pm1,\pm2,\ldots)$

(b) $\log(1) = \log(1e^{2n\pi i}) = \ln 1 + 2n\pi i = 2n\pi i \quad (\quad " \quad)$

(c) $\log(i) = \log(1e^{(\frac{\pi}{2}+2n\pi)i}) = \ln 1 + (\frac{4n+1}{2})\pi i \quad (\quad " \quad)$

(d) $\log(-5i) = \log(5e^{(-\frac{\pi}{2}+2n\pi)i}) = \ln 5 + (\frac{4n-1}{2})\pi i \quad (\quad " \quad)$

(e) $\log(2-i) = \log(\sqrt{5}\, e^{(-.4636+2n\pi)i}) = \frac{1}{2}\ln 5 + (-.4636+2n\pi)i \quad (n=0,\pm1,\pm2,\ldots)$

7. $x_1 = x_2$ and $y_1 = y_2$ gives $\begin{cases} r_1\cos\theta_1 = r_2\cos\theta_2 \\ r_1\sin\theta_1 = r_2\sin\theta_2 . \end{cases}$

Squaring and adding gives $r_1^2 = r_2^2$ so $r_1 = r_2$. Then, $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$ give $\theta_1 = \theta_2$ + arbitrary integer multiple of 2π.

8. (a) $(2i)^{2/3} \overset{\text{not an essential step}}{=} (-4)^{1/3} = (4e^{i(\pi+2k\pi)})^{1/3} = \sqrt[3]{4}\, e^{\pi i/3}, \sqrt[3]{4}\, e^{\pi i}, \sqrt[3]{4}\, e^{5\pi i/3}$

$(2i)^{3/2} = (-8i)^{1/2} = (8e^{i(\frac{3\pi}{2}+2k\pi)})^{1/2} = \sqrt{8}\, e^{3\pi i/4}, \sqrt{8}\, e^{7\pi i/4}$

$(2i)^{\pi} = e^{\pi \log 2i} = \exp\{\pi \log[2e^{(\frac{\pi}{2}+2k\pi)i}]\} = \exp\{\pi[\ln 2 + (\frac{\pi}{2}+2k\pi)i]\}$

$= e^{\pi \ln 2}\, e^{i(1+4k)\pi^2/2} \quad (k=0,\pm1,\pm2,\ldots)$

(b) $3^{2/3} = 9^{1/3} = (9e^{i2k\pi})^{1/3} = \sqrt[3]{9}, \sqrt[3]{9}\, e^{2\pi i/3}, \sqrt[3]{9}\, e^{4\pi i/3}$

$3^{3/2} = 27^{1/2} = (27e^{i2k\pi})^{1/2} = \sqrt{27}, -\sqrt{27}$

$3^{\pi} = e^{\pi \log 3} = e^{\pi(\ln 3 + 2k\pi i)} = e^{\pi \ln 3}\, e^{2k\pi^2 i}$

(e) $(1-i)^{2/3} = (-2i)^{1/3} = \left(2e^{(-\frac{\pi}{2}+2k\pi)i}\right)^{1/3} = \sqrt[3]{2}\, e^{-\pi i/6},\ \sqrt[3]{2}\, e^{\pi i/2},\ \sqrt[3]{2}\, e^{7\pi i/6}$

$(1-i)^{3/2} = (-2-2i)^{1/2} = \left(\sqrt{8}\, e^{(\frac{5\pi}{4}+2k\pi)i}\right)^{1/2} = \sqrt[4]{8}\, e^{5\pi i/8},\ \sqrt[4]{8}\, e^{13\pi i/8}$

$$(1-i)^{\pi} = e^{\pi \log(1-i)} = e^{\pi \log[\sqrt{2}\, e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{\pi \ln\sqrt{2} + \pi(8k\pi-1)i/4}$$
$$= e^{\pi\ln\sqrt{2}}\, e^{\pi(8k\pi-1)i/4} \quad (k=0,\pm1,\ldots)$$

9. (a) $(2i)^{i} = e^{i\log 2i} = e^{i\log(2e^{(\frac{\pi}{2}+2k\pi)i})} = e^{i[\ln 2 + (4k+1)\pi i/2]}$
$$= e^{-(4k+1)\pi/2}[\cos(\ln 2) + i\sin(\ln 2)]$$

$$(2i)^{1-i} = e^{(1-i)\log(2i)} = e^{(1-i)\log[2e^{(\frac{\pi}{2}+2k\pi)i}]} = e^{(1-i)[\ln 2 + (\frac{\pi}{2}+2k\pi)i]}$$
$$= e^{\ln 2 + (\pi/2+2k\pi)}\, e^{i(\frac{\pi}{2}+2k\pi-\ln 2)}$$
$$= 2e^{(4k+1)\pi/2}\left[\cos(\tfrac{\pi}{2}+2k\pi-\ln 2) + i\sin(\tfrac{\pi}{2}+2k\pi-\ln 2)\right] \quad (k=0,\pm1,\ldots)$$

(b) $3^{i} = e^{i\log 3} = e^{i\log[3e^{2k\pi i}]} = e^{i[\ln 3 + 2k\pi i]} = e^{-2k\pi}(\cos(\ln 3) + i\sin(\ln 3)),$
for $k=0,\pm1,\ldots$

$$3^{1-i} = 3e^{-i\log 3} = 3e^{-i[\ln 3 + 2k\pi i]} = 3e^{2k\pi}(\cos(\ln 3) + i\sin(\ln 3)) \text{ for } k=0,\pm1,\ldots$$

(e) $(1-i)^{i} = e^{i\log(1-i)} = e^{i\log[\sqrt{2}\, e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{i[\ln\sqrt{2} + (-\frac{\pi}{4}+2k\pi)i]}$
$$= e^{(\frac{\pi}{4}-2k\pi)}[\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})]$$

$$(1-i)^{1-i} = e^{(1-i)\log(1-i)} = e^{(1-i)\log[\sqrt{2}\, e^{(-\frac{\pi}{4}+2k\pi)i}]} = e^{(1-i)[\ln\sqrt{2} + (-\frac{\pi}{4}+2k\pi)i]}$$
$$= e^{\ln\sqrt{2} - \frac{\pi}{4} + 2k\pi}\, e^{i[2k\pi - \frac{\pi}{4} - \ln\sqrt{2}]}$$
$$= \sqrt{2}\, e^{2k\pi - \pi/4}\left[\cos(2k\pi - \tfrac{\pi}{4} - \ln\sqrt{2}) + i\sin(2k\pi - \tfrac{\pi}{4} - \ln\sqrt{2})\right] \quad (k=0,\pm1,\ldots)$$

10. With $c = 1+\sqrt{3}i$, $|c| = 2$ and $\operatorname{Arg} c = \pi/3$, (10.2) gives, for $z = 2-5i$,
$$c^{z} = e^{(2-5i)(\ln 2 + i\pi/3)} = e^{2\ln 2 + 5\pi/3}\, e^{i(2\pi/3 - 5\ln 2)}$$
$$= 4e^{5\pi/3}\left[\cos(\tfrac{2\pi}{3} - 5\ln 2) + i\sin(\tfrac{2\pi}{3} - 5\ln 2)\right]$$

11. (a) $\log(-3i) = \log(3e^{-\pi i/2}) = \ln 3 - \pi i/2$
$$\sqrt{-3i} = (3e^{-\pi i/2})^{1/2} = \sqrt{3}\, e^{-\pi i/4} = \frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{2}$$

(b) $\log 2 = \log(2e^{i0}) = \ln 2 + i0 = \ln 2$
$$\sqrt{2} = (2e^{i0})^{1/2} = \sqrt{2}\, e^{i0} = \sqrt{2}$$

(c) $\log(-4) = \log(4e^{i\theta})$. Is $\theta = +\pi$ or $-\pi$? If the point -4 is on top of the cut then $\theta = \pi$ and
$$\log(-4) = \log(4e^{i\pi}) = \ln 4 + i\pi,$$
and if -4 is on the bottom of the cut then $\theta = -\pi$ and
$$\log(-4) = \log(4e^{-i\pi}) = \ln 4 - i\pi$$

NOTE: We can't "figure out" whether -4 means the point -4 on top of the cut or the point -4 on the bottom of the cut; we need to specify whether it is on the top or bottom (and it does have to be one or the other!).

(d) $\log(2-i) = \log(\sqrt{5}\, e^{-.4636 i}) = \ln\sqrt{5} - 0.4636 i$

NOTE: What does maple give for $\log(2-i)$?

The command log(2-I); merely gives the output "ln(2-I)",
the command eval(log(2-I)); does the same, but
the command evalf(log(2-I)); does give the principal value,
.8047189562 - .4636476090I.

$\sqrt{2-i} = (\sqrt{5}\, e^{-.4636 i})^{1/2} = \sqrt[4]{5}\, e^{-.2318 i} = \sqrt[4]{5}(\cos .2318 - i \sin .2318)$

Likewise, the Maple command evalf(sqrt(2-I)); gives this same value,
namely, 1.455346690 - .3435607497I.

(e) $\log(1+\sqrt{3} i) = .6931471807 + 1.047197551 i$

$\sqrt{1+\sqrt{3} i} = 1.224744871 + .7071067813 i$

(f) $\log(-1-i) = .3465735903 - 2.356194490 i$

$\sqrt{-1-i} = .4550898606 - 1.098684113 i$

(g) $\log(-5i) = 1.609437912 - 1.570796327 i$

$\sqrt{-5i} = 1.581138830 - 1.581138830 i$

(h) $\log(4-2i) = 1.497866137 - .4636476090 i$

$\sqrt{4-2i} = 2.058171027 - .4858682718 i$

12. (a) $\log(z_1 z_2) = \log(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) = \log(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \ln(r_1 r_2) + i(\theta_1+\theta_2)$

$= (\ln r_1 + i\theta_1) + (\ln r_2 + i\theta_2) = \log(r_1 e^{i\theta_1}) + \log(r_2 e^{i\theta_2}) = \log z_1 + \log z_2$ ✓

(b) Similar to (a).

(c) $\log z^c = \log e^{c \log z}$ (let $c = a+ib$, say)

$= \log e^{(a+ib)(\ln r + i\theta)} = \log(e^{a \ln r - b\theta} e^{i(b \ln r + a\theta)})$

$= (a \ln r - b\theta) + i(b \ln r + a\theta)$

$c \log z = (a+ib)(\ln r + i\theta) = (a \ln r - b\theta) + i(b \ln r + a\theta) = \log z^c$. ✓

13. (a) $z = \sin w = (e^{iw} - e^{-iw})/2i$ gives $(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$ so, by the quadratic formula, $e^{iw} = (2iz \pm \sqrt{-4z^2+4})/2 = iz + \sqrt{1-z^2}$, where there is no loss in dropping the ± since the $\sqrt{\ }$ always gives the ±. Then, log of both sides gives

$$iw = \log(iz + \sqrt{1-z^2})$$

$$w = -i \log(iz + \sqrt{1-z^2})$$

$$\sin^{-1} z = -i \log(iz + \sqrt{1-z^2}).$$

(b) $\sin^{-1}(\frac{1}{2}) = -i \log(\frac{1}{2} i \pm \frac{\sqrt{3}}{2})$

Using the upper (+) sign gives

$$\sin^{-1}(\tfrac{1}{2}) = -i \log(\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}) = -i \log(1 e^{i(\frac{\pi}{6} + 2k\pi)}) = \frac{\pi}{6} + 2k\pi$$

and using the lower sign gives

$$\sin^{-1}(\tfrac{1}{2}) = -i \log(-\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}) = -i \log(1 e^{i(\frac{5\pi}{6} + 2k\pi)}) = \frac{5\pi}{6} + 2k\pi,$$

for $k = 0, \pm 1, \ldots$.

(c) $\sin^{-1} 2 = -i \log(2i \pm \sqrt{3}\, i) = -i \log[(2 \pm \sqrt{3})i]$ both are positive and real

$$= -i \log[(2 \pm \sqrt{3})\, e^{(\frac{\pi}{2} + 2k\pi)i}] = -i \ln \boxed{2 \pm \sqrt{3}} + (\tfrac{\pi}{2} + 2k\pi)$$

$$= (\tfrac{\pi}{2} + 2k\pi) - i \ln(2 \pm \sqrt{3}) \qquad (k = 0, \pm 1, \ldots)$$

(d) $\sin^{-1}(2i) = -i \log[i(2i) \pm \sqrt{5}\,] = -i \log(-2 \pm \sqrt{5})$.

The upper sign gives (since $-2 + \sqrt{5} > 0$)

$$\sin^{-1}(2i) = -i \log[(\sqrt{5} - 2)\, e^{i(0 + 2k\pi)}]$$

$$= -i [\ln(\sqrt{5} - 2) + 2k\pi i] = 2k\pi - i \ln(\sqrt{5} - 2)$$

and the lower sign gives (since $-2 - \sqrt{5} < 0$)

$$\sin^{-1}(2i) = -i \log[(\sqrt{5} + 2)\, e^{i(\pi + 2k\pi)}]$$

$$= -i [\ln(\sqrt{5} + 2) + (2k+1)\pi i] = (2k+1)\pi - i \ln(\sqrt{5} + 2)$$

14. (a) Let $w = \cos^{-1} z$. Then $z = \cos w = (e^{iw} + e^{-iw})/2$ so $(e^{iw})^2 - 2z(e^{iw}) + 1 = 0$ and the quadratic formula gives

$$e^{iw} = (2z \pm \sqrt{4z^2 - 4})/2 = z + \sqrt{z^2 - 1}$$

$$iw = \log(z + \sqrt{z^2 - 1})$$

$$w = -i \log(z + \sqrt{z^2 - 1}) .$$

(b) Let $w = \tan^{-1} z$. Then $z = \tan w = \dfrac{\sin w}{\cos w} = \dfrac{1}{i} \dfrac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \dfrac{1}{i} \dfrac{e^{i2w} - 1}{e^{i2w} + 1}$

and, by algebra, $e^{i2w} = \dfrac{1 + iz}{1 - iz} = \dfrac{i - z}{i + z}$,

so $i2w = \log \dfrac{i - z}{i + z}$, $w = \tan^{-1} z = -\dfrac{i}{2} \log \dfrac{i - z}{i + z}$.

15. (a) Let $w = \sinh^{-1} z$. Then $z = \sinh w = (e^{w} - e^{-w})/2$ so $(e^{w})^2 - 2z(e^{w}) - 1 = 0$ and the quadratic formula gives

$$e^{w} = (2z \pm \sqrt{4z^2 + 4})/2 = z + \sqrt{z^2 + 1}$$

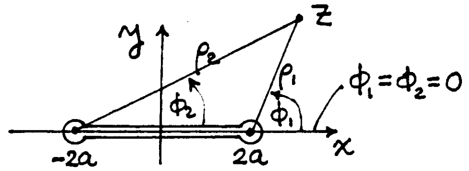
$$w = \sinh^{-1} z = \log(z + \sqrt{z^2 + 1})$$

(c) Let $w = \tanh^{-1} z$. Then $z = \tanh w = \dfrac{\sinh w}{\cosh w} = \dfrac{e^{w} - e^{-w}}{e^{w} + e^{-w}} = \dfrac{e^{2w} - 1}{e^{2w} + 1}$ and, by algebra,

$$e^{2w} = \frac{1 + z}{1 - z} , \quad 2w = \log\left(\frac{1 + z}{1 - z}\right)$$

$$w = \tanh^{-1} z = \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right) .$$

16.



On the top of the plate (i.e., on the top of the cut) $\rho_1 = 2a - x,\ \phi_1 = \pi$
$\rho_2 = 2a + x,\ \phi_2 = 0$

so $$u(x,0+) - iv(x,0+) = \frac{iV_0 x}{\sqrt{(z-2a)(z+2a)}} = \frac{iV_0 x}{\sqrt{\rho_1 e^{i\phi_1} \rho_2 e^{i\phi_2}}} = \frac{iV_0 x}{\sqrt{(4a^2-x^2)\, e^{i(\pi+0)}}}$$
$$= \frac{V_0 x}{\sqrt{4a^2-x^2}}$$

so $u(x,0+) = V_0 x/\sqrt{4a^2-x^2}$ and $v(x,0+) = 0$ (as it should!).

On the bottom of the plate $\rho_1 = 2a - x,\ \phi_1 = -\pi$
$\rho_2 = 2a + x,\ \phi_2 = 0$

so $$u(x,0-) - iv(x,0-) = \frac{iV_0 x}{\sqrt{(z-2a)(z+2a)}} = \frac{iV_0 x}{\sqrt{\rho_1 e^{i\phi_1} \rho_2 e^{i\phi_2}}} = \frac{iV_0 x}{\sqrt{(4a^2-x^2)\, e^{i(-\pi+0)}}}$$
$$= -\frac{V_0 x}{\sqrt{4a^2-x^2}}$$

so $u(x,0-) = -V_0 x/\sqrt{4a^2-x^2}$ and $v(x,0-) = 0$ (as it should).

Observe that $u = v = 0$ both on top of the plate and on the bottom of the plate - at the origin. The x-velocity increases as we move away from the origin and $\to \infty$ as $x \to \pm 2a$, so we say that the flow is "singular" at $z = \pm 2a$. When the flow turns the 180° corners, at $z = \pm 2a$ it slows down as it approaches the origin which, again, is a stagnation point.

Section 21.5

1. Is there a $\delta(\epsilon)$ such that $|3iz - 3i| < \epsilon$ for all $0 < |z-1| < \delta$? Well, $|3iz - 3i| < \epsilon$ gives $|z-1| < \epsilon/3$, so we can choose $\delta = \epsilon/3$ or smaller. Besides $\lim_{z\to 1} 3iz = 3i$ it is also true that $(3iz)|_{z=1} = 3i$. Thus, $w(z) = 3iz$ is continuous at $z = 1$.

2. (a) $|z^2| < \epsilon$ gives $|z|^2 < \epsilon$, $|z| < \sqrt{\epsilon}$, so with $\delta(\epsilon) = \sqrt{\epsilon}$ (or smaller) it follows that $|z^2| < \epsilon$ for all $|z| < \sqrt{\epsilon}$. Further, z^2 is $= 0$ at $z = 0$ so $w(z) = z^2$ is continuous at $z = 0$.

3. (a) $\lim_{z\to z_0} f(z) = A$ implies that for any $\epsilon > 0$, no matter how small, there is a δ_1 such that $|f(z) - A| < \epsilon/2$ for all $0 < |z - z_0| < \delta_1$. Similarly, $|g(z) - B| < \epsilon/2$ for all $0 < |z - z_0| < \delta_2$. Thus, $|f(z) + g(z) - A - B| \le |f(z) - A| + |g(z) - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $0 < |z - z_0| < \min(\delta_1, \delta_2)$.

   (b) $\lim_{z\to z_0} f(z) = A$ implies that for any $\epsilon' > 0$ (no matter how small) there is a δ_1 such that $|f(z) - A| < \epsilon'$ for all $0 < |z - z_0| < \delta_1$. Similarly, $|g(z) - B| < \epsilon'$ for all $0 < |z - z_0| < \delta_2$. Thus,

$$
\begin{aligned}
|f(z)g(z)-AB| &= |(f(z)-A)(g(z)-B)+Ag(z)+Bf(z)-AB-AB| \\
&= |(f(z)-A)(g(z)-B)+B(f(z)-A)+A(g(z)-B)| \\
&\le |f(z)-A||g(z)-B|+|B||f(z)-A|+|A||g(z)-B| \\
&< \epsilon'^2+|B|\epsilon'+|A|\epsilon' \equiv \epsilon
\end{aligned}
$$

Now, for any value of ϵ (no matter how small) we can solve

$$\epsilon'^2+(|A|+|B|)\epsilon'=\epsilon$$

for ϵ', namely, $\epsilon'=\left[-(|A|+|B|)+\sqrt{(|A|+|B|)^2+4\epsilon}\,\right]/2>0$. Thus, for any given value of ϵ (no matter how small) there exists a $\delta=\min(\delta_1,\delta_2)$ such that $|f(z)g(z)-AB|<\epsilon$ for all z in $0<|z-z_0|<\delta$, so $\lim\limits_{z\to z_0} f(z)g(z)=AB$.

4. No. As proof, a single counterexample will suffice. Here is one: $f(x)=1/(1+x^2)$ is continuous for all x (its graph is a bell-shaped curve), but $f(z)=1/(1+z^2)$ is not continuous everywhere, for it is discontinuous at $z=\pm i$, where it $\to\infty$.

5. (a) $\dfrac{d}{dz}z^3=\lim\limits_{\Delta z\to 0}\dfrac{(z+\Delta z)^3-z^3}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{\cancel{z^3}+(\Delta z)^3+3z^2\Delta z+3z(\Delta z)^2-\cancel{z^3}}{\Delta z}=3z^2$

(b) $\dfrac{d}{dz}\dfrac{1}{z}=\lim\limits_{\Delta z\to 0}\dfrac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{z-(z+\Delta z)}{z(z+\Delta z)\Delta z}=-\lim\limits_{\Delta z\to 0}\dfrac{1}{z(z+\Delta z)}=-\dfrac{1}{z^2}$ (if $z\neq 0$)

(c) $\dfrac{d}{dz}\dfrac{1}{z^2}=\lim\limits_{\Delta z\to 0}\dfrac{\frac{1}{(z+\Delta z)^2}-\frac{1}{z^2}}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{z^2-(z^2+2z\Delta z+(\Delta z)^2)}{z^2(z+\Delta z)^2\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{-2z-\Delta z}{z^2(z+\Delta z)^2}=-\dfrac{2}{z^3}$ (if $z\neq 0$)

6. (b) $\lim\limits_{\Delta z\to 0}\dfrac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{f(z+\Delta z)[g(z+\Delta z)-g(z)]+[f(z+\Delta z)-f(z)]g(z)}{\Delta z}$

$=\lim\limits_{\Delta z\to 0}f(z+\Delta z)\lim\limits_{\Delta z\to 0}\dfrac{g(z+\Delta z)-g(z)}{\Delta z}+g(z)\lim\limits_{\Delta z\to 0}\dfrac{f(z+\Delta z)-f(z)}{\Delta z}=f(z)g'(z)+f'(z)g(z).$

(c) $\lim\limits_{\Delta z\to 0}\dfrac{f(g(z+\Delta z))-f(g(z))}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{f(g+\Delta g)-f(g)}{\Delta g}\dfrac{\Delta g}{\Delta z}$ where $g(z+\Delta z)\equiv g+\Delta g$

$=\lim\limits_{\Delta g\to 0}\dfrac{f(g+\Delta g)-f(g)}{\Delta g}\lim\limits_{\Delta z\to 0}\dfrac{\Delta g}{\Delta z}=\underset{\uparrow\, df/dg}{f'}(g(z))\,g'(z)$

7. $\lim\limits_{z\to z_0}f(z)=\lim\limits_{z\to z_0}\left[\dfrac{f(z)-f(z_0)}{z-z_0}(z-z_0)+f(z_0)\right]=f'(z_0)(0)+f(z_0)=f(z_0)$

8. $\lim\limits_{z\to z_0}\dfrac{f(z)}{g(z)}=\lim\limits_{z\to z_0}\dfrac{[f(z)-f(z_0)]/(z-z_0)}{[g(z)-g(z_0)]/(z-z_0)}$ since $f(z_0)=g(z_0)=0$

$=\dfrac{\lim[f(z)-f(z_0)]/(z-z_0)}{\lim[g(z)-g(z_0)]/(z-z_0)}=\dfrac{f'(z)}{g'(z)}$

9. $u(x,y) = \begin{cases} (x^3-y^3)/(x^2+y^2), & z\neq 0 \\ 0, & z=0 \end{cases}$ and $v(x,y) = \begin{cases} (x^3+y^3)/(x^2+y^2), & z\neq 0 \\ 0, & z=0 \end{cases}$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{\substack{x\to 0\\ y=0}} \frac{(x^3-y^3)/(x^2+y^2)-0}{x} = 1, \quad \frac{\partial v}{\partial y}(0,0) = \lim_{\substack{y\to 0\\ x=0}} \frac{(x^3+y^3)/(x^2+y^2)-0}{y} = 1 \ ✓$$

and similarly for $\partial u/\partial y$ and $\partial v/\partial x$: we find that $\frac{\partial u}{\partial y}(0,0)=-1$ and $\frac{\partial v}{\partial x}(0,0)=1$. ✓ (Note that to evaluate these derivatives we *must* use the difference quotient formula; for ex., we can't compute $\partial u/\partial x$ from $u(x,y)=(x^3-y^3)/(x^2+y^2)$ since the latter does not hold *at* $z=0$.) But, consider letting $\Delta z\to 0$ along any ray $y=\alpha x$. Then $\Delta z = x+i\alpha x = (1+i\alpha)x$, so

$$f'(z) = \lim_{x\to 0} \frac{\dfrac{(1-\alpha^3)x^3+i(1+\alpha^3)x^3}{(1+\alpha^2)x^2} - 0}{(1+i\alpha)x} = \frac{(1-\alpha^3)+i(1+\alpha^3)}{(1+\alpha^2)(1+i\alpha)},$$ which is not

independent of α. For example, if $\alpha=0$ it gives $1+i$, but if $\alpha=1$ it gives $(1+i)/2$. Since the result is not unique for all possible paths of approach, f is not differentiable at $z=0$.

10. (a) $f(z) = \cos z = \cos(x+iy) = \cos x\cos iy - \sin x \sin iy = \cos x\cosh y - i\sin x\sinh y$

so $u(x,y) = \cos x\cosh y$, $v(x,y) = -\sin x\sinh y$.

$f'(z) = u_x + iv_x$, say

$= -\sin x\cosh y - i\cos x\sinh y$.

We weren't asked to express the answer in terms of z, but we *can*:

$f'(z) = -(\sin x\cosh y + i\cos x\sinh y) = -\sin(x+iy) = -\sin z$.

(b) $f(z) = e^z = e^x(\cos y + i\sin y)$ so $u = e^x\cos y$, $v = e^x\sin y$

$f'(z) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^z$

11. (a) $f(z) = (1-2z^3)^5$, $f'(z) = 5(-6z^2)(1-2z^3)^4 = -30z^2(1-2z^3)^4$ for all z; f is analytic for all z

(b) $f(z) = \dfrac{x+iy}{x^2+y^2} = \dfrac{z}{z\bar{z}} = \dfrac{1}{\bar{z}}$, but can't express it as a function of z itself, so let us use the $u+iv$ form,

$f(z) = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$ so $u = x/(x^2+y^2)$ and $v = y/(x^2+y^2)$.

$f'(z) = u_x + iv_x$ [by (19)], but (19) holds only if f is indeed differentiable in the first place. Let's check that *first*:

$$u_x = \frac{1}{x^2+y^2} + \frac{x(-1)2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$ (for $z\neq 0$; at $z=0$, $f=(x+iy)/(x^2+y^2)$ does not even exist)

$$v_y = \frac{1}{x^2+y^2} + \frac{y(-1)2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$ (")

so the Cauchy-Riemann condition $u_x = v_y$ is satisfied only along the lines $y=\pm x$ (except at the origin, where f is not even defined). Next,

$u_y = \frac{x(-1)2y}{(x^2+y^2)^2}$ and $v_x = \frac{y(-1)2x}{(x^2+y^2)^2}$ so $u_y = -v_x$ only along the lines $x=0$ and $y=0$. The intersection of these lines with the lines $y=\pm x$ is only the origin. Thus, at best, f is differentiable only at the single point $z=0$ and therefore analytic nowhere. In fact, it is not differentiable at $z=0$ either since f is not even defined uniquely at $z=0$. Thus, f is differentiable nowhere, and analytic nowhere. NOTE: For generalization of this result see Exercise 14(c).

(c) $f(z)=|z|\sin z = (x^2+y^2)(\sin x \cos iy + \sin iy \cos x) = \underbrace{(x^2+y^2)\sin x\cosh y}_{u} + i\underbrace{(x^2+y^2)\sinh y\cos x}_{v}$

$u_x = [2x\sin x + (x^2+y^2)\cos x]\cosh y$

$v_y = [2y\sinh y + (x^2+y^2)\cosh y]\cos x$

so $u_x = v_y$ gives $x\sin x\cosh y = y\sinh y\cos x$ ①

$u_y = [2y\cosh y + (x^2+y^2)\sinh y]\sin x$

$v_x = [2x\cos x - (x^2+y^2)\sin x]\sinh y$

so $u_y = -v_x$ gives $y\cosh y\sin x = -x\cos x\sinh y$ ②

Solving ① and ② for y/x and equating those results gives

$$\frac{\sin x\cosh y}{\sinh y\cos x} = -\frac{\cos x\sinh y}{\cosh y\sin x}$$

or, $\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = 0$.

Since the latter is a sum of squares we need each of the two terms to be zero. Since $\cosh y \neq 0$ for all y, we need $\sin x = 0$ so $x = n\pi$. Then $\cos x\sinh y = 0$ becomes $(-1)^n\sinh y = 0$, so $y=0$. Indeed, $x = n\pi$ $(n=0,\pm1,..)$ and $y=0$ does satisfy ① and ②, and u_x, u_y, v_x, v_y are all continuous at those points, so $f(z)$ is differentiable at those points only. Since it is not differentiable throughout any neighborhood of those points, f is not analytic at those points. Conclusion: f is

differentiable at $(n\pi, 0)$ for $n=0,\pm1,\pm2,\dots$,
analytic nowhere.

(d) Merely differentiating f gives $f'(z) = -\frac{2z+3i}{(z^2+3iz-2)^2}$, which is a unique finite number for all z's except where the denominator vanishes, namely, at $z=-i$ and $-2i$, at which points the numerator is nonzero. Thus, f is analytic for all z except at $z=-i,-2i$.

(e) $f' = \frac{(-1)3z^2}{(z^3+1)^2}$, except where $z^3+1=0$, namely, at $z=-1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$.

(f) $u=x$ and $v=\sin y$, so $u_x = v_y$ gives $1=\cos y$ and $u_y=-v_x$ gives $0=0$. Thus, $f(z)$ is differentiable all along the lines $y=\pm\pi/2, \pm3\pi/2, \dots$, and analytic nowhere.

13. (a) $f(z)=z^{100}=(x+iy)^{100}$ is too cumbersome to express in the form $u(x,y)+iv(x,y)$, so express $f(z)=r^{100}e^{i100\theta}=\underbrace{r^{100}\cos 100\theta}_{u(r,\theta)}+i\underbrace{r^{100}\sin 100\theta}_{v(r,\theta)}$

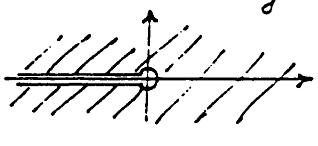
Then,

$u_r=100r^{99}\cos 100\theta,\ v_\theta=100r^{100}\cos 100\theta$

$u_\theta=-100r^{100}\sin 100\theta,\ v_r=100r^{99}\sin 100\theta.$

$u, v, u_r, v_r, u_\theta, v_\theta$ are continuous everywhere, and the Cauchy-Riemann conditions (30) are satisfied everywhere, so z^{100} is analytic everywhere. (More simply, of course, $f'=100z^{99}$ exists, and is unique, everywhere, so f is analytic everywhere.)

(b) $f'(z)=\frac{1}{2}z^{-1/2}=1/2\sqrt{z}$ where the $\sqrt{z}$ is defined by the branch cut in Fig. 6. Thus, f is analytic everywhere in the cut plane:



(c) As in (b), f is analytic everywhere in the cut plane.

14. (a) $f=u+iv$: $u_x=v_y,\ u_y=-v_x$

$\bar{f}=u-iv$: $\underline{u_x=-v_y,\ u_y=v_x}$

Adding gives $2u_x=0$ and $2u_y=0$ so $u(x,y)=$ constant. Likewise, $v_x=v_y=0$ gives $v=$ constant so, at most, f is a constant.

(b) If $f'=0$ then (19) gives $u_x=v_x=u_y=v_y=0$ so u and v are, at most, constants. Thus, $f(z)$ is at most a constant.

(c) Let $f=u+iv=f(z,\bar{z})\equiv F(x,y)$

Note that $x=(z+\bar{z})/2,\ y=(z-\bar{z})/2i$.

Then

$$f_{\bar{z}}=F_x\frac{\partial x}{\partial \bar{z}}+F_y\frac{\partial y}{\partial \bar{z}}=(u_x+iv_x)(\tfrac{1}{2})+(u_y+iv_y)(-\tfrac{1}{2i})$$

$$=\tfrac{1}{2}[\underbrace{(u_x-v_y)}_{0\text{ by CR}}+i\underbrace{(v_x+u_y)}_{0\text{ by CR}}]=0,\ \text{so } f_{\bar{z}}=0,\ f=f(z).$$

15. (a) $\nabla^2(e^x\cos y)=e^x\cos y-e^x\cos y=0$ so u is harmonic. To find v,

$u_x=e^x\cos y=v_y$ gives $v=\int e^x\cos y\,\partial y=e^x\sin y+A(x)$

$u_y=-e^x\sin y=-v_x=-e^x\sin y-A'(x)$ gives $A'(x)=0,\ A(x)=C$ so

$f(z)=u+iv=e^x\cos y+ie^x\sin y+C=e^z+$ Constant.

(b) $\nabla^2(e^{2x}\sin 2y)=4e^{2x}\sin 2y-4e^{2x}\sin 2y=0$ so u is harmonic. To find v,

$u_x=2e^{2x}\sin 2y=v_y$ gives $v=\int 2e^{2x}\sin 2y\,\partial y=-e^{2x}\cos 2y+A(x)$,

$u_y=2e^{2x}\cos 2y=-v_x=2e^{2x}\cos 2y-A'(x)$ gives $A'=0,\ A(x)=$ constant,

$f(z)=u+iv=e^{2x}\sin 2y-ie^{2x}\cos 2y+$ const.

$=-ie^{2x}(\cos 2y+i\sin 2y)+$ const. $=-ie^{2x}e^{i2y}+$ const. $=-e^{2z}+$ const

(c) $\nabla^2(x^3-3xy^2)=6x-6x=0$ so u is harmonic. To find v,

$u_x=3x^2-3y^2=v_y$ gives $v=\int(3x^2-3y^2)\partial y=3x^2y-y^3+A(x)$,

$u_y=-6xy=-v_x=-6xy-A'(x)$ gives $A'(x)=0,\ A(x)=$ const., so

$f(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3) + \text{const.} = (x+iy)^3 + \text{const.} = z^3 + \text{const.}$

(d) $\nabla^2 u = \nabla^2(r^3 \sin 3\theta) = 6r\sin 3\theta + 3r\sin 3\theta - 9r\sin 3\theta = 0$ so, by (30),

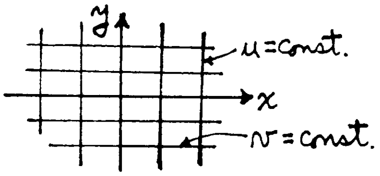
$u_r = 3r^2 \sin 3\theta = \frac{1}{r} v_\theta$ gives $v = \int 3r^3 \sin 3\theta \, \partial\theta = -r^3 \cos 3\theta + A(r)$

$-\frac{1}{r} u_\theta = -3r^2 \cos 3\theta = v_r = -3r^2 \cos 3\theta + A'(r)$ gives $A' = 0$, $A(r) = \text{const.}$,

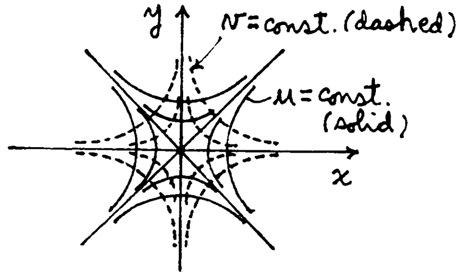
$f(z) = u + iv = r^3 \sin 3\theta - ir^3 \cos 3\theta + \text{const} = -ir^3(\cos 3\theta + i \sin 3\theta) = -ir^3 e^{i3\theta}$
$= -iz^3.$

16.(a) Normals to the $u = $ const. curves are given by $\underset{\sim}{n} = \underset{\sim}{\nabla} u = u_x \hat{\imath} + u_y \hat{\jmath}$ (if $u_x^2 + u_y^2 \neq 0$)
" " " $v =$ " " " " " $\underset{\sim}{n} = \underset{\sim}{\nabla} v = v_x \hat{\imath} + v_y \hat{\jmath}$ (if $v_x^2 + v_y^2 \neq 0$)

(b) Simple: $u = x$, $v = y$, so:

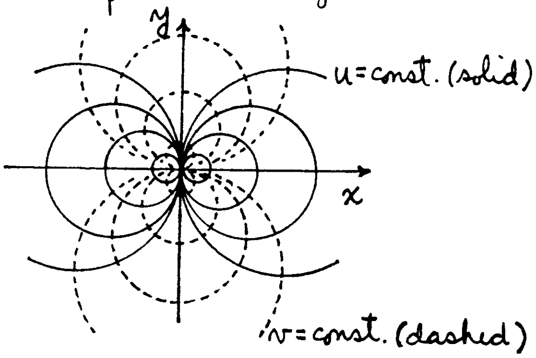


(c) $u = x^2 - y^2$, $v = 2xy$ so the u=const and v=const curves are hyperbolas:



Note that orthogonality does break down at $z = 0$, where $f'(z) = 2z = 0$

(d) $u = x/(x^2+y^2)$ so $(x - \frac{1}{2u})^2 + y^2 = (\frac{1}{2u})^2$
$v = -y/(x^2+y^2)$ so $x^2 + (y - \frac{1}{2v})^2 = (\frac{1}{2v})^2$
so the u= const and v = const curves are families of circles through $z=0$:



17. $u_x = v_y \Rightarrow v(x,y) = \int_{y_0}^{y} u_x(x,y')\,\partial y' + A(x)$

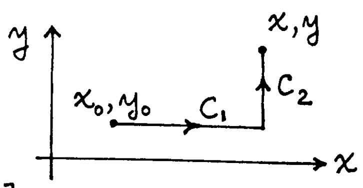
$u_y = -v_x = -\int_{y_0}^{y} u_{xx}(x,y')\,\partial y' - A'(x) = \int_{y_0}^{y} u_{y'y'}(x,y')\,\partial y' - A'(x)$ since $\nabla^2 u = 0$

or, $\cancel{u_y(x,y)} = \cancel{u_y}(x,y) - u_y(x,y_0) - A'(x)$, so $A(x) = -\int_{x_0}^{x} u_y(x',y_0)\,dx'$ and

$v(x,y) = \int_{y_0}^{y} u_x(x,y')\,\partial y' - \int_{x_0}^{x} u_y(x',y_0)\,dx'$

$= \int_{x_0}^{x} -u_y(x',y_0)\,dx' + \int_{y_0}^{y} u_x(x,y')\,\partial y'$

$= \int_{C_1 + C_2} \left[-\frac{\partial u}{\partial y}(x',y')\,dx' + \frac{\partial u}{\partial x}(x',y')\,dy' \right]$



Finally, the Cauchy-Riemann conditions imply that the vector field $-u_y \hat{\imath} + u_x \hat{\jmath}$ is irrotational so, by Theorem 16.10.1, the line integral is independent of path. That is, a unique value is obtained for any path C from x_0, y_0 to x, y, within D:

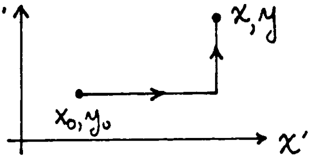
$$v(x,y) = \int_{x_0,y_0}^{x,y} \left[-\frac{\partial u}{\partial y}(x',y')\,dx' + \frac{\partial u}{\partial x}(x',y')\,dy' \right], \qquad 웃$$

which result is unique only up to an arbitrary additive constant since the initial point x_0, y_0 is arbitrary. That is, if we change x_0, y_0 in 웃 to x_1, y_1 then the difference between the two expressions for v is the line integral from x_0, y_0 to x_1, y_1, which is a constant.

To illustrate the use of 웃 let us use it to find v in Exercise 15(a).

$$u(x,y) = e^x \cos y,$$

$$v(x,y) = \int_{x_0,y_0}^{x,y} (e^{x'} \sin y'\,dx' + e^{x'} \cos y'\,dy')$$



We can use the simple path shown at the right. Then

$$v(x,y) = \int_{x_0}^{x} e^{x'} \sin y_0\,dx' + 0 + 0 + \int_{y_0}^{y} e^{x} \cos y'\,dy'$$

$$= e^x \cancel{\sin y_0} - e^{x_0} \sin y_0 + e^x (\sin y - \cancel{\sin y_0}) = e^x \sin y - e^{x_0} \sin y_0$$

$$= e^x \sin y + \text{constant}.$$

18. $$df/dz = (u_x + i v_x)\frac{1}{1+iK} + (u_y + i v_y)\frac{K}{1+iK}$$

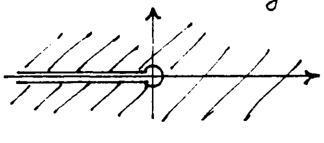
$$= (u_x + i v_x)\frac{1}{1+iK} + (-v_x + i u_x)\frac{K}{1+iK}$$

$$= \frac{u_x(1+iK) + i v_x (1+iK)}{1+iK} = u_x + i v_x$$

independent of K. However, we are still short of a proof of part (ii) of the Theorem 21.5.1 since we have not allowed for an arbitrary path of approach, only linear paths.

13. (a) $f(z)=z^{100}=(x+iy)^{100}$ is too cumbersome to express in the form $u(x,y)+iv(x,y)$, so express $f(z)=r^{100}e^{i100\theta}=\underbrace{r^{100}\cos 100\theta}_{u(r,\theta)}+i\underbrace{r^{100}\sin 100\theta}_{v(r,\theta)}$
Then,
$u_r=100r^{99}\cos 100\theta$, $v_\theta=100r^{100}\cos 100\theta$
$u_\theta=-100r^{100}\sin 100\theta$, $v_r=100r^{99}\sin 100\theta$.
$u, v, u_r, v_r, u_\theta, v_\theta$ are continuous everywhere, and the Cauchy-Riemann conditions (30) are satisfied everywhere, so z^{100} is analytic everywhere. (More simply, of course, $f'=100z^{99}$ exists, and is unique, everywhere, so f is analytic everywhere.)

(b) $f'(z)=\frac{1}{2}z^{-1/2}=1/2\sqrt{z}$ where the $\sqrt{z}$ is defined by the branch cut in Fig. 6. Thus, f is analytic everywhere in the cut plane:



(c) As in (b), f is analytic everywhere in the cut plane.

14. (a) $f=u+iv$: $u_x=v_y$, $u_y=-v_x$
$\bar{f}=u-iv$: $\underline{u_x=-v_y,\ u_y=v_x}$
adding gives $2u_x=0$ and $2u_y=0$ so $u(x,y)=$ constant. Likewise, $v_x=v_y=0$ gives $v=$ constant so, at most, f is a constant.

(b) If $f'=0$ then (19) gives $u_x=v_x=u_y=v_y=0$ so u and v are, at most, constants. Thus, $f(z)$ is at most a constant.

(c)
$$\left.\begin{aligned} u_x&=u_z z_x+u_{\bar z}\bar z_x=u_z+u_{\bar z}\\ u_y&=u_z z_y+u_{\bar z}\bar z_y=iu_z-iu_{\bar z}\\ v_x&=v_z z_x+v_{\bar z}\bar z_x=v_z+v_{\bar z}\\ v_y&=v_z z_y+v_{\bar z}\bar z_y=iv_z-iv_{\bar z}\end{aligned}\right]\ \text{so the Cauchy-Riemann conditions give}\ \begin{aligned} u_z+u_{\bar z}&=iv_z-iv_{\bar z} \quad ①\\ iu_z-iu_{\bar z}&=-v_z-v_{\bar z} \quad ②\end{aligned}$$

①+ i times ② gives $2u_{\bar z}=-2iv_{\bar z}$ or, $u_{\bar z}+iv_{\bar z}=0$ or, $f_{\bar z}=0$.

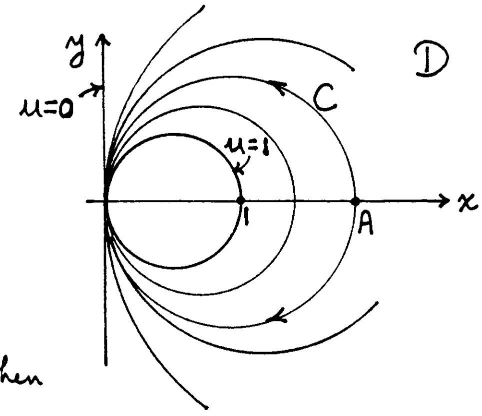
15. (a) $\nabla^2(e^x\cos y)=e^x\cos y-e^x\cos y=0$ so u is harmonic. To find v,
$u_x=e^x\cos y=v_y$ gives $v=\int e^x\cos y\,\partial y=e^x\sin y+A(x)$
$u_y=-e^x\sin y=-v_x=-e^x\sin y-A'(x)$ gives $A'(x)=0$, $A(x)=C$ so
$f(z)=u+iv=e^x\cos y+ie^x\sin y+C=e^z+$ Constant.

(b) $\nabla^2(e^{2x}\sin 2y)=4e^{2x}\sin 2y-4e^{2x}\sin 2y=0$ so u is harmonic. To find v,
$u_x=2e^{2x}\sin 2y=v_y$ gives $v=\int 2e^{2x}\sin 2y\,\partial y=-e^{2x}\cos 2y+A(x)$,
$u_y=2e^{2x}\cos 2y=-v_x=2e^{2x}\cos 2y-A'(x)$ gives $A'=0$, $A(x)=$ constant,
$f(z)=u+iv=e^{2x}\sin 2y-ie^{2x}\cos 2y+$ const.
$=-ie^{2x}(\cos 2y+i\sin 2y)+$ const. $=-ie^{2x}e^{i2y}+$ const. $=-e^{2z}+$ const

(c) $\nabla^2(x^3-3xy^2)=6x-6x=0$ so u is harmonic. To find v,
$u_x=3x^2-3y^2=v_y$ gives $v=\int(3x^2-3y^2)\partial y=3x^2y-y^3+A(x)$,
$u_y=-6xy=-v_x=-6xy-A'(x)$ gives $A'(x)=0$, $A(x)=$ const., so

CHAPTER 22

Section 22.2



1. (a) $u = x/(x^2+y^2)$ gives

$$(x-\frac{1}{2u})^2 + y^2 = (\frac{1}{2u})^2$$

so the u = const. curves in the x,y plane are the circles shown at the right. Thus, the strip $0<u<1$ is the image of D.

(c) Consider a u = const. circle, such as C, in the x,y plane. If we begin at A, where $v=0$, and move on it counterclockwise then $v = -y/(x^2+y^2) \to -\infty$ as we approach the origin (from above). If, instead, we move clockwise from A then $v \to +\infty$ as we approach the origin (from below). Thus, $|\psi| \to \infty$ as we approach the origin from above or below (from within D of course).

2. $\dfrac{1}{\frac{dw}{dz}} = \dfrac{1}{\lim\limits_{\Delta z\to 0}\frac{\Delta w}{\Delta z}} = \lim\limits_{\Delta z\to 0}\dfrac{1}{\frac{\Delta w}{\Delta z}} = \lim\limits_{\Delta z\to 0}\dfrac{\Delta z}{\Delta w} = \lim\limits_{\Delta w\to 0}\dfrac{\Delta z}{\Delta w} = \dfrac{dz}{dw}$, where the fourth equality follows from the fact that if $w = f(z)$ is analytic then it is necessarily continuous, so $\Delta z \to 0 \Rightarrow \Delta w \to 0$.

3. z is an analytic function of w and F(z) is an analytic function of z, so the composite function $F(z(w))$ is an analytic function of w. (Recall the discussion below Example 6 in Sec. 21.5.) Thus, its real and imaginary parts, in particular its real part $\Psi(u,v)$, are harmonic.

4. $f'(z) = \lim\limits_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}$ so $|f'(z_0)| = \left|\lim\limits_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}\right| = \lim\limits_{\Delta z\to 0}\left|\dfrac{\Delta w}{\Delta z}\right| = \lim\limits_{\Delta z\to 0}\dfrac{|\Delta w|}{|\Delta z|}$ at any point z_0.

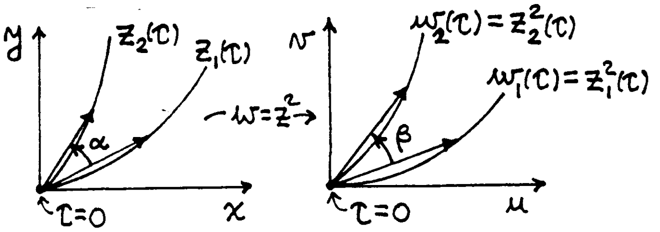
5. (a) e^z is analytic at $z=0$ (in fact, for all z) and $de^z/dz = e^z = e^0 = 1 \neq 0$ so yes, conformal at $z=0$

(b) ze^z is analytic at $z=0$ and $d(ze^z)/dz = (z+1)e^z = 1 \neq 0$ there so yes, it is conformal there

(c) $iz+3$ is analytic at $z=0$ and $d(iz+3)/dz = i \neq 0$ there so yes, it is.

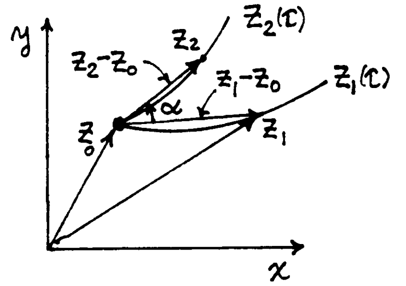
(d) iz^2 is analytic at $z=0$ and $d(iz^2)/dz = 2iz = 0$ there so no, it is not conformal there (though it is conformal for each point $z \neq 0$).

6. (a)



$$\beta = \lim_{\tau\to 0}[\arg w_2(\tau) - \arg w_1(\tau)]$$
$$= \lim_{\tau\to 0}[\arg z_2^2(\tau) - \arg z_1^2(\tau)]$$
$$= \lim_{\tau\to 0} 2[\arg z_2(\tau) - \arg z_1(\tau)]$$
$$= 2\alpha$$

(c)



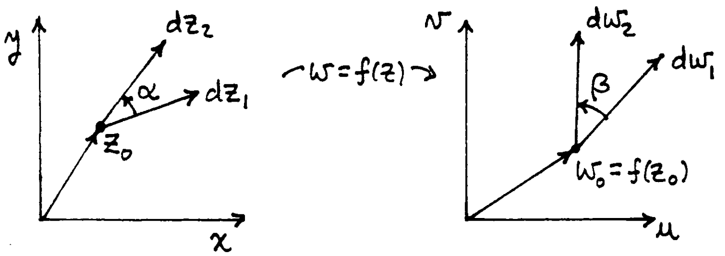
$$\beta = \lim_{\tau\to 0}\{\arg[w_2(\tau) - \overset{0}{\cancel{w_0}}] - \arg[w_1(\tau) - \overset{0}{\cancel{w_0}}]\}$$
$$= \lim_{\tau\to 0}\{\arg (z_2-z_0)^n g(z_2) - \arg (z_1-z_0)^n g(z_1)\}$$
$$= \lim_{\tau\to 0}\{n \arg (z_2-z_0) + \arg g(z_2) - n\arg(z_1-z_0) - \arg g(z_1)\}$$
$$= n\lim_{\tau\to 0}[\arg(z_2-z_0) - \arg(z_1-z_0)] + \cancel{\arg g(z_0)} - \cancel{\arg g(z_0)}$$
$$= n\alpha$$

7. Return to (9) and change the RHS (right-hand side) from 0 to Q. Then

$$\Psi_{uu} + \Psi_{vv} = \frac{1}{u_x^2+u_y^2}\ ,\ \text{or},\ \frac{1}{|f'(z)|^2}Q,$$

where Q is $Q(x(u,v), y(u,v))$, u_x is $u_x(x(u,v), y(u,v))$, and similarly for u_y.

8.

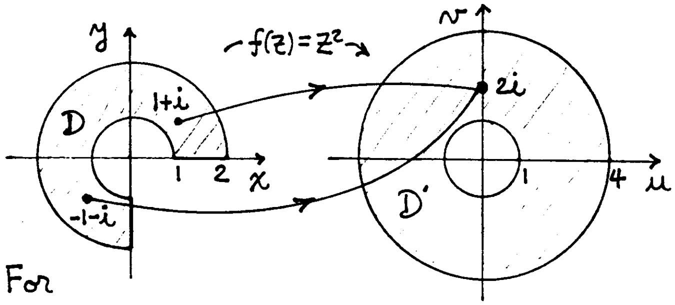


$dw_1 = f'(z_0)dz_1$ so
$\arg dw_1 = \arg f'(z_0) + \arg dz_1$ ①

$dw_2 = f'(z_0)dz_2$ so
$\arg dw_2 = \arg f'(z_0) + \arg dz_2$ ②

Subtracting ① from ② gives $\underbrace{\arg dw_2 - \arg dw_1}_{\beta} = \underbrace{\arg dz_2 - \arg dz_1}_{\alpha}$ so $\beta = \alpha$

NOTE: I find this heuristic approach much easier for the students to follow, so I use this in class.



9. Let it suffice to answer by means of an example, $f(z)=z^2$ on the domain D shown. Clearly, f is analytic in D and $f'(z)=2z\neq 0$ in D, yet the map is not one-to-one. For example it sends both $1+i$ and $-1-i$ into $2i$. Yet, the implicit function theorem, on which we have relied, has not let us down for it asserts that if

$$\left.\begin{aligned} F(x,y,u,v) &= u - x^2 + y^2 = 0 \\ G(\quad \cdot\cdot \quad) &= v - 2xy = 0 \end{aligned}\right\} \star$$

for $x=1, y=1, u=0, v=2$, then $\star$ uniquely implies the existence of single-valued functions $x = x(u,v), y = y(u,v)$ in some neighborhood of $(u,v)^T = (0,2)$ <u>such that $x(0,2)=1$ and $y(0,2)=1$</u>. By virtue of the underlined condition, there is no contradiction. Thus, the conditions that $f'(z) \neq 0$ and that the mapping be one-to-one, in Theorem 22.2.1, are not redundant.

10. If $x = u\cos v$ and $y = \sin v$ then (see page 650), with $r=u$ and $\theta = v$,

$$\begin{aligned} & u_x = \cos v, \; u_y = \sin v, \; v_x = -\sin v/u, \; v_y = \cos v/u, \\ & u_{xx} = (-\sin v)v_x = \sin^2 v/u, \\ & u_{yy} = (\cos v)\, v_y = \cos^2 v/u, \\ & v_{xx} = \tfrac{\partial}{\partial u}(-u^{-1}\sin v)\,u_x + \tfrac{\partial}{\partial v}(-u^{-1}\sin v)\,v_x \\ & \quad\;\; = u^{-2}\sin v\cos v + u^{-2}\cos v\sin v = 2\sin v\cos v/u^2, \\ & v_{yy} = \tfrac{\partial}{\partial u}(u^{-1}\cos v)\,u_y + \tfrac{\partial}{\partial v}(u^{-1}\cos v)\,v_y \\ & \quad\;\; = -u^{-2}\cos v\sin v - u^{-2}\sin v\cos v = -2\sin v\cos v/u^2, \end{aligned}$$

so (7) becomes

$$(\cos^2 v + \sin^2 v)\Psi_{uu} + \left(-\cancel{\frac{\cos v\sin v}{u}} + \cancel{\frac{\sin v\cos v}{u}}\right)(\Psi_{uv} + \Psi_{vu}) + \left(\frac{\sin^2 v}{u^2} + \frac{\cos^2 v}{u^2}\right)\Psi_{vv}$$

$$+ \left(\frac{\sin^2 v}{u} + \frac{\cos^2 v}{u}\right)\Psi_u + \left(\cancel{\frac{2\sin v\cos v}{u^2}} - \cancel{\frac{2\sin v\cos v}{u^2}}\right)\Psi_v = 0$$

or, $\Psi_{uu} + \frac{1}{u}\Psi_u + \frac{1}{u^2}\Psi_{vv} = 0$. (Not surprising since it is merely the Laplace equation $\Psi_{rr} + \frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta} = 0$ <u>in polar coordinates</u>.)

Thus, the map $w(z) = u + iv = \sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$ can <u>not</u> be conformal because it does not give $\Psi_{uu} + \Psi_{vv} = 0$.

Section 22.3

1. With $w = u + iv$, $A = a - ib$, and $B = c^2 - a^2 - b^2$, (8) becomes

$$u^2 + v^2 + \frac{2au - 2bv}{B} = \frac{1}{B}$$

or

$$\begin{aligned} \left(u + \frac{a}{B}\right)^2 + \left(v - \frac{b}{B}\right)^2 &= \frac{1}{c^2-a^2-b^2} + \frac{a^2+b^2}{(c^2-a^2-b^2)^2} \\ &= \left(\frac{c}{c^2-a^2-b^2}\right)^2 \end{aligned}$$

so $(u-\alpha)^2 + (v-\beta)^2 = \gamma^2$ with $\alpha = -a/(c^2-a^2-b^2)$, $\beta = b/(c^2-a^2-b^2)$, and $\gamma = c/(c^2-a^2-b^2)$.

2. (6) gives the general equation of a circle in the z-plane as

$$z\bar{z}-Az-\bar{A}\bar{z}=B. \quad (B \text{ real}) \quad ♀$$

(a) $w(z)=Pz$ with P nonzero and, in general, complex. Then ♀ becomes

$$\frac{w}{P}\frac{\bar{w}}{\bar{P}}-A\frac{w}{P}-\bar{A}\frac{\bar{w}}{\bar{P}}=B$$

or, $$w\bar{w}-(A\bar{P})w-(\bar{A}P)\bar{w}=(P\bar{P}B),$$
which is of the same form as ♀ since $\bar{A}P$ is the conjugate of $A\bar{P}$ and $P\bar{P}B$ is real. Thus, the image is a circle too.

(b) $w(z)=z+Q$ with Q nonzero and, in general, complex. Then ♀ becomes

$$(w-Q)(\bar{w}-\bar{Q})-A(w-Q)-\bar{A}(\bar{w}-\bar{Q})=B$$

or, $$w\bar{w}-(\bar{Q}+A)w-(Q+\bar{A})\bar{w}=B+Q\bar{Q}-(AQ+\bar{A}\bar{Q}),$$
which is of the form ♀ since $Q+\bar{A}$ is the conjugate of $\bar{Q}+A$ and $B+Q\bar{Q}-\overline{(A\bar{Q}+\bar{A}\bar{Q})}=B+Q\bar{Q}-(AQ+\overline{AQ})$ is real. Thus, the image is a circle.

3. (a) $$w=u+iv=\frac{z-a}{az-1}=\frac{(x-a)+iy}{(ax-1)+iay}\,\frac{(ax-1)-iay}{(ax-1)-iay}$$

so $$u=\frac{(x-a)(ax-1)+ay^2}{(ax-1)^2+a^2y^2},\quad v=\frac{(ax-1)y-(x-a)ay}{(ax-1)^2+a^2y^2}=\frac{(a^2-1)y}{(ax-1)^2+a^2y^2}$$

and $$\rho=\sqrt{\frac{[(x-a)(ax-1)+ay^2]^2+[(a^2-1)y]^2}{[(ax-1)^2+a^2y^2]^2}}=\text{etc.},$$

but it's easier to proceed as follows:

$$\rho=\sqrt{w\bar{w}}=\sqrt{\frac{(x-a)+iy}{(ax-1)+iay}\,\frac{(x-a)-iy}{(ax-1)-iay}}=\sqrt{\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}}$$

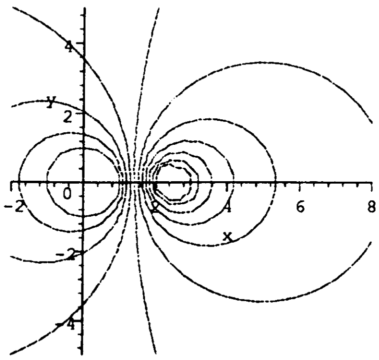
and putting this into (23) gives (24).

(b) Maple:

```
> with(plots):
> P:=50+10.9055*ln(((x-2.3798)^2+y^2)/((2.3798*x-1)^2+5.6634*y^2));
```

$$P := 50+10.9055\ln\left(\frac{(x-2.3798)^2+y^2}{(2.3798\,x-1)^2+5.6634\,y^2}\right)$$

```
> implicitplot({P=0,P=5,P=10,P=15,P=20,P=25,P=30,P=35,P=40,P=45,P=50
  },x=-2..8,y=-5..5,view=[-2..8,-5..5]);
```



4. Let $f(z)=\frac{az+b}{cz+d}$ $(ad-bc\neq 0)$ and $g(z)=\frac{ez+h}{jz+k}$ $(ek-hj\neq 0)$. Then

$$f(g(z))=\frac{a\left(\frac{ez+h}{jz+k}\right)+b}{c\left(\frac{ez+h}{jz+k}\right)+d}=\frac{(ae+bj)z+(ah+kb)}{(ce+dj)z+(ch+kd)},$$ which is of bilinear form. It remains to verify that $ad-bc\neq 0$ and $ek-hj\neq 0$ imply that

$$\star=(ae+bj)(ch+kd)-(ce+dj)(ah+kb)\neq 0$$

Well, $\star=(ad-bc)(ek-hj)+a\cancel{ec}h+b\cancel{jk}d-a\cancel{ec}h-b\cancel{jk}d\neq 0.$ ✓

5. $z=\frac{az+b}{cz+d}$ gives $cz^2+(d-a)z-b=0$, which gives at most two roots and at least one, <u>unless</u> $c=b=0$ and $d=a$ (i.e., unless $w=z$ is the identity transformation), in which case <u>every</u> z is a fixed point.

6. (a) (6.1) is of the form $\frac{w-w_1}{w-w_3}=A\frac{z-z_1}{z-z_3}$. Solving for w gives $w=-\frac{Aw_3(z-z_1)}{(1-A)z+(Az_1-z_3)}$, which <u>is</u> of bilinear form. It is easy to see from (6.1) that $w(z_1)=w_1$ and $w(z_3)=w_3$, but less obvious is that $w(z_2)=w_2$. If $z=z_2$ then the RHS of (6.1) is 1 and (6.1) becomes

$$(w-w_1)(w_2-w_3)=(w-w_3)(w_2-w_1)$$

or, $$(\cancel{w_2}-w_3-\cancel{w_2}+w_1)w=w_1(w_2-\cancel{w_3})-w_3(w_2-\cancel{w_1})$$

or, $$(w_1-w_3)w=w_2(w_1-w_3).$$

Since w_1, w_2, w_3 are assumed distinct, $w_1-w_3\neq 0$ so the latter gives $w=w_2$. ✓

(b) Follow the hint.

7. Clearly, if

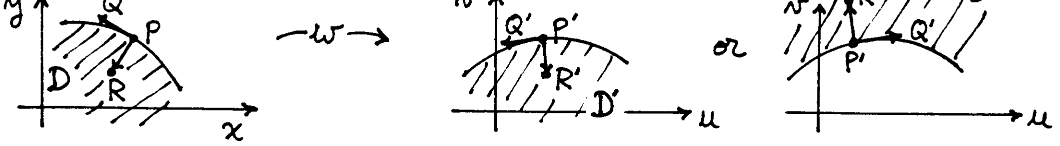
$$w=\frac{z-a}{1-\bar{a}z}$$

Then $w(a)=0$. To show that $|z|=1$ maps into $|w|=1$ write

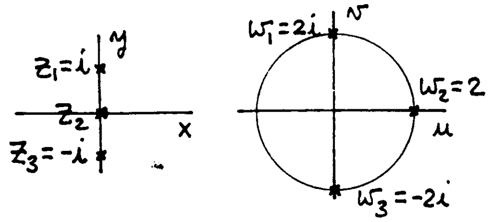
$$|w|^2=w\bar{w}=\frac{z-a}{1-\bar{a}z}\,\frac{\bar{z}-\bar{a}}{1-a\bar{z}}=\frac{|z|^2-\bar{a}z-a\bar{z}+|a|^2}{1-\bar{a}z-a\bar{z}+|a|^2|z|^2}=\frac{1-\bar{a}z-a\bar{z}+|a|^2}{1-\bar{a}z-a\bar{z}+|a|^2}=1.$$

Further, $|w(0)|=\left|\frac{0-a}{1-0}\right|=|a|<1$, so the interior $|z|<1$ maps into the interior $|w(z)|<1$.

8. Conformality implies that the angle from PQ to PR will be preserved, both in magnitude and sense. Thus,



9. (a)



Then (6.1) gives

$$\frac{(w-2i)(2+2i)}{(w+2i)(2-2i)}=\frac{(z-i)(0+i)}{(z+i)(0-i)}$$

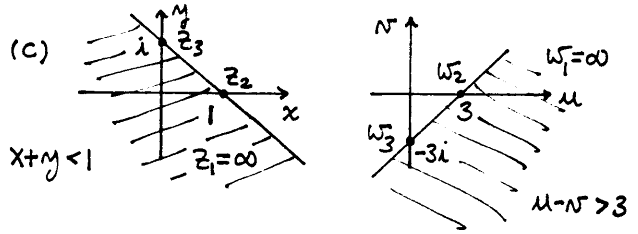
and, solving for w, we obtain $w=\dfrac{2z+2}{-z+1}$.

As we move along $x=0$ from z_3 to z_2 to z_1, the region $(x<0)$ is at our left. Likewise, as we move along the image curve $|w|=2$ from w_3 to w_2 to w_1 the region must be to our left and is therefore the <u>interior</u> $|w|<2$, not the exterior $|w|>2$. Another choice: $z_1=\infty,\ z_2=0,\ z_3=2i$

$w_1=2,\ w_2=2i,\ w_3=-2$

Then (6.1) gives $\dfrac{w-2}{w+2}\,\dfrac{2i+2}{2i-2}=\dfrac{\cancel{z-\infty}}{z-2i}\,\dfrac{0-2i}{\cancel{0-\infty}}$ and, solving for w, gives $w=\dfrac{2z+(4-4i)}{z-(2+2i)}$

(c)



$z_1=\infty,\ w_1=\infty$

$z_2=3,\ w_2=3$

$z_3=i,\ w_3=-3i$

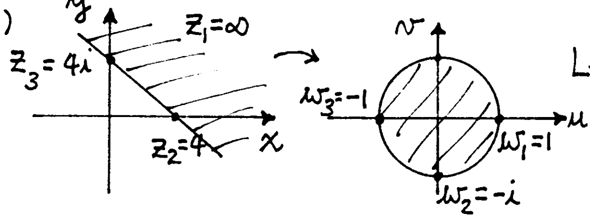
Then, moving from z_2 to z_3 the region is at our left so, moving from w_2 to w_3, the region will again be on our left.

(6.1) gives $\dfrac{\cancel{w-\infty}}{w+3i}\,\dfrac{3+3i}{\cancel{3-\infty}}=\dfrac{\cancel{z-\infty}}{z-i}\,\dfrac{1-i}{\cancel{1-\infty}}$ or, $w=3iz+3-3i$

NOTE: We don't <u>have</u> to choose one of the z_j's and one of the w_j's $=\infty$. For example, we could use $z_1=1,\ z_2=(1+i)/2,\ z_3=i$

$w_1=3,\ w_2=(3-3i)/2,\ w_3=-3i$.

(d)



Let $z_1=\infty,\ z_2=4,\ z_3=4i$,

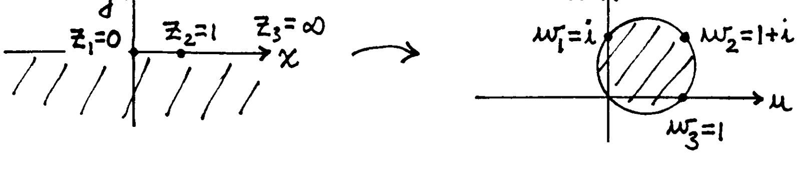
$w_1=1,\ w_2=-i,\ w_3=-1$, say.

Then (6.1) gives

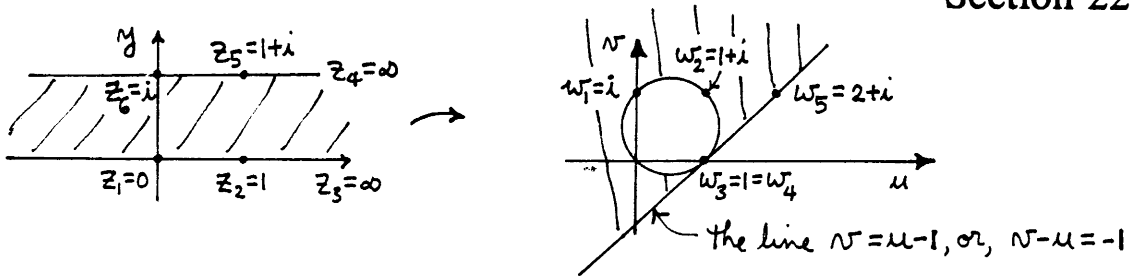
$$\frac{w-1}{w+1}\,\frac{-i+1}{-i-1}=\frac{\cancel{z-\infty}}{z-4i}\,\frac{4-4i}{\cancel{4-\infty}}$$ or, solving for w, $w=\dfrac{z-(4+8i)}{z+4}$

10. $w=(z+1)/(z-i)$. The pole is $z=1$, so any circle or straight line through $z=1$ will be sent into a straight line.

(a)

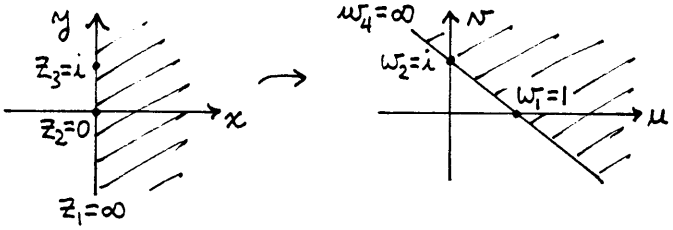


(b)

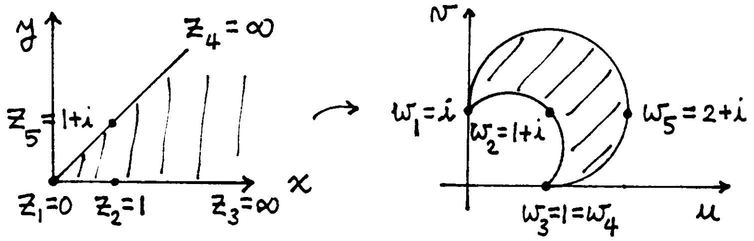


(c) From (a) and (b) we see that the image of $y>1$ is the region $v-u<-1$

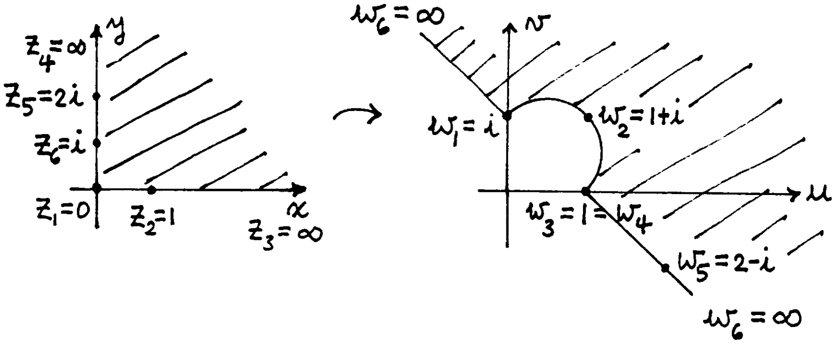
(d)



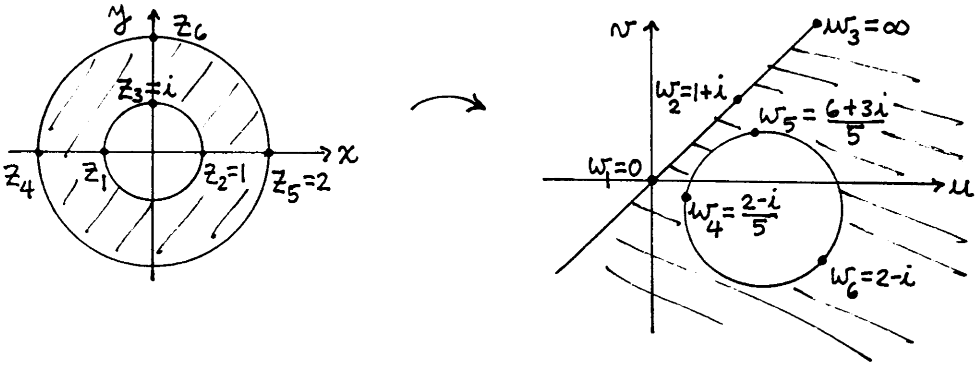
(e)



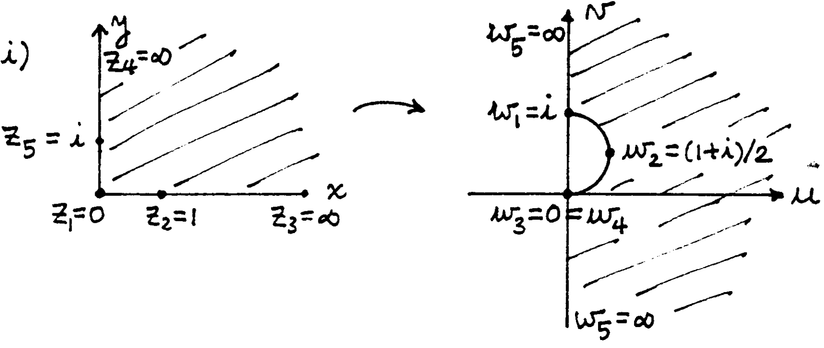
(f)



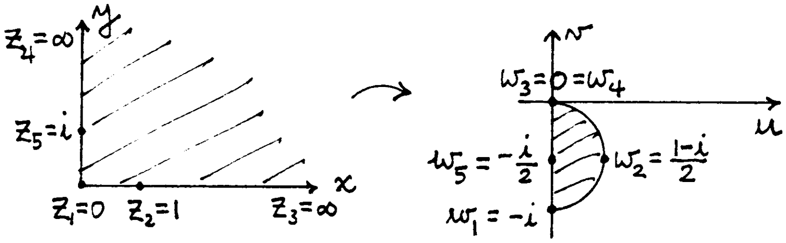
(g)



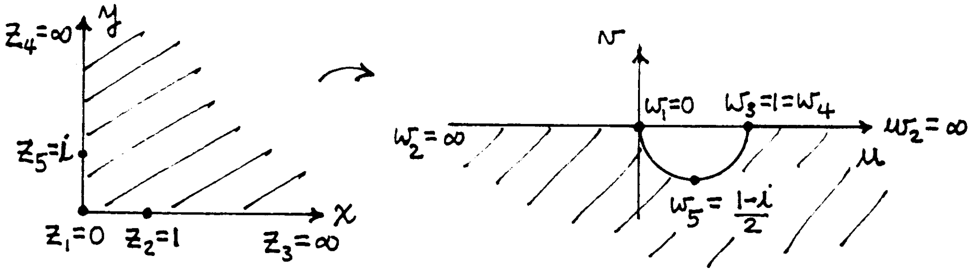
11. (a) $w = 1/(z-i)$
pole@i



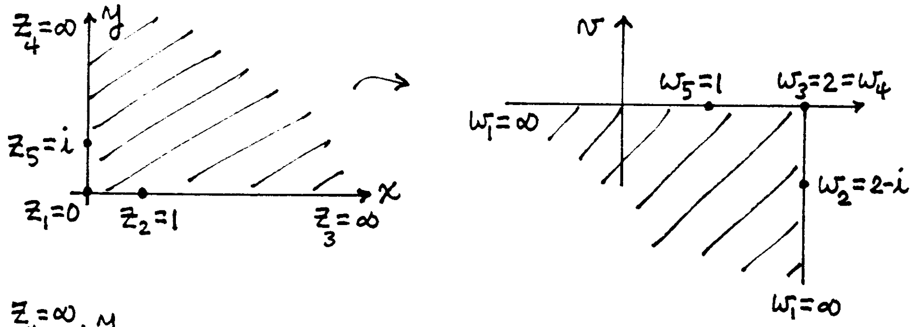
(b) $w = 1/(z+i)$
pole@$-i$



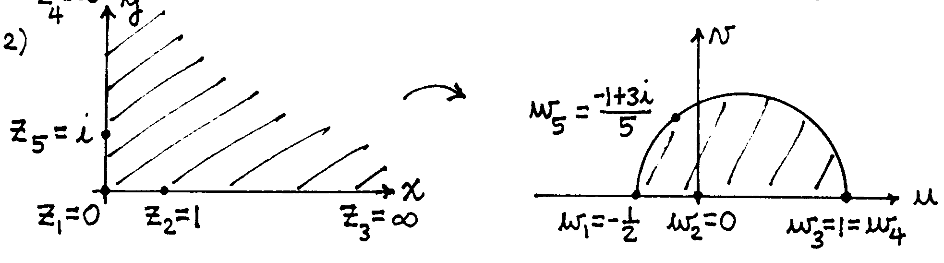
(c) $w = z/(z-1)$
pole@1



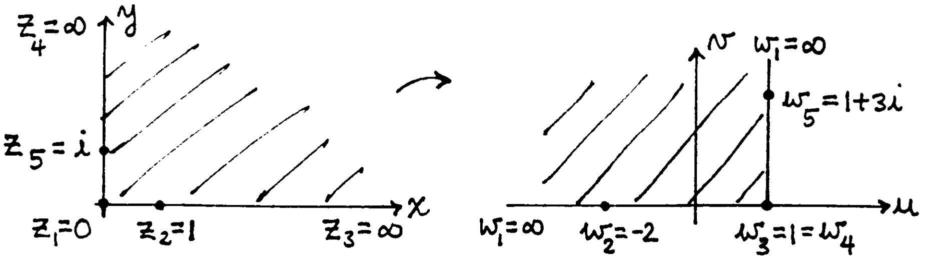
(d) $w = 2-\frac{i}{z}$
$=(2z-i)/z$
pole@0



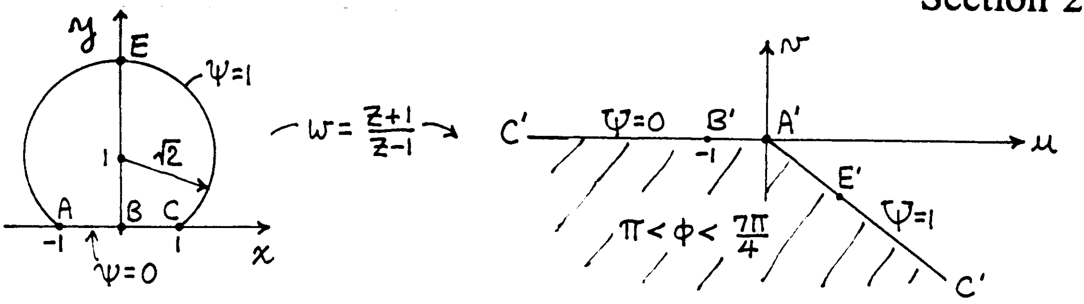
(e) $w = (z-1)/(z+2)$
pole@-2



(f) $w = (z-3)/z$
pole@0



12.



$$\rightarrow \Psi = \frac{4}{3\pi}\phi - \frac{4}{3} = \frac{4}{3}\left(\frac{1}{\pi}\phi - 1\right) -$$

$$w = u + iv = \frac{z+1}{z-1} = \frac{x+1+iy}{x-1+iy}\,\frac{x-1-iy}{x-1-iy}$$

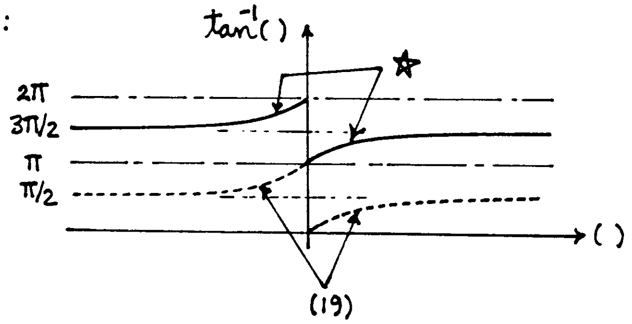
so $u = (x^2 - 1 + y^2)/((x-1)^2 + y^2)$

$v = (-2y)/((x-1)^2 + y^2)$

so

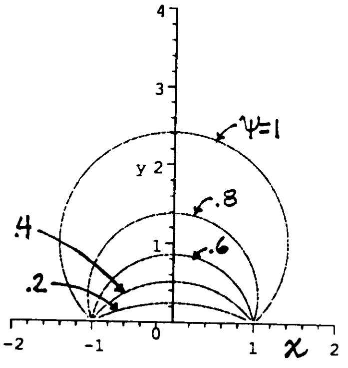
$$\Psi = \frac{4}{3}\left(\frac{1}{\pi}\tan^{-1}\frac{v}{u} - 1\right) = \frac{4}{3}\left(\frac{1}{\pi}\tan^{-1}\left(\frac{-2y}{x^2+y^2-1}\right) - 1\right) \quad ✩$$

✩ looks different from (19) but does agree with it because in ✩ we have $\pi < \tan^{-1}(\ ) < 7\pi/4$ and in (19) we have $\pi/4 < \tan^{-1}(\ ) < \pi$. That is, we are using these different branches of $\tan^{-1}(\ )$:

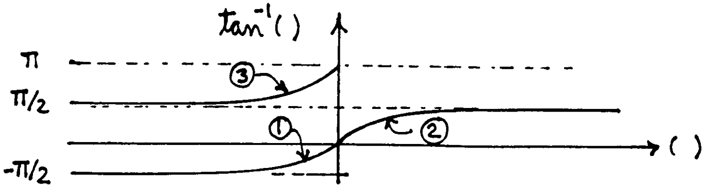


13. NOTE: This problem is useful in emphasizing the care that is necessary when working with inverse trigonometric function computation.

```
> with(plots):
> P:=(4/3)*(1-(1/Pi)*(Pi*Heaviside(-2*y/(x^2+y^2-1))+arctan(2*y/(x^2
  +y^2-1)))):
> implicitplot({P=.2,P=.4,P=.6,P=.8,P=1},x=-3..3,y=0..6,grid=[150,15
  0],view=[-2..2,0..4]);
```



The problem is that the Maple arctan function is the continuous branch between $-\pi/2$ and $+\pi/2$, whereas (19) calls for the discontinuous branch between 0 and π:

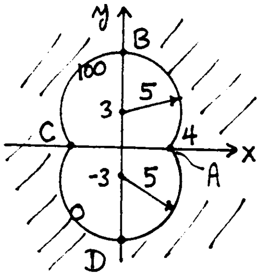


That is, Maple gives ① and ② whereas we want ③ and ②. If ( ) > 0 we're OK, but if ( ) < 0 we need to add π to get from ① to ③. Hence, in place of (19) we write

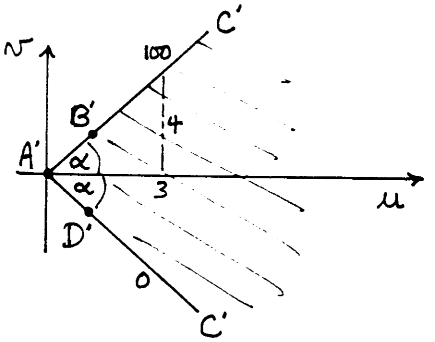
$$\Psi(x,y)=\frac{4}{3}\left[1-\frac{1}{\pi}\left(\pi H(-\text{arg})+\tan^{-1}(\text{arg})\right)\right]$$

where arg is $2y/(x^2+y^2-1)$. We used the grid option to refine the grid from the default of [25,25]. Also, we took the same values of $x_{max}-x_{min}$ and $y_{max}-y_{min}$, and used the view option to force that rectangle, to avoid one axis getting squashed relative to the other.

14. (a)



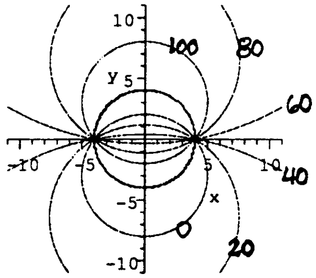
Use $w=\frac{z-4}{z+4}$, say, so as to put the pole at one of the points A, C and a zero at the other



$$\Psi=50\left(1+\frac{\phi}{\alpha}\right)=50+50\,\frac{\tan^{-1}\frac{8y}{x^2+y^2-16}}{\tan^{-1}\frac{4}{3}}$$

where $-\pi/2<\tan^{-1}<\frac{\pi}{2}$. Though not asked for, here is a Maple plot of the Ψ= 0,20,40,60,80,100 contours. (Ignore the ones inside the cutout region.)

```
> with(plots):
> P:=50*(1+arctan(8*y/(x^2+y^2-16))/arctan(4/3)):
> implicitplot({P=0,P=20,P=40,P=60,P=80,P=100},x=-11..11,y=-11..11,g
  rid=[200,200],view=[-11..11,-11..11]);
```

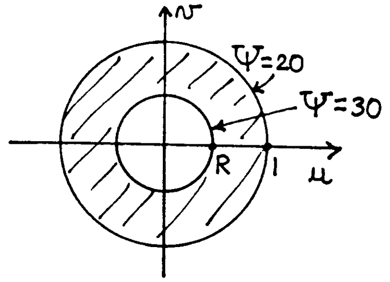


(b) In Entry 4 let $x_2=2$ and $x_1\to\infty$. Then $a\sim(2x_1+\sqrt{3}x_1)/x_1=2+\sqrt{3}$ and $R\sim(2x_1-\sqrt{3}x_1)/x_1=2-\sqrt{3}$. Then the image is as shown at the right.

$$\Psi=A+B\ln\rho$$

gives $\Psi=20+10\ln\rho/\ln R$.

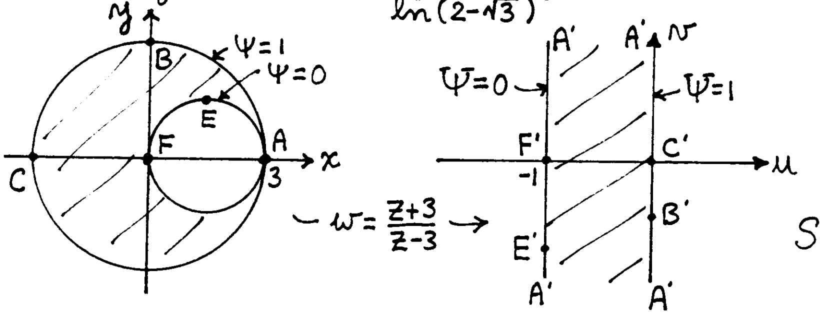
For ρ we can use $\rho=\sqrt{u^2+v^2}$ but it's a bit easier to use



$$\rho = |w| = \sqrt{w\bar{w}} = \sqrt{\frac{x-a+iy}{ax-1+iay}\,\frac{x-a-iy}{ax-1-iay}} = \sqrt{\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}}$$

so $\Psi(x,y) = 20 + 5\,\dfrac{\ln\left(\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}\right)}{\ln(2-\sqrt{3})}$

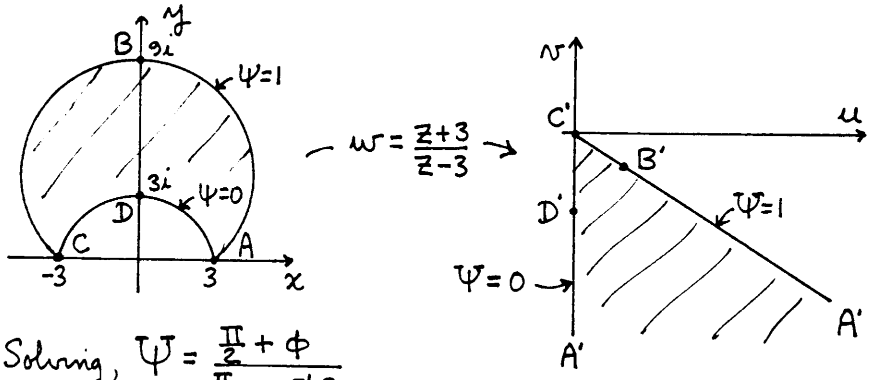
(c)



B' is at $-i$ and E' is at $-1-2i$

Solving, $\Psi = A + Bu$
$= u+1$
$= 1 + \dfrac{x^2+y^2-9}{(x-3)^2+y^2}$

(d)



B' is at $(4-3i)/5$, D' is at. Looking ahead to preferring the $-\pi/2 < \tan^{-1} < \pi/2$ branch of $\tan^{-1}$, let C'D'A' be at $\phi = -\pi/2$ rather than $3\pi/2$, and hence C'B'A' be at $-\tan^{-1}\frac{3}{4}$.

Solving, $\Psi = \dfrac{\frac{\pi}{2}+\phi}{\frac{\pi}{2}-\tan^{-1}\frac{3}{4}}$

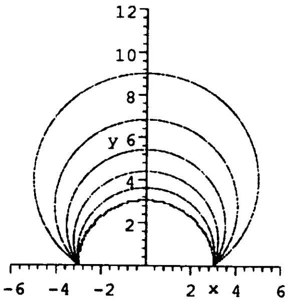
$= \dfrac{\frac{\pi}{2}-\tan^{-1}\left(\frac{6y}{x^2+y^2-9}\right)}{\frac{\pi}{2}-\tan^{-1}\frac{3}{4}}$ where $-\pi/2 < \tan^{-1} < \pi/2$.

Though not asked for, here is a Maple plot of the $\Psi = 0, 0.2, 0.4, 0.6, 0.8, 1$ level curves:

```
> with(plots):
> P:=(.5*Pi-arctan(6*y/(x^2+y^2-9)))/(.5*Pi-arctan(.75));
```

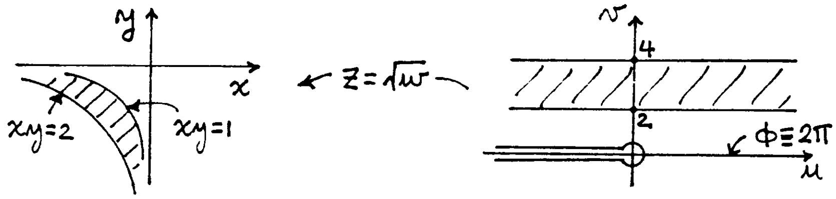
$$P := \frac{.5\,\pi - \arctan\left(6\,\dfrac{y}{x^2+y^2-9}\right)}{.5\,\pi - .6435011088}$$

```
> implicitplot({P=0,P=.2,P=.4,P=.6,P=.8,P=1},x=-6..6,y=0..12,grid=[1
  50,150],view=[-6..6,0..12]);
```



Section 20.4

1.



← φ≡2π will send us to 3rd quadrant (φ≡0 would send us to 1st quadrant)

2. D has two corners. By conformality, those corners will yield two 90° corners in the image D', so D' cannot be a circular annulus or an infinite strip. Of course, we can we can get rid of one corner by making that corner point in the z plane a pole of the transformation, but we can't do that with both corners because a bilinear transformation has only one pole.

3. $\Psi(x,y) = (Ax+B)(Cy+D) + (E\sin kx + F\cos kx)(Ge^{ky}+He^{-ky})$

Boundedness as $y\to\infty \Rightarrow C=G=0$ so

$\Psi(x,y) = A'x+B' + (E'\sin kx + F'\cos kx)e^{-ky}$

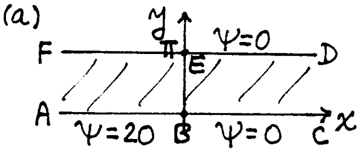
$\Psi(0,y)=0 = B'+F'e^{-ky} \Rightarrow B'=F'=0$ so $\Psi(x,y) = A'x + E'\sin kx\, e^{-ky}$.

$\Psi(1,y)=0 = A'+E'\sin k\, e^{-ky} \Rightarrow A'=0,\ k=n\pi,$

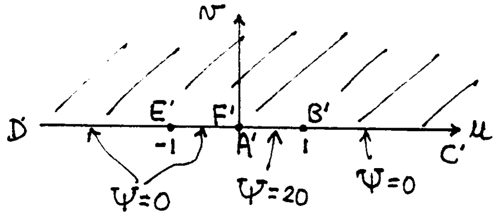
$\Psi(x,y) = \sum_1^\infty E_n' \sin n\pi x\, e^{-n\pi y}$

$$\Psi(x,0) = 100 = \sum_1^\infty E_n' \sin n\pi x, \quad E_n' = \frac{2}{1}\int_0^1 100 \sin n\pi x\,dx = \begin{cases} 400/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

4. (a)



Use Entry 6 in Appendix F, $w = e^z$



Using (12) in Section 20.4,

$$\Psi(u,v) = \frac{v}{\pi}\int_0^1 \frac{20}{(u-\xi)^2+v^2}\,d\xi = \frac{20v}{\pi v}\tan^{-1}\left(\frac{\xi-u}{v}\right)\Big|_{\xi=0}^{\xi=1} = \frac{20}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) - \tan^{-1}\left(\frac{-u}{v}\right)\right],$$

where the choice of the branch of $\tan^{-1}$ is irrelevant since we are differencing two $\tan^{-1}$'s, so let us use $-\pi/2 < \tan^{-1}(\ ) < \pi/2$, say. Finally, $u+iv = e^x(\cos y + i\sin y)$ gives

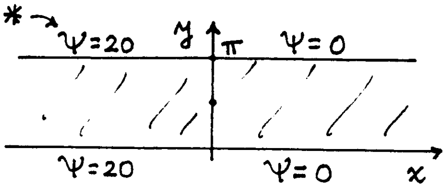
$$u = e^x\cos y,\ v = e^x \sin y,$$

so

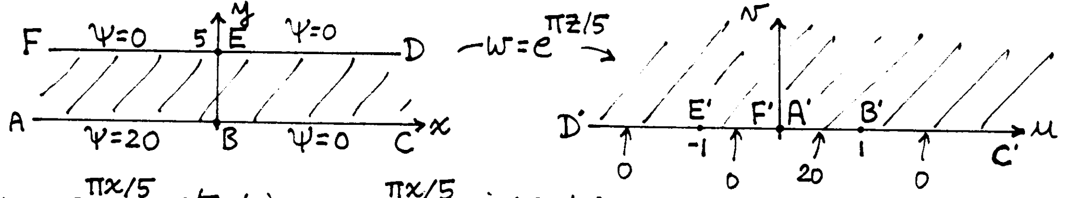
$$\Psi(x,y) = \frac{20}{\pi}\left[\tan^{-1}\left(\frac{1-e^x\cos y}{e^x\sin y}\right) - \tan^{-1}(-\cot y)\right]$$

$$= \frac{20}{\pi}\left[\tan^{-1}(\cot y) + \tan^{-1}\left(\frac{1-e^x\cos y}{e^x\sin y}\right)\right]$$

For ex., $\Psi(0,\pi/2) = \frac{20}{\pi}(0+\tan^{-1}1) = 5$. Does this *look* correct? Well, if the boundary conditions were as shown at the right, then we'd have $\Psi(0,y)=10$, so if we change * to $\Psi=0$, then we *should* have $\Psi(0,\pi/2) = \frac{1}{2}(10) = 5$.



(b) Try $w = e^{az}$. Then $u = e^{ax}\cos ay$, $v = e^{ax}\sin ay$. To map the upper edge $y=5$ into part of the u axis, as in (a), we want $\sin 5a = 0$, so choose $a=\pi/5$. Thus,



$$u = e^{\pi x/5}\cos(\pi y/5),\quad v = e^{\pi x/5}\sin(\pi y/5)$$

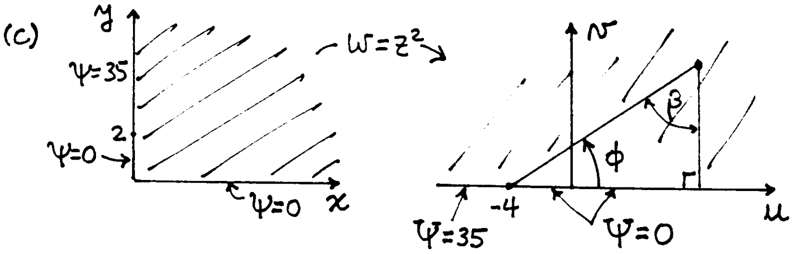
so, again,

$$\Psi(u,v) = \frac{20}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) + \tan^{-1}\left(\frac{u}{v}\right)\right],$$

hence

$$\Psi(x,y) = \frac{20}{\pi}\left[\tan^{-1}\left(\frac{1-e^{\pi x/5}\cos \pi y/5}{e^{\pi x/5}\sin \pi y/5}\right) + \tan^{-1}\left(\cot\frac{\pi y}{5}\right)\right], \qquad -\frac{\pi}{2} < \tan^{-1}(\,) < \frac{\pi}{2}.$$

(c)



Using a polar coordinate system ρ,ϕ centered not at $u=v=0$ but at $u=-4$, $v=0$, we obtain

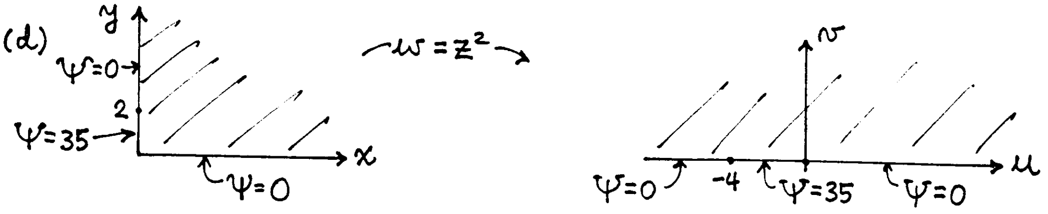
$$\Psi = \frac{35}{\pi}\phi$$

We can express $\phi = \tan^{-1}(v/(u+4))$, but then we will be using the branch $0 < \tan^{-1}(\,) < \pi$. Since Maple and other softwares favor $-\pi/2 < \tan^{-1}(\,) < \pi/2$, let us re-express ϕ as $\pi/2 - \beta = \pi/2 - \tan^{-1}\frac{u+4}{v}$, where *this* $\tan^{-1}$ is between $-\pi/2$ and $\pi/2$. Thus,

$$\Psi = \frac{35}{\pi}\phi = \frac{35}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{u+4}{v}\right) = \frac{35}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{x^2-y^2+4}{2xy}\right],$$

where $-\pi/2 < \tan^{-1}(\,) < \pi/2$. For example, $\Psi(2,2) = 17.5 - \frac{35}{\pi}\tan^{-1}(1/2) = 12.33$, which seems reasonable.

(d)



Using (12) in Sec. 20.4,

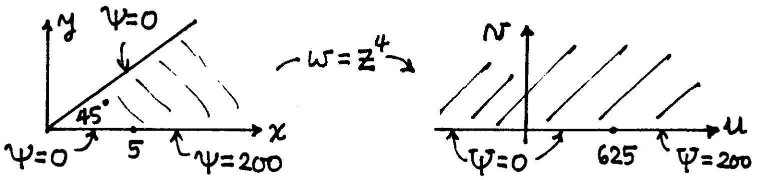
$$\Psi = \frac{v}{\pi}\int_{-4}^{0}\frac{35}{(u-\xi)^2+v^2}\,d\xi = \frac{35v}{\pi v}\tan^{-1}\left(\frac{\xi-u}{v}\right)\Big|_{\xi=-4}^{\xi=0} = \frac{35}{\pi}\left[\tan^{-1}\left(\frac{-u}{v}\right)-\tan^{-1}\left(\frac{-4-u}{v}\right)\right]$$

$$= \frac{35}{\pi}\left[\tan^{-1}\left(\frac{4+u}{v}\right)-\tan^{-1}\left(\frac{u}{v}\right)\right]$$

so

$$\Psi(x,y) = \frac{35}{\pi}\left[\tan^{-1}\left(\frac{x^2-y^2+4}{2xy}\right)-\tan^{-1}\left(\frac{x^2-y^2}{2xy}\right)\right], \text{ where } -\pi/2<\tan^{-1}(\;)<\pi/2$$

(e)



To solve for Ψ we could either introduce a polar coordinate system at u=625, v=0, as we did in (c), or we can use (12) from Sec. 20.4, as we did in (a) and (d). Let us do the latter:

$$\Psi = \frac{v}{\pi}\int_{625}^{\infty}\frac{200}{(u-\xi)^2+v^2}\,d\xi = \frac{200v}{\pi v}\tan^{-1}\left(\frac{\xi-u}{v}\right)\Big|_{\xi=625}^{\xi=\infty}$$

Using $-\pi/2<\tan^{-1}(\;)<\pi/2$ (the choice of the branch won't matter, since we are differencing two $\tan^{-1}$'s), we have

$$\Psi = \frac{200}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{625-u}{v}\right)\right]$$

so

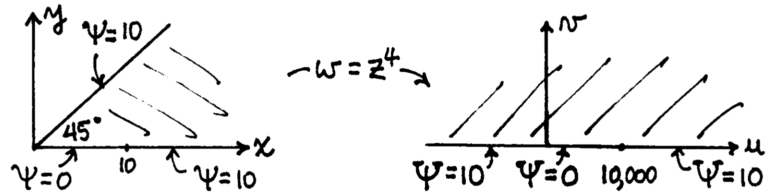
$$\Psi(x,y) = 100-\frac{200}{\pi}\tan^{-1}\left(\frac{625-[(x^2-y^2)^2-4x^2y^2]}{4xy(x^2-y^2)}\right), \quad -\pi/2<\tan^{-1}(\;)<\pi/2.$$

For ex.,

$$\Psi(5,1) = 100-\frac{200}{\pi}\tan^{-1}\left(\frac{149}{480}\right) = 80.8,$$

which looks reasonable.

(f)



Using (12) from Sec. 20.4,

$$\Psi = \frac{v}{\pi}\int_{-\infty}^{0}\frac{10\,d\xi}{(u-\xi)^2+v^2}+\frac{v}{\pi}\int_{10000}^{\infty}\frac{10\,d\xi}{(u-\xi)^2+v^2}$$

$$= \frac{10v}{\pi v}\left[\tan^{-1}\frac{\xi-u}{v}\Big|_{\xi=-\infty}^{\xi=0}+\tan^{-1}\frac{\xi-u}{v}\Big|_{\xi=10000}^{\xi=\infty}\right]$$

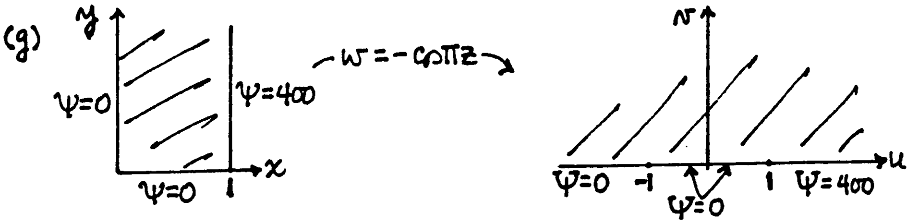
$$= \frac{10}{\pi}\left[\tan^{-1}\left(\frac{-u}{v}\right)-\left(-\frac{\pi}{2}\right)+\left(\frac{\pi}{2}\right)-\tan^{-1}\left(\frac{10000-u}{v}\right)\right] = 10+\frac{10}{\pi}[\text{lan}$$

$$= 10-\frac{10}{\pi}\left[\tan^{-1}\frac{u}{v}+\tan^{-1}\left(\frac{10000-u}{v}\right)\right]$$

so

$$\Psi(x,y) = 10-\frac{10}{\pi}\left[\tan^{-1}\left(\frac{(x^2-y^2)^2-4x^2y^2}{4xy(x^2-y^2)}\right)+\tan^{-1}\left(\frac{10000-(x^2-y^2)^2+4x^2y^2}{4xy(x^2-y^2)}\right)\right], \quad -\frac{\pi}{2}<\tan^{-1}(\;)<\pi/2.$$

(g)



Using (12) from Sec. 20.4,

$$\psi = \frac{v}{\pi}\int_1^\infty \frac{400\,d\xi}{(u-\xi)^2+v^2} = \frac{400v}{\pi v}\tan^{-1}\frac{\xi-u}{v}\Big|_{\xi=1}^{\xi=\infty} = \frac{400}{\pi}\left(\frac{\pi}{2}-\tan^{-1}\frac{1-u}{v}\right)$$

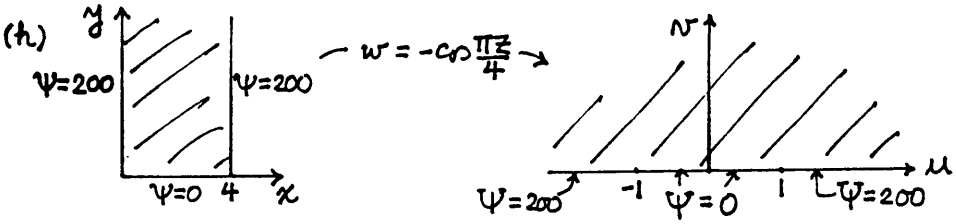
$$w = u+iv = -\cos\pi(x+iy) = -\cos\pi x\cos i\pi y + \sin\pi x\sin i\pi y$$

so $u = -\cos\pi x\cosh\pi y,\quad v = \sin\pi x\sinh\pi y$

so $\psi(x,y) = 200 - \frac{400}{\pi}\tan^{-1}\left(\frac{1+\cos\pi x\cosh\pi y}{\sin\pi x\sinh\pi y}\right)$, $\quad -\pi/2 < \tan^{-1}(\ ) < \frac{\pi}{2}$

For ex., $\psi(.5,.1) \approx 39.4$.

(h)



Using (12) from Sec. 20.4,

$$\psi = \frac{v}{\pi}\int_{-\infty}^{-1}\frac{200\,d\xi}{(u-\xi)^2+v^2} + \frac{v}{\pi}\int_1^\infty \frac{200\,d\xi}{(u-\xi)^2+v^2}$$

$$= \frac{200v}{\pi v}\left[\tan^{-1}\frac{\xi-u}{v}\Big|_{-\infty}^{-1} + \tan^{-1}\frac{\xi-u}{v}\Big|_1^\infty\right] = \frac{200}{\pi}\left[\tan^{-1}\frac{-1-u}{v}+\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}\frac{1-u}{v}\right]$$

so $\psi(x,y) = 200 - \frac{200}{\pi}\left[\tan^{-1}\left(\frac{1-\cos\frac{\pi x}{4}\cosh\frac{\pi y}{4}}{\sin\frac{\pi x}{4}\sinh\frac{\pi y}{4}}\right) + \tan^{-1}\left(\frac{1+\cos\frac{\pi x}{4}\cosh\frac{\pi y}{4}}{\sin\frac{\pi x}{4}\sinh\frac{\pi y}{4}}\right)\right]$,

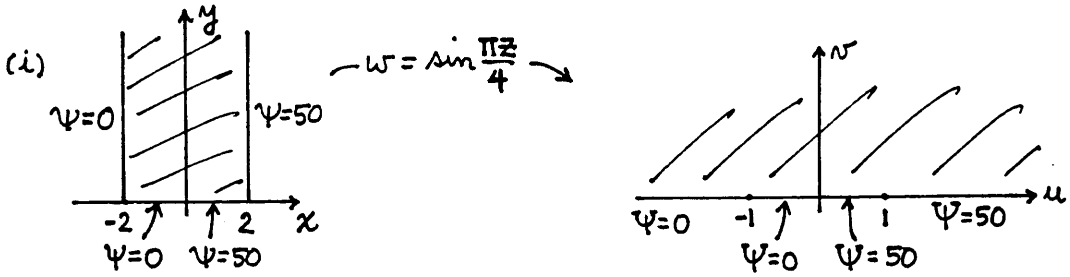
for $-\pi/2 < \tan^{-1}(\ ) < \pi/2$.

$$\psi(2,y) = 200 - \frac{400}{\pi}\tan^{-1}\left(\frac{1}{\sinh\frac{\pi y}{4}}\right)$$

so

$\psi(2,0)=0,\ \psi(2,2)=147.8,\ \psi(2,4)=189.0,\ \psi(2,6)=197.7,\ \psi(2,8)=199.5,\ \psi(2,10)=199.9$

(i)



Using (12) from Sec 20.4,

$$\Psi = \frac{v}{\pi}\int_0^\infty \frac{50\,d\xi}{(u-\xi)^2+v^2} = \frac{50v}{\pi v}\left[\tan^{-1}\left(\frac{\xi-u}{v}\right)\right]_{\xi=0}^{\xi=\infty} = 25 - \frac{50}{\pi}\tan^{-1}\left(-\frac{u}{v}\right)$$

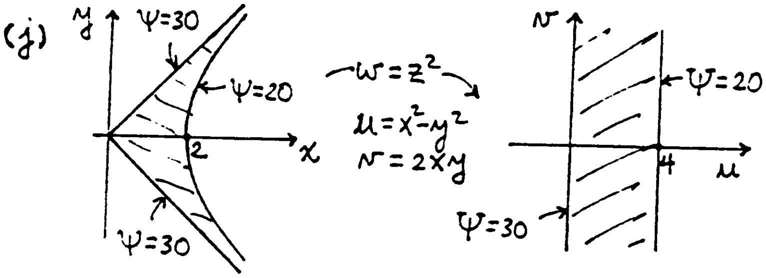
$$w = u+iv = \sin\frac{\pi}{4}(x+iy) = \sin\frac{\pi x}{4}\cos\frac{i\pi y}{4} + \sin\frac{i\pi y}{4}\cos\frac{\pi x}{4}$$

$$= \underbrace{\sin\frac{\pi x}{4}\cosh\frac{\pi y}{4}}_{u} + i\underbrace{\sinh\frac{\pi y}{4}\cos\frac{\pi x}{4}}_{v}$$

so $\Psi(x,y) = 25 + \frac{50}{\pi}\tan^{-1}\left(\tan\frac{\pi x}{4}\coth\frac{\pi y}{4}\right)$, $-\pi/2 < \tan^{-1}(\ ) < \pi/2$.

$\Psi(0,y) = 25 + \frac{50}{\pi}\tan^{-1}0 = 25$ for all y.

(j)



so $\Psi = 30 - \frac{10}{4}u$

$\Psi = 30 - \frac{5}{2}(x^2-y^2)$

Section 22.5

1. Heuristically, if $w = f(z)$ and f is analytic at z_0 with $f'(z_0) \neq 0$, then $dw = f'(z)dz$ or, at z_0, $dw = f'(z_0)dz$. Thus, the amplification is $\frac{|\Delta w|}{|\Delta z|} = |f'(z_0)|$. To explain, if only heuristically, why we need $f'(z_0) \neq 0$, consider the Taylor series of an analytic function of a real variable x, $f(x)$: $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$ or,

$$f(x) - f(x) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots \quad \text{or,}$$

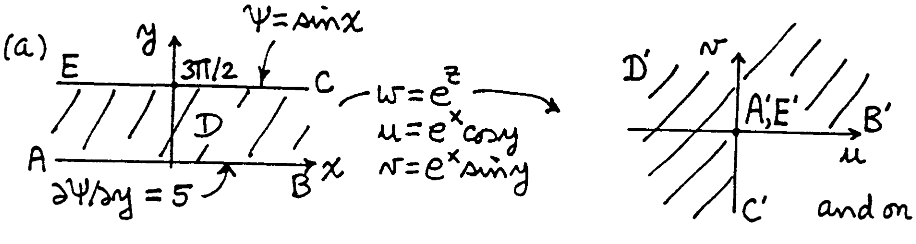
$$\Delta f = f'(a)\,\Delta x + \frac{1}{2!}f''(a)(\Delta x)^2 + \cdots$$

If $f'(a) \neq 0$ then we can say $\Delta f \sim f'(a)\Delta x$; if $f'(a) = 0$ and $f''(a) = 0$ we write $\Delta f \sim \frac{1}{2!}f''(a)(\Delta x)^2$ instead, and so on. Likewise with an analytic function of a complex variable z, although we won't study Taylor series for functions of a complex variable until Chapter 24.

2. $|f'(z)| = |2z| = 2\sqrt{x^2+y^2}$, and $u = x^2 - y^2$ gives $y = \sqrt{-u}$ on FE, so (9) gives

$$\frac{\partial\Psi}{\partial N} = \frac{1}{|f'(z)|}\frac{\partial\Psi}{\partial n} = \frac{1}{2y}\frac{25}{1+y} = \frac{25}{2}\frac{1}{(\sqrt{-u}-u)}$$

3. (a)

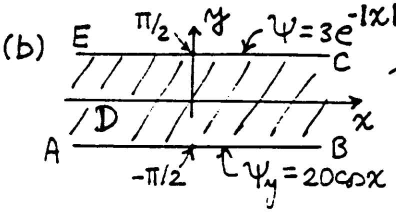


On A'B', $\frac{\partial\Psi}{\partial N} = \frac{1}{|e^z|}\Big|_{y=0}(-5)$

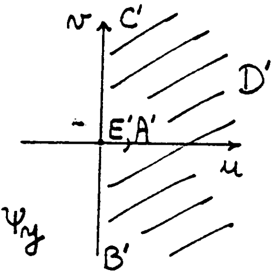
$= -5e^{-x} = -5/u$

and on E'C', $\Psi = \sin(\ln(-v))$.

(b)



$\xrightarrow{w = e^z}$
$u = e^x \cos y$
$v = e^x \sin y$



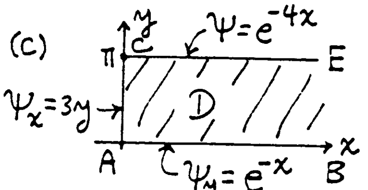
On A'B': $\frac{\partial \Psi}{\partial N} = \frac{1}{|e^z|}\Big|_{z = x - i\pi/2} (-20\cos x)$ ↙since $\Psi_n = -\Psi_y$

$= -\frac{20\cos x}{e^x}$. Now, on AB $y = -\pi/2$ so $u = 0$ and $v = -e^x$ so $x = \ln(-v)$,

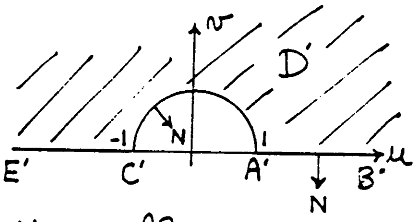
so $\frac{\partial \Psi}{\partial N} = \frac{20\cos[\ln(-v)]}{v}$

On E'C': $\Psi = 3e^{-|x|}$. Now, on EC $y = \pi/2$ so $u = 0$ and $v = e^x$ so $x = \ln v$,
so $\Psi = 3e^{-|\ln v|}$. (Note that $e^{-|x|}$ is NOT $= |e^{-x}|$.)

(c)



$\xrightarrow{w = e^z}$
$u = e^x \cos y$
$v = e^x \sin y$



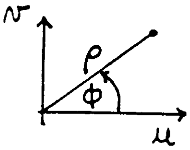
On A'B': $\partial\Psi/\partial N = \frac{1}{|e^z|}\Big|_{y=0} (-e^{-x}) = -\frac{1}{e^{2x}}$. Now, on AB $y = 0$ so $v = 0$ and $u = e^x$

so $\partial\Psi/\partial N = -1/u^2$.

On E'C': $\Psi = e^{-4x}$. Now, on EC $y = \pi$ so $v = 0$ and $u = -e^x$
so $\Psi = (e^x)^{-4} = (-u)^{-4} = u^{-4}$.

On A'C': $\partial\Psi/\partial N = \frac{1}{|e^z|}\Big|_{x=0} (-3y) = -3y$. Now, on AC $x = 0$ so $y = \cos^{-1}u$, say

so $\partial\Psi/\partial N = -3\cos^{-1}u$.



NOTE: Observing that the Ψ problem will best be expressed in terms of polar variables ρ, φ, let us re-express the results, above, in terms of ρ, φ:

A'B': $\partial\Psi/\partial N = -\hat{e}_\phi \cdot \underset{\sim}{\nabla}\Psi = -\frac{1}{\rho}\frac{\partial \Psi}{\partial \phi}$

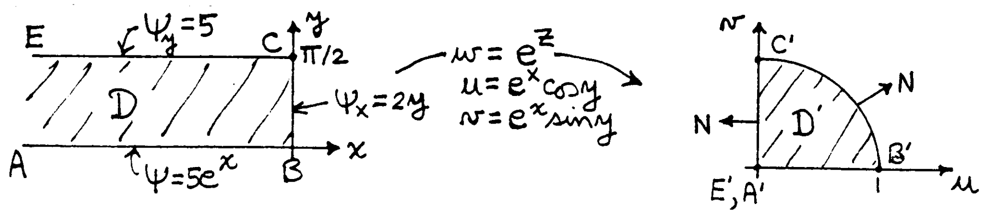
so $-\frac{1}{\rho}\frac{\partial \Psi}{\partial \phi} = -\frac{1}{\rho^2}$ or, $\frac{\partial \Psi}{\partial \phi}(\rho, 0) = \frac{1}{\rho}$

E'C': $\Psi(\rho, \pi) = \frac{1}{\rho^4}$

A'C': There, $x = 0$ so $u = \cos y$ and $v = \sin y$. That is, y is the polar angle ϕ, so

$$\frac{\partial \Psi}{\partial \rho}(1, \phi) = -\frac{\partial \Psi}{\partial N} = -(-3\phi) = 3\phi.$$

(d)



On A'B': $u=e^x$ and $v=0$, so $\Psi(u,0)=5e^x=5u$.

On B'C': $u=\cos y$ and $v=\sin y$, so $\frac{\partial\Psi}{\partial N}=\frac{1}{|e^z|}\Big|_{x=0}(2y)=2\cos^{-1}u$ (or $2\sin^{-1}v$)

On C'E': $u=0$ and $v=e^x$, so $\frac{\partial\Psi}{\partial N}=\frac{1}{|e^z|}\Big|_{y=\pi/2}(5)=\frac{5}{e^x}=\frac{5}{v}$.

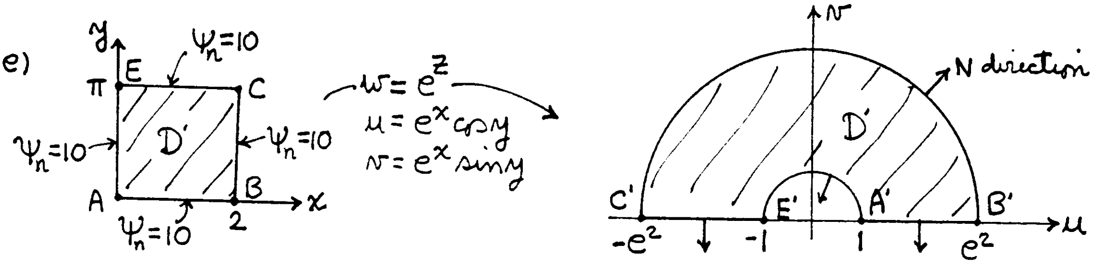
NOTE: Surely, the Ψ problem is better expressed in terms of polar variables ρ,ϕ, so let us re-express the results, above, in terms of ρ,ϕ:

A'B': $\Psi(\rho,0)=5\rho$

B'C': $\frac{\partial\Psi}{\partial N}=\frac{\partial\Psi}{\partial\rho}(1,\phi)=2\phi$

E'C': $\frac{\partial\Psi}{\partial N}=\frac{1}{\rho}\frac{\partial\Psi}{\partial\phi}(\rho,\pi/2)=\frac{5}{\rho}$ or, $\frac{\partial\Psi}{\partial\phi}(\rho,\pi/2)=5$.

(e)



Once again, the Ψ problem is most conveniently expressed in terms of polar variables ρ,ϕ, so let us do so.

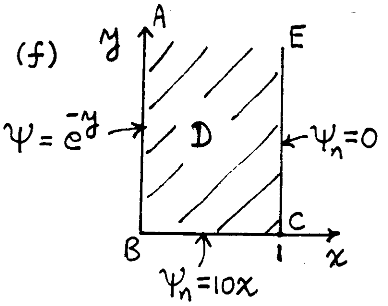
A'B': $\frac{\partial\Psi}{\partial N}=-\frac{1}{\rho}\frac{\partial\Psi}{\partial\phi}=\frac{1}{|e^z|}\Big|_{y=0}(10)=\frac{10}{e^x}=\frac{10}{u}=\frac{10}{\rho}$, so $\frac{\partial\Psi}{\partial\phi}(\rho,0)=-10$

B'C': $\frac{\partial\Psi}{\partial N}=\frac{\partial\Psi}{\partial\rho}=\frac{1}{|e^z|}\Big|_{x=2}(10)=\frac{10}{e^2}$, so $\frac{\partial\Psi}{\partial\rho}(e^2,\phi)=\frac{10}{e^2}$

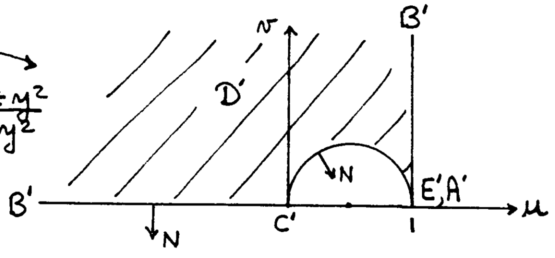
C'E': $\frac{\partial\Psi}{\partial N}=\frac{1}{\rho}\frac{\partial\Psi}{\partial\phi}=\frac{1}{|e^z|}\Big|_{y=\pi}(10)=\frac{10}{e^x}=\frac{10}{-u}=\frac{10}{\rho}$, so $\frac{\partial\Psi}{\partial\phi}(\rho,\pi)=10$

E'A': $\frac{\partial\Psi}{\partial N}=-\frac{\partial\Psi}{\partial\rho}=\frac{1}{|e^z|}\Big|_{x=0}(10)=10$, so $\frac{\partial\Psi}{\partial\rho}=-10$

(f)



$$w = \frac{z-1}{z} \qquad u = \frac{x(x-1)+y^2}{x^2+y^2} \qquad v = \frac{y}{x^2+y^2}$$

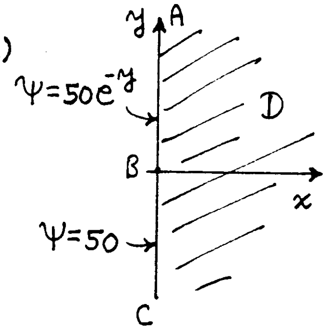


On B'C': $u=(x-1)/x$ and $v=0$ so $\frac{\partial\Psi}{\partial N} = \frac{1}{|1/z^2|}\Big|_{y=0} (10x) = 10x^3 = \frac{10}{(1-u)^3}$

On C'E': $u = y^2/(1+y^2)$ and $v = y/(1+y^2)$ so $\frac{\partial\Psi}{\partial N} = (\text{etc})(0) = 0$

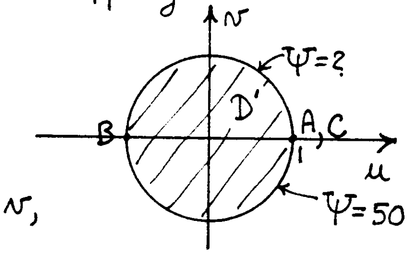
On A'B': $u=1$ and $v = 1/y$ so $\Psi(1,v) = e^{-y} = e^{-1/v}$

(g)



$$w = \frac{z-1}{z+1}$$

We find that AB maps into the upper half of the unit circle and BC into the bottom half, with D mapping into the interior of that circle



We will need y as a function of u and v, for the b.c. $\Psi(0,y) = 50e^{-y}$, so solve $w = (z-1)/(z+1)$ for z: $z = (1+w)/(1-w)$

$$x+iy = \frac{(u+1)+iv}{(-u+1)-iv}\,\frac{(-u+1)+iv}{(-u+1)+iv} = \frac{1-u^2-v^2}{(1-u)^2+v^2} + i\underbrace{\frac{2v}{(1-u)^2+v^2}}_{y}$$

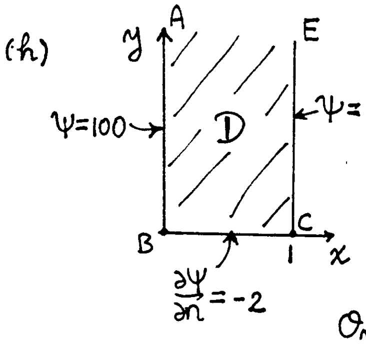
Thus, on the upper semicircle

$$\Psi = 50e^{-y} = 50\exp\left[-\frac{2v}{1-2u+u^2+v^2}\right] = 50\exp\left[\frac{v}{u-1}\right]$$

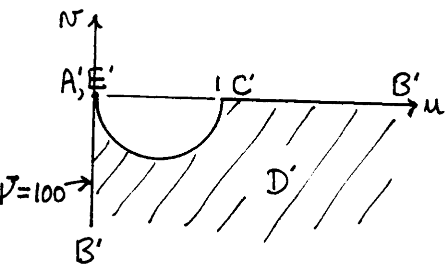
or, in terms of polar coordinates ρ, ϕ,

$$\Psi(1,\phi) = 50\exp\left[\frac{\sin\phi}{\cos\phi - 1}\right] \qquad (0<\phi<\pi)$$

(h)



$$w = \frac{1}{z} \qquad x = u/(u^2+v^2) \qquad y = -v/(u^2+v^2)$$

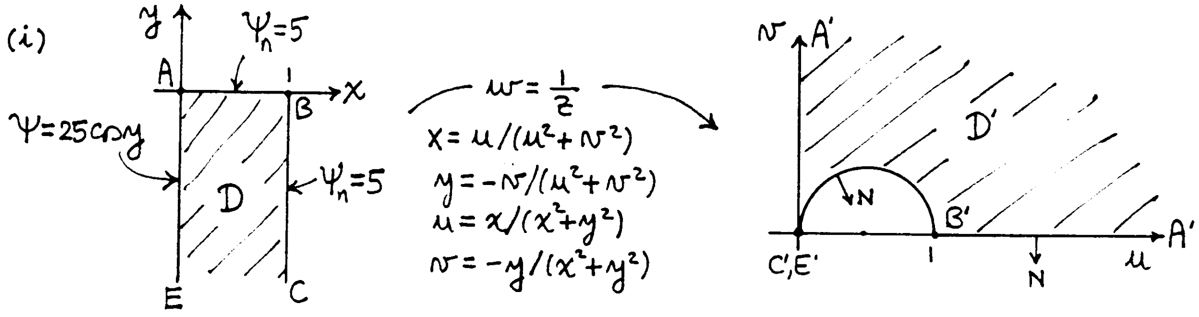


On B'C': $\partial\Psi/\partial N = \frac{1}{|-1/z^2|}\Big|_{y=0}(-2) = -2x^2 = -2/u^2$

On C'E': $\Psi = 3e^{v/(u^2+v^2)}$. To express this in terms of u alone note that $x=1$ on CE, so $1 = u/(u^2+v^2)$, so $v = -\sqrt{u-u^2}$, the negative root chosen since $v<0$ on C'E'. Thus,

$$\Psi = 3\exp\left(-\frac{\sqrt{u-u^2}}{u}\right). \quad (0<u<1)$$

(i)



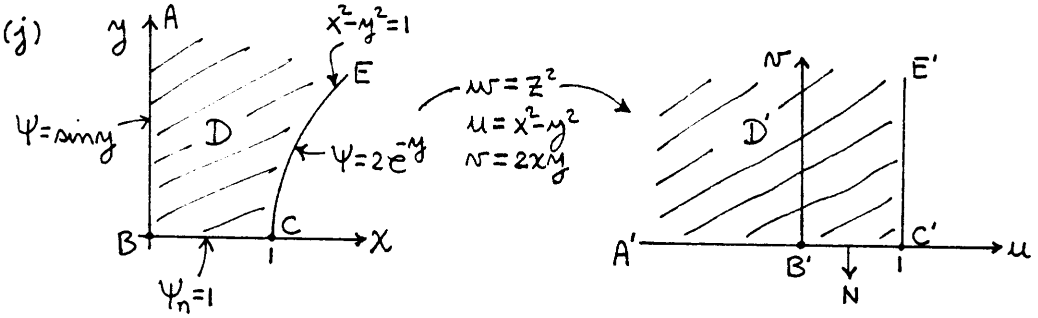
A'E': $y=-1/v$, so $\Psi(0,v) = 25\cos(-\frac{1}{v}) = 25\cos(\frac{1}{v})$.

B'A': $x=1/u$ so $\frac{\partial\Psi}{\partial N} = \frac{1}{|-1/z^2|}\Big|_{y=0}(5) = 5x^2 = 5/u^2$

B'C': $\frac{\partial\Psi}{\partial N} = \frac{1}{|-1/z^2|}\Big|_{x=1}(5) = 5|z^2|\Big|_{x=1} = 5(1+y^2) = 5\left[1+\frac{v^2}{(u^2+v^2)^2}\right]$

$= 5(1+\frac{v^2}{u^2}) = 5(1+\frac{u-u^2}{u^2}) = 5/u.$

(j)



On A'B': $\Psi(u,0) = \sin\sqrt{-u}$

On B'C': $\frac{\partial\Psi}{\partial N}(u,0) = \frac{1}{|2z|}\Big|_{y=0}(1) = \frac{1}{2x} = \frac{1}{2\sqrt{u}}$

On C'E': $\Psi(1,v) = 2e^{-y} = ?$ To express y as a function of v on C'E' eliminate x between $u=1=x^2-y^2$ and $v=2xy$, giving

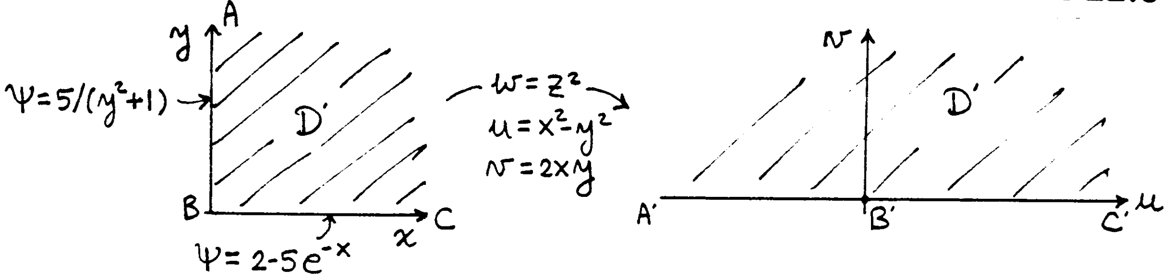
$v = 2y\sqrt{1+y^2}$ or $y^4+y^2-\frac{v^2}{4}=0$, $y^2 = (-1 \pm \sqrt{1+v^2})/2$

Choose + since y is real

$$y = \sqrt{(\sqrt{1+v^2}-1)/2}$$

so $\Psi(1,v) = 2\exp\left[-\sqrt{(\sqrt{1+v^2}-1)/2}\,\right]$

(k)



On A'B': $\Psi = 5/(1-u)$

On B'C': $\Psi = 2-5e^{-\sqrt{u}}$

Section 22.6

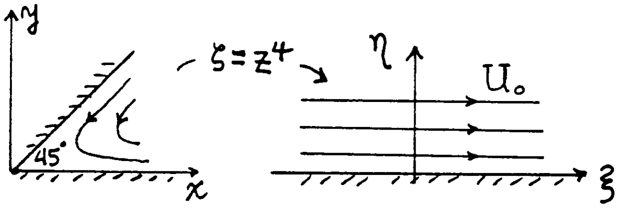
1. $\phi(x,y) = U_0(x^2-y^2)$ so

$\Psi_y = u = \phi_x = 2xU_0 \rightarrow \Psi = \int 2xU_0\,\partial y = 2xyU_0 + A(x)$

$\Psi_x = -v = -\phi_y = 2yU_0 = 2yU_0 + A'(x)$ so $A'(x)=0$, $A(x) = \overset{0,\text{ say}}{\cancel{\text{const}}}.$

Thus, $\Psi(x,y) = 2U_0xy$.

2.(a)



$W=U_0\zeta$

so $w = U_0z^4$, $dw/dz =$

$u-iv = 4U_0z^3$

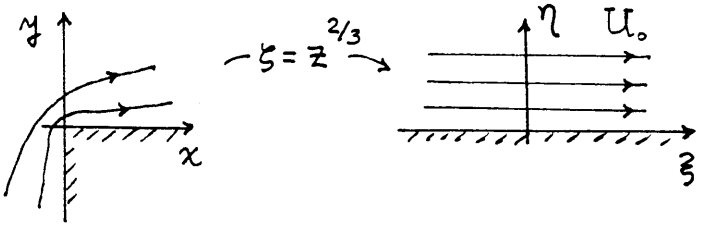
$= 4U_0(x+iy)^3$

$= 4U_0x(x^2-3y^2) + i4U_0y(3x^2-y^2)$,

so $u=4U_0x(x^2-3y^2)$, $v=4U_0y(y^2-3x^2)$. Since $\phi_x=u$, $\phi_y=v$, $\Psi_x=-v$, and $\Psi_y=u$, integration of these gives

$$\phi(x,y)=U_0[(x^2-y^2)^2-4x^2y^2],\quad \Psi(x,y)=4U_0xy(x^2-y^2)$$

(b)



$W=U_0\zeta$

so $w = U_0z^{2/3}$

$\frac{dw}{dz} = u-iv = \frac{2}{3}U_0z^{-1/3}$

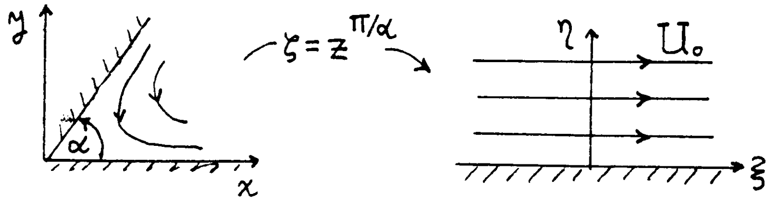
$= \frac{2}{3}U_0(re^{i\theta})^{-1/3}$

$= \frac{2}{3}U_0r^{-1/3}(\cos\frac{\theta}{3}-i\sin\frac{\theta}{3})$

so $u = \frac{2U_0}{3r^{1/3}}\cos\frac{\theta}{3}$, $v = \frac{2U_0}{3r^{1/3}}\sin\frac{\theta}{3}$.

In this case $u,v \rightarrow \infty$ as $r\rightarrow 0$ so the flow is singular at this "outside corner", whereas in (a) we see that $u=v=0$ at $r=0$, which is a stagnation point.

(c)



$$W = U_0\,\zeta = U_0 z^{\pi/\alpha} = w$$
$$\frac{dw}{dz} = u - iv = \frac{\pi U_0}{\alpha} z^{\frac{\pi}{\alpha}-1}$$
$$= \frac{\pi U_0}{\alpha}(re^{i\theta})^{\frac{\pi-\alpha}{\alpha}}$$

so $u = \frac{\pi U_0}{\alpha} r^{\frac{\pi-\alpha}{\alpha}} \cos\left(\frac{\pi-\alpha}{\alpha}\theta\right)$, $v = -\frac{\pi U_0}{\alpha} r^{\frac{\pi-\alpha}{\alpha}} \sin\left(\frac{\pi-\alpha}{\alpha}\theta\right)$.

Let $\frac{\pi-\alpha}{\alpha} \equiv \beta$. Then, to find $\phi(r,\theta)$ write

$$\left.\begin{aligned} u &= \frac{\pi U_0}{\alpha} r^\beta \cos\beta\theta = \phi_x = \phi_r r_x + \phi_\theta \theta_x = \cos\theta\,\phi_r - \frac{\sin\theta}{r}\phi_\theta \\ v &= -\frac{\pi U_0}{\alpha} r^\beta \sin\beta\theta = \phi_y = \phi_r r_y + \phi_\theta\theta_y = \sin\theta\,\phi_r + \frac{\cos\theta}{r}\phi_\theta \end{aligned}\right\} \text{by (41) and (42) on page 650}$$

Solve for ϕ_r and ϕ_θ by Cramer's rule:

$$\phi_r = \frac{\begin{vmatrix} (\pi U_0/\alpha) r^\beta \cos\beta\theta & -\sin\theta/r \\ -(\pi U_0/\alpha) r^\beta \sin\beta\theta & \cos\theta/r \end{vmatrix}}{\begin{vmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{vmatrix}} = \frac{\pi U_0}{\alpha} r^\beta (\cos\beta\theta\cos\theta - \sin\beta\theta\sin\theta)$$
$$= \frac{\pi U_0}{\alpha} r^\beta \cos(\beta+1)\theta$$
$$= \frac{\pi U_0}{\alpha} r^\beta \cos\frac{\pi\theta}{\alpha} \qquad ①$$

and, after some algebra,

$$\phi_\theta = -\frac{\pi U_0}{\alpha} r^{\pi/\alpha} \sin\frac{\pi\theta}{\alpha} \qquad ②$$

Integrating ①, $\phi = \int \frac{\pi U_0}{\alpha} r^\beta \cos\frac{\pi\theta}{\alpha}\,\partial r = U_0 r^{\pi/\alpha} \cos\frac{\pi\theta}{\alpha} + A(\theta)$. ③

Then put ③ into ②:

$$-\frac{\pi U_0}{\alpha} r^{\pi/\alpha} \sin\frac{\pi\theta}{\alpha} = -\frac{\pi U_0}{\alpha} r^{\pi/\alpha} \sin\frac{\pi\theta}{\alpha} + A'(\theta)$$

so $A'(\theta) = 0$, $A(\theta) = \text{const.} = 0$, say. Thus,

$$\phi(r,\theta) = U_0 r^{\pi/\alpha} \cos\frac{\pi\theta}{\alpha}.$$

To find ψ, the simplest approach is to use the fact that ψ and ϕ are conjugate harmonic functions. In polar coordinates the Cauchy-Riemann equations are $\phi_r = \frac{1}{r}\psi_\theta$

$$\psi_r = -\frac{1}{r}\phi_\theta$$

from page 1144. Thus,

$$\psi_\theta = r\phi_r = \frac{\pi}{\alpha} U_0 r^{\frac{\pi}{\alpha}} \cos\frac{\pi\theta}{\alpha} \quad \text{so } \psi = U_0 r^{\frac{\pi}{\alpha}} \sin\frac{\pi\theta}{\alpha} + B(r).$$

and

$$\psi_r = -\frac{1}{r}\phi_\theta = +\frac{\pi}{\alpha}\cancel{U_0 r^{\frac{\pi}{\alpha}-1}} \sin\frac{\pi\theta}{\alpha} = \frac{\pi}{\alpha}\cancel{U_0 r^{\frac{\pi}{\alpha}-1} \sin\frac{\pi\theta}{\alpha}} + B'(r)$$

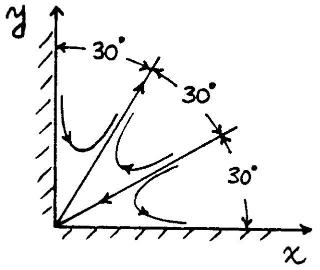
so $B'(r) = 0$, $B = \text{const.} = 0$.

Thus,

$$\psi(r,\theta) = U_0 r^{\pi/\alpha} \sin\frac{\pi\theta}{\alpha}.$$

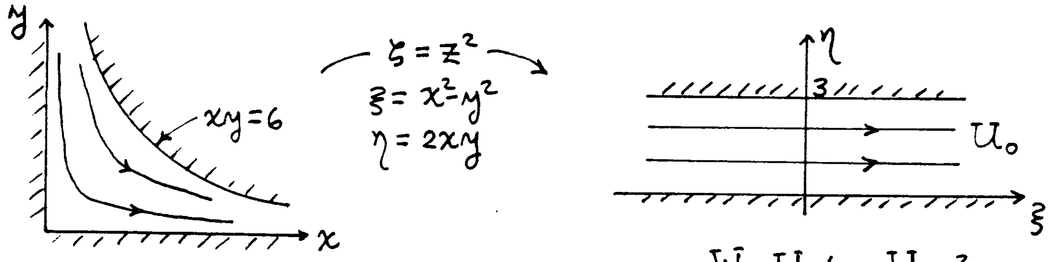
As a check, observe that $\psi(r,0) = \psi(r,\alpha) = 0$ so the physical boundary is a streamline, as it should be.

(d) No, it is not unique, as suggested in the hint. For instance, another possible corner flow is as sketched at the right.



3. We already did this in Exercise 6(b) of Section 20.3.

4.(a)



$$W = U_0 \zeta = U_0 z^2 = w.$$

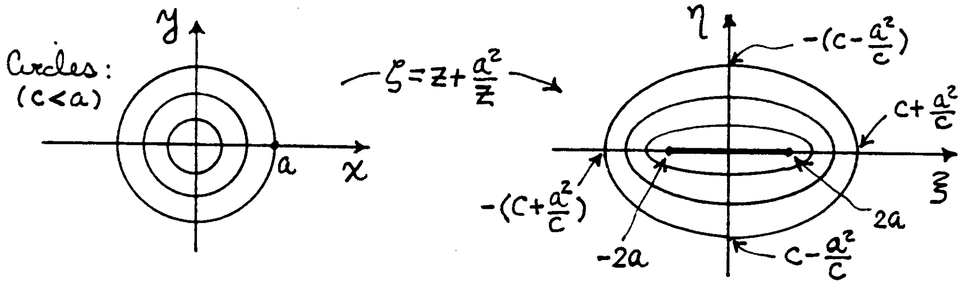
This is the same flow as in Example 1 except that instead of being in the whole quarter plane $xy > 0$ it is between $0 < xy < 6$.

5. If $z = ce^{i\theta}$ (circle of radius c) then

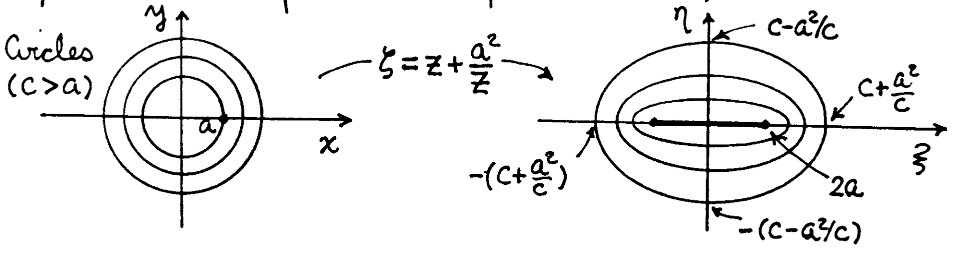
$$\zeta = ce^{i\theta} + \frac{a^2}{c}e^{-i\theta} = (c+\frac{a^2}{c})\cos\theta + i(c-\frac{a^2}{c})\sin\theta$$

so $\left.\begin{array}{l} \xi = (c+\frac{a^2}{c})\cos\theta \\ \eta = (c-\frac{a^2}{c})\sin\theta \end{array}\right\} \rightarrow \frac{\xi^2}{(c+\frac{a^2}{c})^2} + \frac{\eta^2}{(c-\frac{a^2}{c})^2} = 1$ (ellipses)

Thus,



Thus, the image of $|z| < a$ is the whole ζ plane less the line $|\xi| \leq 2a, \eta = 0$; if we include the circle $|z| = a$ then it maps to that line segment and completes the ζ plane. Next, consider circles with $c > a$:

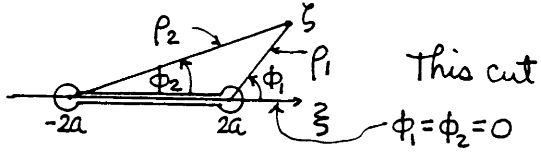


Again the image of $|z|>c$ is the whole ζ plane less the line $|\xi|\le 2a, \eta=0$; if we include the circle $|z|=c$ it maps to that line segment and completes the ζ plane.

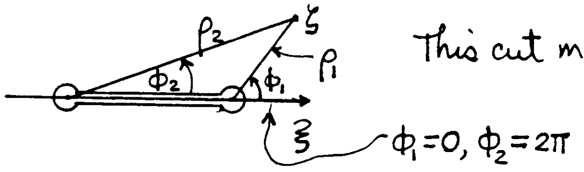
Solving $\zeta = z + a^2/z$ for z, by the quadratic formula, gives

$$z = (\zeta + \sqrt{\zeta^2-4a^2})/2,$$

where there's no point in writing $(\zeta \pm \sqrt{\ })/2$ since it still remains to define the $\sqrt{\ }$. To do that, introduce a branch cut:



This cut maps the slit ζ plane onto $|z|>a$.

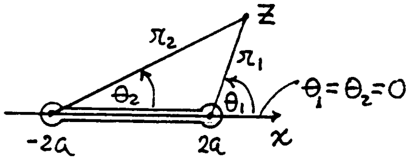


This cut maps the slit ζ plane onto $|z|<a$.

6. (a) $w = V_0 z''' = V_0\left(z'' + \frac{a^2}{z''}\right) = iV_0\left(z' - \frac{a^2}{z'}\right) = iV_0\left(\frac{z+\sqrt{z^2-4a^2}}{2} - \frac{2a^2}{z+\sqrt{z^2-4a^2}}\right)$

so $\frac{dw}{dz} = u - iv = \text{etc} = \frac{iV_0 z}{\sqrt{z^2-4a^2}}$

where the cut (this time going from the z plane to a z' plane) is as shown at the right.



(b) On top of the plate $\theta_1=\pi, r_1=2a-x, \theta_2=0, r_2=2a+x$ so

$$\begin{aligned}\sqrt{z^2-4a^2} &= \sqrt{(z-2a)(z+2a)} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} = \sqrt{(2a-x)(2a+x)}\, e^{i\pi/2} \\ &= i\sqrt{4a^2-x^2}\end{aligned}$$

and $z=x$, so

$$u(x,0+) - iv(x,0+) = \frac{iV_0 x}{i\sqrt{4a^2-x^2}} \quad \text{so} \quad \begin{aligned} u(x,0+) &= V_0 x/\sqrt{4a^2-x^2} \\ v(x,0+) &= 0\end{aligned}$$

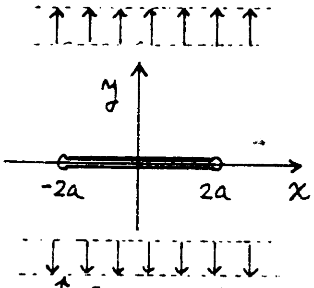
and on the bottom of the plate $\theta_1=-\pi, r_1=2a-x, \theta_2=0, r_2=2a+x$ so

$$\begin{aligned}\sqrt{z^2-4a^2} &= \sqrt{(z-2a)(z+2a)} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} = \sqrt{(2a-x)(2a+x)}\, e^{-i\pi/2} \\ &= \sqrt{4a^2-x^2}\,(-i)\end{aligned}$$

and $z=x$, so

$$u(x,0-) - iv(x,0-) = \frac{iV_0 x}{-i\sqrt{4a^2-x^2}} \quad \text{so} \quad \begin{aligned} u(x,0-) &= -V_0 x/\sqrt{4a^2-x^2} \\ v(x,0+) &= 0.\end{aligned}$$

NOTE: It is important to observe that the flow is "singular" at the ends of the plate ($z=\pm 2a$) since $u - iv = iV_0 z/\sqrt{z^2-4a^2} \to \infty$ there. Analogous mathematics governs the stress and strain in an infinite sheet with a slit (i.e. a crack) in it, from $z=-2a$ to $z=+2a$, subjected to a uniform tensile stress in the y direction. Likewise, the stress and strain fields will be singular at the crack tips ($\pm 2a$), with the same square-root type singularity.



Uniform stress applied at $y=\pm\infty$.

(c) $$w = \phi + i\psi = iV_0\left(\frac{z+\sqrt{z^2-4a^2}}{2} - \frac{2a^2}{z+\sqrt{z^2-4a^2}}\right)$$

$$= iV_0\left(\frac{z^2+2z\sqrt{\;}+z^2-4a^2-4a^2}{2(z+\sqrt{\;})}\right) = iV_0\,\frac{(z^2-4a^2)+z\sqrt{\;}}{z+\sqrt{\;}}\quad\frac{z-\sqrt{\;}}{z-\sqrt{\;}}$$

$$= iV_0\,\frac{z(z^2-4a^2)-(z^2-4a^2-z^2)\sqrt{\;}-z(z^2-4a^2)}{z^2-(z^2-4a^2)}$$

$$= iV_0\sqrt{(z-2a)(z+2a)} = iV_0\sqrt{r_1 r_2}\,e^{i\frac{\theta_1+\theta_2}{2}}$$

$$= iV_0\sqrt{r_1 r_2}\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right]$$

so $$\psi = V_0\sqrt{r_1 r_2}\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)$$

$$= V_0\sqrt{r_1 r_2}\left(\sqrt{\frac{1+\cos\theta_1}{2}}\sqrt{\frac{1+\cos\theta_2}{2}}-\sqrt{\frac{1-\cos\theta_1}{2}}\sqrt{\frac{1-\cos\theta_2}{2}}\right)$$

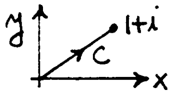
$$= \frac{V_0}{2}\left[\sqrt{(r_1+r_1\cos\theta_1)(r_2+r_2\cos\theta_2)}-\sqrt{(r_1-r_1\cos\theta_1)(r_2-r_2\cos\theta_2)}\right],$$

which is now in a convenient form to change over to x and y because $r_1\cos\theta_1 = x-2a$, $r_2\cos\theta_2 = x+2a$, $r_1=\sqrt{(x-2a)^2+y^2}$, $r_2=\sqrt{(x+2a)^2+y^2}$.

CHAPTER 23

Section 23.2

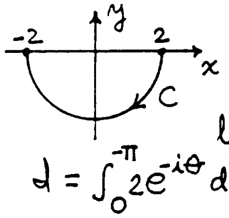
1\. (a)



$I = \int_C |z|^2 dz = \int_C (x^2+y^2)(dx+idy)$ but $y=x$ on C, so this $= \int_0^1 2x^2(1+i)dx = (2+2i)/3$

(b) $I = \int_C \bar{z}\, dz = \int_C (x-iy)(dx+idy) = \int_0^1 (1-i)x(1+i)dx = 2x^2/2\big|_0^1 = 1$

(c)

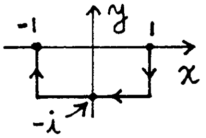


$I = \int_C \bar{z}\, dz = \int_C (x-iy)(dx+idy)$ $(x=2\cos\theta,\ y=2\sin\theta)$ $= \int_0^{-\pi} 2(\cos\theta - i\sin\theta)(-2\sin\theta+i2\cos\theta)\,d\theta$,

but easier to use $z=2e^{i\theta}$, $\bar{z}=2e^{-i\theta}$ on C:

$I = \int_0^{-\pi} 2e^{-i\theta}d(2e^{i\theta}) = 4\int_0^{-\pi} e^{-i\theta} i e^{i\theta} d\theta = 4i\int_0^{-\pi} d\theta = -4\pi i$

(d)



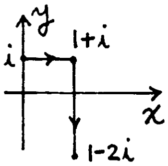
$I = \int_C \frac{1}{z}dz = \int_0^{-1} \frac{i\,dy}{1+iy}\frac{1-iy}{1-iy} + \int_1^{-1}\frac{dx}{x-i}\frac{x+i}{x+i} + \int_{-1}^{0}\frac{i\,dy}{-1+iy}\frac{-1-iy}{-1-iy}$

$= \int_0^{-1}\frac{i+y}{1+y^2}dy + \int_1^{-1}\frac{i+x}{1+x^2}dx - \int_{-1}^{0}\frac{i-y}{1+y^2}dy$

$= (i\tan^{-1}y + \ln\sqrt{1+y^2})\big|_0^{-1} + (i\tan^{-1}x + \ln\sqrt{1+x^2})\big|_1^{-1} - (i\tan^{-1}y - \ln\sqrt{1+y^2})\big|_{-1}^{0}$

$= -i\frac{\pi}{4} + \ln\sqrt{2} - 0 - 0 - i\frac{\pi}{4} + \ln\sqrt{2} - i\frac{\pi}{4} - \ln\sqrt{2} - 0 - 0 - i\frac{\pi}{4} - \ln\sqrt{2} = -\pi i$

(e)



$I = \int_C e^z dz = \int_C e^x(\cos y + i\sin y)(dx+idy)$

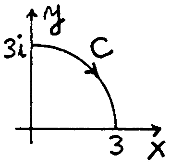
$= \int_C e^x(\cos y\,dx - \sin y\,dy) + i\int_C e^x(\sin y\,dx + \cos y\,dy)$

$= \int_0^1 e^x\cos 1\,dx + i\int_0^1 e^x \sin 1\,dx + \int_1^{-2} -e^1\sin y\,dy + i\int_1^{-2} e^1\cos y\,dy$

$= (\cancel{e}-1)\cos 1 + i(\cancel{e}-1)\sin 1 + e(\cos 2 - \cancel{\cos 1}) + ie(-\sin 2 - \cancel{\sin 1})$

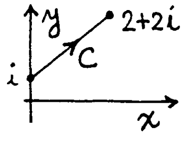
$= (e\cos 2 - \cos 1) - i(\sin 1 + e\sin 2)$

(f)



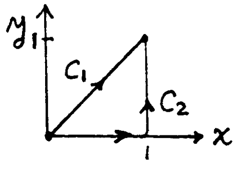
$I = \int_C (\mathrm{Re}\, z)\,dz = \int_C x(dx+idy) = \int_0^3 x\,dx + i\int_{\pi/2}^{0}(3\cos\theta)(3\cos\theta\,d\theta = \frac{9}{2} - \frac{9\pi}{4}i$

(g)



$I = \int_C (\mathrm{Im}\, z)\,dz = \int_C y(dx+idy) = \int_0^2 (\frac{1}{2}x+1)dx + i\int_1^2 y\,dy = 1+2+i\frac{3}{2} = 3+\frac{3}{2}i$

2\.

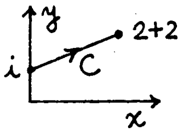


$\int_{C_1}(x-iy)(dx+idy) = (1-i)(1+i)\int_0^1 x\,dx = 1$

$\int_{C_2}(x-iy)(dx+idy) = \int_0^1 x\,dx + \int_0^1 (1-iy)i\,dy + \frac{1}{2}+i+\frac{1}{2} = 1+i.$

$\int_{C_1} \neq \int_{C_2}$ so the integral is path dependent.

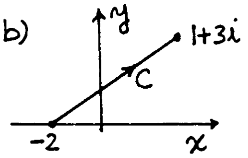
3.(a)



$\left|\int_C z^5 dz\right| \le \underbrace{(\sqrt{8})^5}_{M}\,\underbrace{\sqrt{4+1}}_{L} = 128\sqrt{10}$ (or larger, of course)

↑ since the maximum $|z|$ on C is $|2+2i| = \sqrt{8}$

(b)



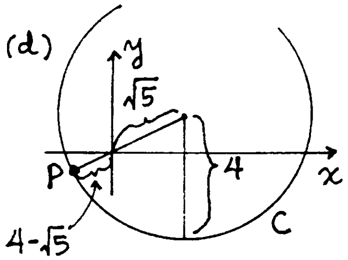
$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \le e^1$ on C and $L = \sqrt{9+9} = 3\sqrt{2}$, so

$$\left|\int_C e^z dz\right| \le 3\sqrt{2}\,e.$$

(c) $|e^{-z}| = |e^{-x-iy}| = |e^{-x}||e^{-iy}| = e^{-x} \le e^{-(-2)} = e^2$ on C, and $L = 3\sqrt{2}$ again, so

$$\left|\int_C e^{-z} dz\right| \le 3\sqrt{2}\,e^2.$$

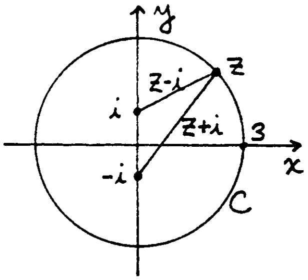
(d)



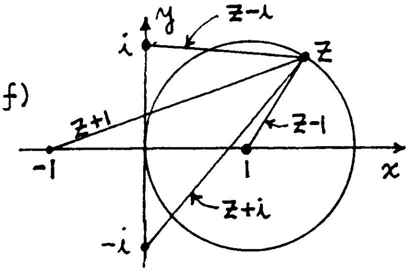
$\max\left|\frac{1}{z}\right| = \max\frac{1}{|z|} = \frac{1}{\min|z|} = \frac{1}{4-\sqrt{5}}$ since the point on C that is closest to the origin (to minimize $|z|$) is P. Also, $L = (2\pi)(4)$, so $\left|\int_C \frac{dz}{z}\right| \le \frac{8\pi}{4-\sqrt{5}}$

(e) $\max\left|\frac{1}{z^2+1}\right| = \frac{1}{\min|z^2+1|} \le \frac{1}{(\min|z-i|)(\min|z+i|)} = \frac{1}{2}\frac{1}{2}$

and $L = (2\pi)(3)$, so $\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{6\pi}{4} = \frac{3\pi}{2}$



(f)



$$\max\left|\frac{z^2+1}{z^2-1}\right| \le \frac{\max|(z-i)(z+i)|}{\min|(z-1)(z+1)|}$$

$$\le \frac{\max|z-i|\,\max|z+i|}{\min|z+1|} \longleftarrow \text{since } |z-1| = 1 \text{ everywhere on C}$$

$$= \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{1} = 3+2\sqrt{2}$$

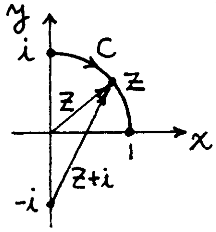
Alternatively, we could say $\max\left|\frac{z^2+1}{z^2-1}\right| \le \frac{\max|z^2+1|}{\min|z^2-1|} \overset{\text{by triangle inequality}}{\le} \frac{\max(|z^2|+|1|)}{\min|z+1|}$

$$= \frac{\max|z|^2+1}{1} = \frac{5}{1} = 5.$$

Let's use the latter in place of the former since 5 is smaller than $3+2\sqrt{2}$. Also, $L = (2\pi)(1)$, so we have

$$\left|\int_C \frac{z^2+1}{z^2-1}dz\right| \le (5)(2\pi) = 10\pi.$$

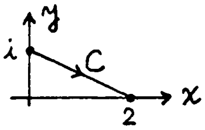
(g)



$\max\left|\frac{1}{z(z+i)}\right| = \max\left(\frac{1}{|z|}\frac{1}{|z+i|}\right) = \max\frac{1}{|z+i|}$ since $|z| = 1$ on C

$= 1/\min|z+i| = 1/\sqrt{2}$. Also, $L = (2\pi)(1)/4$, so

$$\left|\int_C \frac{dz}{z(z+i)}\right| \le \frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{4}.$$

(h) 

$$\max\left|\frac{e^z}{z}\right| \le \frac{\max|e^z|}{\min|z|} = \frac{\max|e^{x+iy}|}{1} = \max|e^x| = e^2$$

and $L=\sqrt{5}$, so $\left|\int_C e^z dz/z\right| \le (e^2)(\sqrt{5}) = \sqrt{5}\,e^2$

(i) $$\max\left|\frac{\cos z}{z}\right| \le \frac{\max|\cos x\cosh y - i\sin x\sinh y|}{\min|z|} = \max\sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \le \sqrt{\cosh^2 1 + \sinh^2 1}$$

and $L=\sqrt{5}$, so $\left|\int_C \frac{\cos z}{z}dz\right| \le \sqrt{\cosh^2 1+\sinh^2 1}\,\sqrt{5} \approx 4.34$

4. $$\left|\frac{\sin z}{z(z^2+9)}\right| = \frac{\sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}}{5\,|(x+iy)-3i|\,|(x+iy)+3i|}$$

$$= \frac{1}{5}\sqrt{\frac{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}{[x^2+(y-3)^2][x^2+(y+3)^2]}} \equiv f(\theta), \text{ since } x=5\cos\theta,\ y=5\sin\theta \text{ on } C.$$

Let us use maple to find $\max\limits_{0\le\theta\le 2\pi} f(\theta)$:

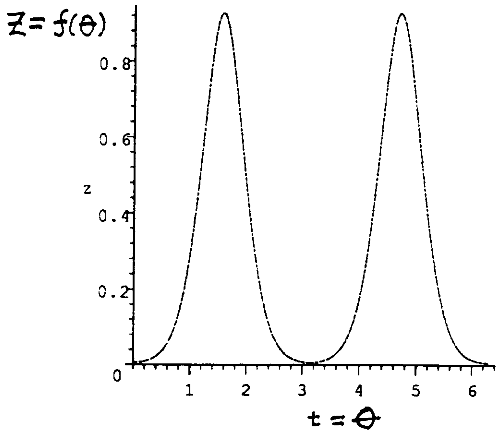
```
> x:=5*cos(t):
> y:=5*sin(t):
> f:=.2*sqrt((sin(x)^2*cosh(y)^2+cos(x)^2*sinh(y)^2)/((x^2+(y-3)^2)*
  (x^2+(y+3)^2)));
```

$$f := .2\sqrt{\frac{\sin(5\cos(t))^2\cosh(5\sin(t))^2+\cos(5\cos(t))^2\sinh(5\sin(t))^2}{(25\cos(t)^2+(5\sin(t)-3)^2)\,(25\cos(t)^2+(5\sin(t)+3)^2)}}$$

```
> g:=diff(f(t),t):
> fsolve(g=0,t,0..2*Pi);
```

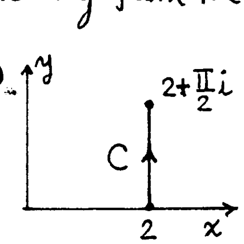
Unfortunately, maple does not give a response, so let us plot f:

```
> with(plots):
> implicitplot(z=f,t=0..2*Pi,z=0..10,grid=[200,200]);
```



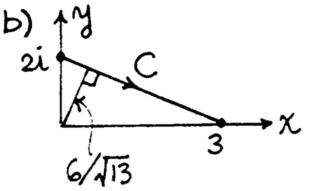
Perhaps if we go back to the fsolve command and narrow the search by changing the 0..2*Pi search interval to 1.4..1.6 (as seen from the plot), but, unfortunately, it still doesn't work. So let it suffice to observe, directly from the plot, that $\max f(\theta) \approx 0.72$

5. (a)



$$|e^z+1| = |e^2(\cos y + i\sin y)+1| = \sqrt{(e^2\cos y+1)^2+(e^2\sin y)^2}$$
$$= \sqrt{e^4+2e^2\cos y+1} \geq \sqrt{e^4+1} \text{ on } 0 \leq y \leq \pi/2$$
so $\left|\frac{1}{e^z+1}\right| \leq \frac{1}{\sqrt{e^4+1}}$. Also, $L = \pi/2$, so $\left|\int_C \frac{dz}{e^z+1}\right| \leq \frac{\pi}{2\sqrt{e^4+1}}$.

(b)

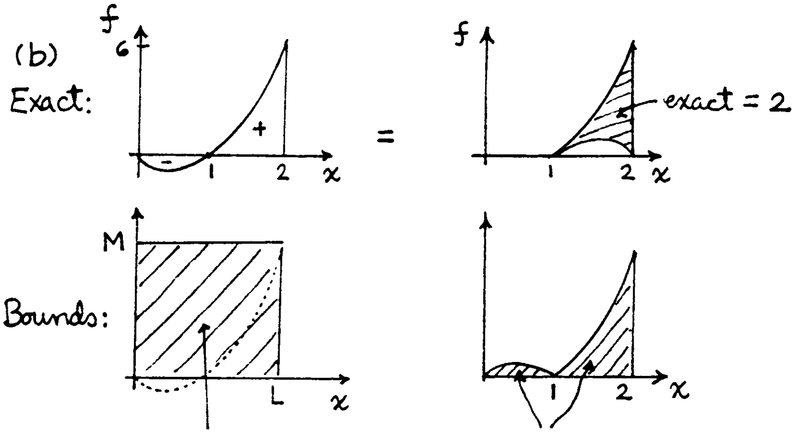


$$|\cos z| = |\cos(x+iy)| = |\cos x\cosh y - i\sin x\sinh y|$$
$$= \sqrt{(\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y)}$$
$$= \sqrt{[\cos^2 x\cosh^2 y + (1-\cos^2 x)\sinh^2 y]}$$
$$= \sqrt{[\cos^2 x\underbrace{(\cosh^2 y - \sinh^2 y)}+\sinh^2 y]}$$
$$= \sqrt{(\cos^2 x+\sinh^2 y)} \leq \sqrt{1+\sinh^2 2} = \sqrt{\cosh^2 2} = \cosh 2$$

and $L = \sqrt{13}$, so
$$\left|\int_C \frac{\cos z}{z}\,dz\right| \leq \frac{\max|\cos z|}{\min|z|}\sqrt{13} \leq \frac{\cosh 2}{6/\sqrt{13}}\sqrt{13} = \frac{13}{6}\cosh 2.$$

6. A counterexample will suffice. For example, if C is a closed curve of length L then $\oint_C dz = 0$. Thus, with $m=1$, (6.1) gives $0 \geq L$, which is false.

7. (b)



$ML = (6)(2) = 12$

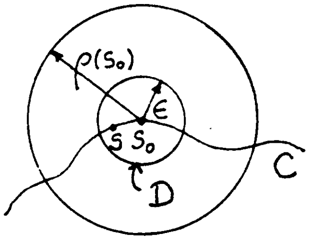
$\int_C |f(z)||dz| = \int_0^1 (x-x^3)dx + \int_1^2 (x^3-x)dx$
$= 2.5$, which is sharper than the ML bound.

Section 23.3

1. Cauchy's theorem says, essentially, that if $f(z)$ is analytic inside C, then $\oint_C f(z)dz = 0$; it does not say that if $f(z)$ is not analytic inside C then $\oint_C f(z)dz \neq 0$. That is, the theorem does not contain a converse.

2. Cauchy's theorem, 23.3.1, calls for D to be simply connected, but the D in this exercise is not.

3. If f is analytic on C, then at each point on C there is a disk of radius $\rho(s)$ throughout which f is analytic, where s is arclength from some initial point on C to that point. We wish to show that $\rho(s)$ is continuous. If $|s-s_0|<\epsilon$, then s must fall in the disk D and, clearly, $\rho(s)$ is at least $\rho(s_0)-\epsilon$; i.e., $\rho(s)\geq\rho(s_0)-\epsilon$, or,

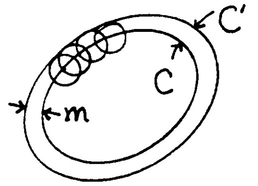


$$\rho(s_0)-\rho(s)\leq\epsilon.$$

Using the same argument, with s_0 and s switched, gives

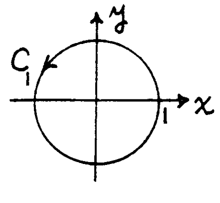
$$\rho(s)-\rho(s_0)\leq\epsilon,$$

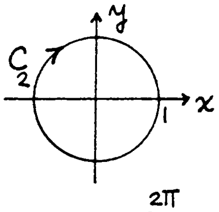
so $|\rho(s)-\rho(s_0)|\leq\epsilon$ for all s's such that $|s-s_0|<\delta(\epsilon)=\epsilon$, for $\epsilon>0$ arbitrarily small. Thus, $\rho(s)$ is a continuous function of s. Since C is rectifiable, $0\leq s\leq L<\infty$. It is known from the Calculus that if $\rho(s)$ is continuous on a closed interval then it has an absolute minimum (and maximum) on the interval. That minimum cannot be zero because then f would not be analytic at that point, as assumed. Let the minimum $\rho(s)$ be m. Then f is analytic in the domain D bounded by C′ and hence $\oint_C f(z)\,dz=0$ by Cauchy's theorem.

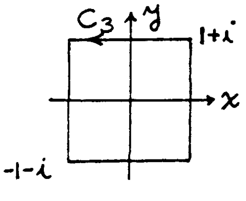


Actually, we should have begun by distinguishing two cases: (i) If $f(z)$ is analytic for all z then $\rho(s)=\infty$ is not continuous – but no matter since if $f(z)$ is analytic for all z then Cauchy's theorem gives $\oint_C f(z)\,dz=0$. (ii) If $f(z)$ is not analytic for all z, then $\rho(s)<\infty$ for all s and the above argument applies.

4.







(a) $\oint_{C_1}\operatorname{Re}z\,dz=\oint_C x(dx+i\,dy)=\int_0^{2\pi}\cos\theta(-\sin\theta+i\cos\theta)\,d\theta=0+\pi i=\pi i$

(b) $\oint_{C_1}\operatorname{Im}z\,dz=\oint_{C_1}y(dx+i\,dy)=\int_0^{2\pi}\sin\theta(-\sin\theta+i\cos\theta)\,d\theta=-\pi+0i=-\pi$

(c) $\oint_{C_3}\operatorname{Im}z\,dz=\oint_{C_3}y\,dx+iy\,dy=\oint_{C_3}y\,dx+i\cancel{\frac{y^2}{2}}\Big|_0^0=\int_1^{-1}1\,dx+0+\int_{-1}^{1}-1\,dx+0=-4$

(d) $\oint_{C_3}\frac{dz}{z^2-3}=0$ by Cauchy's theorem since the singular points, $\pm\sqrt{3}$, are

(e) $\oint_{C_1} dz/z^4 = 0$ according to the "important little integral" result in Example 2.

(f) $\oint_{C_1} \frac{dz}{z(z-2)} = -\frac{1}{2}\oint_{C_1} \frac{dz}{z} + \frac{1}{2}\oint_{C_2} \frac{dz}{z-2} = -\frac{1}{2}\underset{\uparrow \text{by "important little integral"}}{(2\pi i)} + \overset{\text{by Cauchy's theorem}}{\overset{\downarrow}{0}} = -\pi i$

(g) $\oint_{C_2} \frac{dz}{z(z+5)} = \frac{1}{5}\oint_{C_2} \frac{dz}{z} - \frac{1}{5}\oint_{C_2} \frac{dz}{z+5} = \frac{1}{5}\overset{\downarrow}{(-2\pi i)} - \frac{1}{5}\overset{\text{Cauchy theorem}}{\overset{\downarrow}{(0)}} = -2\pi i/5$

(h) $\oint_{C_1} e^{\sin z}\, dz = 0$ by Cauchy's theorem since $e^{\sin z}$ is analytic everywhere.

(i) $\oint_{C_2} \sin(\cos z)\, dz = 0$ by Cauchy's theorem since $\sin(\cos z)$ is analytic everywhere.

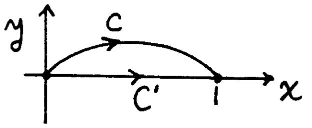
(j) $\oint_{C_3} \frac{dz}{|z|} = \oint_{C_3} \frac{dx+idy}{\sqrt{x^2+y^2}} = \int_{-1}^{1} \cancel{\frac{i\,dy}{\sqrt{1+y^2}}} + \int_{1}^{-1} \cancel{\frac{dx}{\sqrt{x^2+1}}} + \int_{1}^{-1} \cancel{\frac{i\,dy}{\sqrt{1+y^2}}} + \int_{-1}^{1} \cancel{\frac{dx}{\sqrt{x^2+1}}} = 0$, <u>not</u> by Cauchy's theorem (which does not apply), but by "chance" canellations.

(k) $\oint_{C_1} \bar{z}\, dz = \int_0^{2\pi} e^{-i\theta}(ie^{i\theta}\, d\theta) = 2\pi i$

(l) $\oint_{C_3} \bar{z}\, dz = \oint_{C_3} (x-iy)(dx+idy) = \int_{-1}^{1}(1-iy)\,i\,dy + \int_{1}^{-1}(x-i)\,dx + \int_{1}^{-1}(-1-iy)\,i\,dy + \int_{-1}^{1}(x+i)\,dx$
$= 8i$

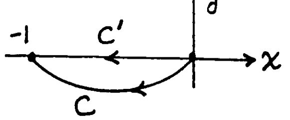
5. No, the conditions of Theorem 23.3.2 are not met because $\bar{z} = x-iy$ is not analytic (<u>anywhere</u>, in fact). In fact, Exercises 4(k) and 4(l), above, show that the results are different for the two paths.

6. (a) z^{20} is analytic everywhere, so we can deform the path to a straight line on the x-axis:

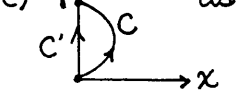


$$\int_C z^{20}\, dz = \int_{C'} z^{20}\, dz = \int_0^1 x^{20}\, dx = 1/21$$

(b) as in (a), $\int_C z^{20}\, dz = \int_0^{-1} x^{20}\, dx = -1/21$

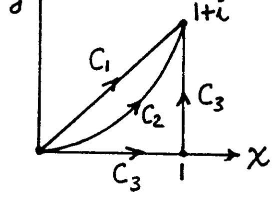


(c)



as in (a), $\int_C z^{20}\, dz = \int_{C'} z^{20}\, dz$
$= \int_0^1 (iy)^{20}\, i\,dy = i/21$

7.



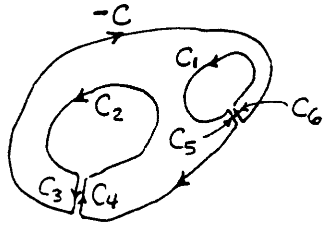
$\int_{C_2} \bar{z}\, dz = \int_{C_2} (x-iy)(dx+idy) = \int_0^1 (x-ix^2)(1+2xi)\, dx = 1 + \frac{i}{3}$,

$\int_{C_1} \bar{z}\, dz = \int_{C_1} (x-iy)(dx+idy) = \int_0^1 (1-i)(1+i)\, x\, dx = 1$,

$\int_{C_3} \bar{z}\, dz = \int_{C_3} (x-iy)(dx+idy) = \int_0^1 x\, dx + \int_0^1 (1-iy)\, i\, dy = 1+i.$

No violation (of course; the theorem is true and cannot be contradicted).

8\. Introduce slits so that the slit domain D' is simply connected (i.e., has no holes): The slit-contour integrals C_3 and C_4 cancel, as do C_5, C_6, so

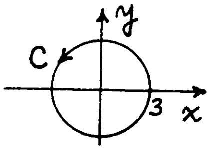


$$\oint_{C_1+C_5+(-C)+C_4+C_2+C_3+C_6} f dz = 0 \text{ by Cauchy's theorem}$$

or, $\oint_{C_1} f dz + \cancel{\int_{C_5} f dz} + \oint_{-C} f dz + \cancel{\int_{C_4} f dz} + \oint_{C_2} f dz + \cancel{\int_{C_3} f dz} + \cancel{\int_{C_6} f dz} = 0$

or, $\oint_{C_1} f dz - \oint_C f dz + \oint_{C_2} f dz = 0$, so $\oint_C f dz = \oint_{C_1} f dz + \oint_{C_2} f dz$

9\.



Let us use partial fractions in each case.

(a) $\oint_C \frac{dz}{z(z-1)} = -\oint_C \frac{dz}{z} + \oint_C \frac{dz}{z-1} = -2\pi i + 2\pi i$ per (16)
$= 0$

(b) $\oint_C \frac{dz}{z(z-5)} = -\frac{1}{5}\oint_C \frac{dz}{z} + \frac{1}{5}\oint \frac{dz}{z-5} = -\frac{1}{5}(2\pi i) + \frac{1}{5}(2\pi i)$ per (16)
$= 0$

(c) $\oint_C \frac{z dz}{z^2+1} = \frac{1}{2}\oint_C \frac{dz}{z+i} + \frac{1}{2}\oint_C \frac{dz}{z-i} = \frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i)$ per (16)
$= 2\pi i$

(d) $\oint_C \frac{z dz}{z^2-3z+2} = 2\oint_C \frac{dz}{z-2} - \oint_C \frac{dz}{z-1} = 2(2\pi i) - 1(2\pi i)$ per (16)
$= 2\pi i$

(e) $\oint_C \frac{dz}{z^3(z^2-1)} = \oint_C \left(\frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1} + \frac{E}{z-1}\right) dz$

$= -\oint_C \frac{dz}{z} + 0\oint_C \frac{dz}{z^2} - \oint_C \frac{dz}{z^3} + \frac{1}{2}\oint_C \frac{dz}{z+1} + \frac{1}{2}\oint \frac{dz}{z-1}$

$= -2\pi i + 0 - 0 + \frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i)$ per (16)

$= 0$

(f)
$$\begin{array}{r} z^2+z+1 \\ z-1\,\overline{)\,z^3-1} \\ \underline{z^3-z^2} \\ z^2-1 \\ \underline{z^2-z} \\ z-1 \\ z-1 \end{array}$$

and $z^2+z+1=0$ gives $z = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \equiv z_\pm$ for short.

(or, we could get the 3 roots of $z^3-1=0$ as the 3 cube roots of 1)

Then, $\frac{z}{z^3-1} = \frac{z}{(z-1)(z-z_+)(z-z_-)} = \frac{A}{z-1} + \frac{B}{z-z_+} + \frac{C}{z-z_-}$ gives $A = \frac{1}{3}$,

$B = -(1+\sqrt{3}i)/6$, $C = -(1-\sqrt{3}i)/6$

Now, the roots $1, z_+, z_-$ are on the unit circle so all lie inside C. Thus,

$$\oint_C \frac{z\,dz}{z^3-1} = A\oint_C \frac{dz}{z-1} + B\oint_C \frac{dz}{z-z_+} + C\oint_C \frac{dz}{z-z_-}$$

$$= A(2\pi i) + B(2\pi i) + C(2\pi i) \quad \text{per (16)}$$

$$= \left(\frac{1}{3} - \frac{1+\sqrt{3}i}{6} - \frac{1-\sqrt{3}i}{6}\right)2\pi i = 0.$$

Section 23.4

1. If $F_1'(z) = f(z)$ and $F_2'(z) = f(z)$ then, with $G(z) \equiv F_1(z) - F_2(z)$, $G'(z) = f(z) - f(z) = 0$. If $G(z) = u + iv$ and $G'(z) = u_x + iv_x = v_y - iu_y = 0$ gives $u_x = v_x = u_y = v_y = 0$ so $u(x,y) = $ constant and $v(x,y) = $ constant. Thus, $F_1(z)$ and $F_2(z)$ differ by at most a constant (i.e., a complex constant).

2. (a) $z^2/2, \quad z^2/2 + 6, \quad z^2/2 + 1 - 4i$

   (b) $z^6/6, \quad z^6/6 - 14i, \quad z^6/6 + 5.73$

   (c) $(e^{2z} - z^2)/2, \quad (e^{2z} - z^2)/2 - 14.3 - 2i, \quad (e^{2z} - z^2)/2 + 10^5$

   (d) $\sin(z-2), \quad \sin(z-2) - 2 + i, \quad \sin(z-2) - 4.13 + 6.75i$

3. (a) $\int_0^i z\,dz = z^2/2\Big|_0^i = -1/2$

   (d) $\int_i^0 \cos 3z\,dz = \frac{\sin 3z}{3}\Big|_i^0 = 0 - \frac{\sin 3i}{3} = -\frac{i}{3}\sinh 3$

   (e) $\int_{1-i}^{1+i} ze^z\,dz = (z-1)e^z\Big|_{1-i}^{1+i} = ie^{1+i} + ie^{1-i} = ie(e^i + e^{-i}) = 2ie\cos 1$

   (g) $\int_0^{3i} ze^{z^2}\,dz = \frac{1}{2}\int_0^{3i} e^{z^2}(2z\,dz) = \frac{1}{2}e^{z^2}\Big|_0^{3i} = \frac{1}{2}(e^{-9} - 1)$

4. $$I = \int_{1-i}^{1+i} \frac{dz}{z(z-1)} = \int_{1-i}^{1+i} \frac{dz}{z-1} - \int_{1-i}^{1+i} \frac{dz}{z} = \left(\log(z-1) - \log z\right)\Big|_{1-i}^{1+i}$$

   $$= \log i - \log(1+i) - \log(-i) + \log(1-i)$$

   $$= i\left(\frac{\pi}{2} + 2m\pi\right) - \left[\cancel{\ln\sqrt{2}} + i\left(\frac{\pi}{4} + 2n\pi\right)\right] - i\left(-\frac{\pi}{2} + 2p\pi\right) + \left[\cancel{\ln\sqrt{2}} + i\left(-\frac{\pi}{4} + 2q\pi\right)\right]$$

   $$= i\left(\frac{\pi}{2} + 2r\pi\right) \text{ for } r = 0, \pm 1, \pm 2, \ldots$$

5. Let $\tan^{-1} z \equiv t$. Then $z = \tan t = \frac{\sin t}{\cos t} = \frac{1}{i}\frac{e^{it} - e^{-it}}{e^{it} + e^{-it}}$ so

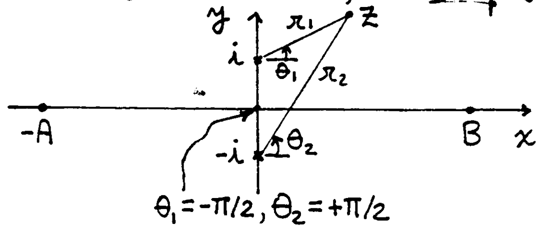
   $iz(e^{it} + e^{-it}) = e^{it} - e^{-it}$, $\quad iz(\varphi^2 + 1) = \varphi^2 - 1$ where $\varphi = e^{it}$, $\quad \varphi^2 = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$,

   $e^{i2t} = \frac{i-z}{i+z}$, $\quad i2t = \log\left(\frac{i-z}{i+z}\right)$, $\quad t = \tan^{-1} z = \frac{1}{2i}\log\frac{i-z}{i+z}$, as in Exercise 14(b) of Sec 21.4.

Thus, $I = \int_C \frac{dz}{z^2+1} = \lim_{\substack{A\to\infty \\ B\to\infty}} \tan^{-1} z \Big|_{-A}^{B}$

$= \lim_{\substack{A\to\infty \\ B\to\infty}} \frac{1}{2i} \log\left(\frac{i-z}{i+z}\right)\Big|_{-A}^{B}$

Use these branch cuts, for example:



$$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\log\left(\frac{z-i}{z+i}\right) + \log(-1)\right]\Big|_{-A}^{B}$$

$$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\log(z-i) - \log(z+i) + \log(-1)\right]\Big|_{-A}^{B}$$

$$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2 + \log(-1)\right]\Big|_{-A}^{B}$$

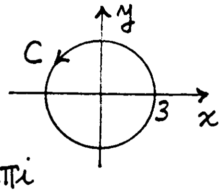
$$= \frac{1}{2i} \lim_{\substack{A\to\infty \\ B\to\infty}} \left[\ln \frac{r_1}{r_2} + i(\theta_1 - \theta_2) + \log(-1)\right]\Big|_{-A}^{B}$$

As $A, B \to \infty$, $r_1/r_2 \to 1$. At B, $\theta_1 - \theta_2 \to (0-0) = 0$. At A, $\theta_1 - \theta_2 \to (-\pi - \pi) = -2\pi$, and the constant $\log(-1)$ term cancels between the two limits, so

$$I = \frac{1}{2i}\Big((\overset{0}{\cancel{\ln 1}} + i0 + \cancel{\log(-1)}) - (\overset{0}{\cancel{\ln 1}} - i2\pi + \cancel{\log(-1)})\Big) = \frac{2\pi i}{2i} = \pi. \ ✓$$

Section 23.5

1. In each case C is

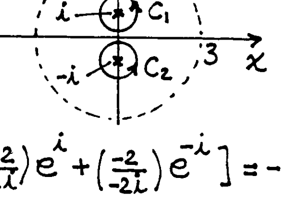


(a) $I = 2\pi i \cos z\big|_{z=0} = 2\pi i$

(b) $I = 2\pi i \sin z\big|_{z=0} = 0.$

(c) $I = \oint_C \overbrace{\left(\frac{1}{z-5}\right)}^{\text{analytic within } C} \frac{dz}{z} = 2\pi i \left(\frac{1}{z-5}\right)\Big|_{z=0} = -2\pi i/5$

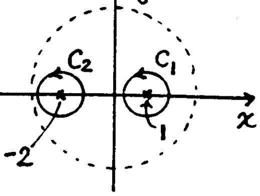
(d) $I = \oint_{C_1} \underbrace{\left(\frac{z^2-1}{z+i} e^z\right)}_{\text{analytic in } C_1} \frac{dz}{z-i} + \oint_{C_2} \underbrace{\left(\frac{z^2-1}{z-i} e^z\right)}_{\text{analytic in } C_2} \frac{dz}{z+i}$



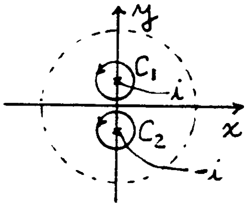
$$= 2\pi i \left(\frac{z^2-1}{z+i} e^z\right)\Big|_{z=i} + 2\pi i \left(\frac{z^2-1}{z-i} e^z\right)\Big|_{z=-i} = 2\pi i \left[\left(\frac{-2}{2i}\right)e^{i} + \left(\frac{-2}{-2i}\right)e^{-i}\right] = -4\pi i \sin 1$$

(e) $I = \oint_{C_1} \left(\frac{z+1}{(z+2)^3}\right) \frac{dz}{z-1} + \oint_{C_2} \left(\frac{z+1}{z-1}\right) \frac{dz}{(z+2)^3}$

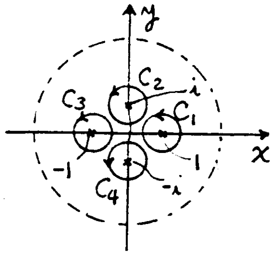
$$= 2\pi i \frac{z+1}{(z+2)^3}\Big|_{z=1} + \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left(\frac{z+1}{z-1}\right)\Big|_{z=-2} = 2\pi i\left(\frac{2}{27} - \frac{2}{27}\right) = 0$$



(f) $$I = \oint_C \frac{e^{2z}}{z^5}\,dz = \frac{2\pi i}{4!}\frac{d^4}{dz^4}(e^{2z})\Big|_{z=0} = 4\pi i/3$$



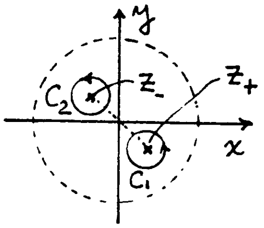
(g) $$I = \oint_{C_1} \frac{\sinh 3z}{(z+i)^2}\frac{dz}{(z-i)^2} + \oint_{C_2}\frac{\sinh 3z}{(z-i)^2}\frac{dz}{(z+i)^2}$$
$$= 2\pi i \frac{d}{dz}\left(\frac{\sinh 3z}{(z+i)^2}\right)\Big|_{z=i} + 2\pi i\frac{d}{dz}\left(\frac{\sinh 3z}{(z-i)^2}\right)\Big|_{z=-i}$$
$$= 2\pi i(-3\cos 3+\sin 3)/4 + 2\pi i(-3\cos 3+\sin 3)/4 = \pi i(\sin 3 - 3\cos 3)$$



(h) $z^4-1=0$ has the roots ± 1 and $\pm i$, so
$$I = \oint_{C_1}\frac{(z+2)(z-1)}{z^4-1}\frac{dz}{z-1} + \oint_{C_2}\frac{(z+2)(z+1)}{z^4-1}\frac{dz}{z+1}$$
$$+\oint_{C_3}\frac{(z+2)(z-i)}{z^4-1}\frac{dz}{z-i} + \oint_{C_4}\frac{(z+2)(z+i)}{z^4-1}\frac{dz}{z+i}$$
$$= 2\pi i\left[\frac{(3)(1)}{4} + \frac{(1)(1)}{-4} + \frac{(i+2)(1)}{4i^3} + \frac{(-i+2)(1)}{4(-i)^3}\right] = 0$$

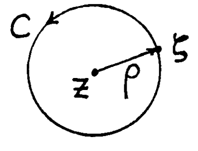
(i) $\cos(z/2)=0$ at $z=\pm\pi, \pm 3\pi, \dots$, which are outside of C, so
$$I = \oint_C \frac{e^{z^2}}{\cos(z/2)}\frac{dz}{z} = 2\pi i\,\frac{e^{z^2}}{\cos(z/2)}\Big|_{z=0} = 2\pi i$$

(j) $I=0$



(k) $z^2+i=0$ at $z=\sqrt{-i} = \pm\left(\frac{1-i}{\sqrt{2}}\right) \equiv z_\pm$ so
$$I = \oint_{C_1}\left(\frac{z^3}{z-z_-}\right)\frac{dz}{z-z_+} + \oint_{C_2}\left(\frac{z^3}{z-z_+}\right)\frac{dz}{z-z_-}$$
$$= 2\pi i\,\frac{z_+^3}{z_+-z_-} + 2\pi i\,\frac{z_-^3}{z_--z_+} = 2\pi i\left(\frac{z_+^3}{2z_+}+\frac{z_-^3}{2z_-}\right) = \pi i(z_+^2+z_-^2) = \pi i(-i-i) = 2\pi$$

2. If $n=0,1,2,\dots$ then $(z-a)^n$ is analytic for all z so $I=\oint_C (z-a)^n\,dz = 0$. If $n=-1$, Cauchy's integral formula gives
$$I = \oint_C \frac{dz}{z-a} = 2\pi i(1) = 2\pi i,$$
and if $n=-2,-3,\dots$ then the generalized Cauchy integral formula gives
$$I = \oint_C\frac{dz}{(z-a)^m} = \frac{2\pi i}{(m-1)!}\underbrace{\frac{d^{(m-1)}}{dz^{(m-1)}}(1)\Big|_{z=a}}_{0} = 0,$$ where m is $-n$.



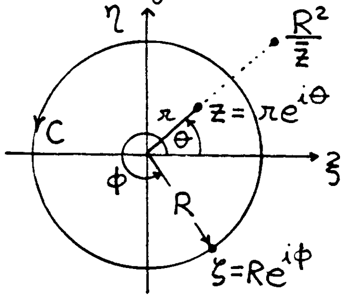
3. (a) (22) says $\displaystyle\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta = \frac{2\pi i}{n!}f^{(n)}(z)$,

so "ML" bound gives
$$\left|\frac{2\pi i}{n!}f^{(n)}(z)\right| = \left|\oint_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\right| \le \frac{M}{\rho^{n+1}}2\pi\rho, \text{ or, } |f^{(n)}(z)| \le \frac{n!M}{\rho^n}.$$

(b) Let $n=0$. Then (3.1) is $|f(z)| \le M$, which gives no information.
Let $n=1$. Then (3.1) is $|f'(z)| \le M/\rho$. Since ρ is arbitrarily large, it follows that $|f'(z)|$ is arbitrarily small. Thus $f'(z)=0$ — for each z, so $f(z)=$ constant.

(c) On imaginary axis $\sin z = \sin iy = i \sinh y$ is unbounded.

(d) Suppose $P(z)$ is nonzero everywhere. Then surely $f(z) = 1/P(z)$ is analytic for all z and is therefore at most a constant. But $1/P(z)$ is not a constant (unless $n=0$ of course), so $P(z)$ must not be nonzero everywhere.



4. (a) The term $-\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - R^2/\bar{z}}\,d\zeta$ can be inserted in (4.1), to give (4.2), because it is 0 by Cauchy's theorem since $f(\zeta)$ is analytic inside and on C and $R^2/\bar{z} = (R^2/r)e^{i\theta}$ lies outside of C since $R^2/r = (R/r)R > R$. Then (4.2) gives

$$\boxed{d\zeta} = d(Re^{i\phi}) = iRe^{i\phi}\,d\phi = i\zeta\,d\phi$$

$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}}\right) f(\zeta)\boxed{d\zeta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} - \boxed{\frac{\bar{z}\zeta}{\zeta\bar{z} - R^2}}\right) f(\zeta)\,d\phi$$

$$\boxed{\frac{\bar{z}\zeta}{\zeta\bar{z} - R^2}} = \frac{\bar{z}}{\bar{z} - \frac{R^2}{Re^{i\phi}}} = \frac{\bar{z}}{\bar{z} - Re^{-i\phi}} = \frac{\bar{z}}{\bar{z} - \bar{\zeta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\boxed{\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right)} f(\zeta)\,d\phi$$

$$\boxed{\left(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right)} = \frac{\zeta\bar{\zeta} - \cancel{\zeta\bar{z}} + \cancel{\bar{z}\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} = \frac{R^2 - r^2}{|\zeta - z|^2}$$

$$= \frac{R^2 - r^2}{\zeta\bar{\zeta} - \zeta\bar{z} - \bar{\zeta}z + z\bar{z}} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

and equating real parts gives (4.4).

(b) The term $\oint_C \frac{f(\zeta)\,d\zeta}{\zeta - \bar{z}}$ can be inserted in (4.1), to give (4.5), because it is 0 by Cauchy's theorem since $f(\zeta)$ is analytic inside and on C and $\bar{z}$ lies outside of C. Next,

$$\left|\int_{\curvearrowleft} \frac{f(\zeta)}{\zeta - z}\,d\zeta\right| \le \frac{M\pi R}{R - r} \sim \pi M \to 0 \text{ as } R \to \infty.$$

Similarly,

$$\left|\int_{\curvearrowleft} \frac{f(\zeta)}{\zeta - \bar{z}}\,d\zeta\right| \le \frac{M\pi R}{R - r} \sim \pi M \to 0 \text{ as } R \to \infty.$$

Thus, letting $R \to \infty$ in (4.5) does give (4.6), namely,

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{\xi - z} - \frac{1}{\xi - \bar{z}}\right) f(\xi)\,d\xi = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{z-\bar{z}}{(\xi - x - iy)(\xi - x + iy)}\, f(\xi)\,d\xi$$

or, $u(x,y) + i v(x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i2y}{(\xi - x)^2 + y^2}\left[u(\xi,0) + i v(\xi,0)\right] d\xi$

and equating real parts gives (4.7).

CHAPTER 24

Section 24.2

1. First, suppose Σa_n and Σb_n both converge, say to A and B, respectively. Then, to each $\epsilon > 0$ there correspond an N_1 and N_2 such that

$$|\sum_1^N a_n - A| < \epsilon/2 \quad \text{for all } N > N_1,$$
$$|\sum_1^N b_n - B| < \epsilon/2 \quad \text{" " } N > N_2.$$

Let $N_0 = \max\{N_1, N_2\}$. Then

$$\begin{aligned} |\sum_1^N (a_n + ib_n) - (A+iB)| &= |(\sum_1^N a_n - A) + i(\sum_1^N b_n - B)| \\ &\leq |\sum_1^N a_n - A| + |\sum_1^N b_n - B| \qquad \text{since if } z = a+ib, |z| \leq |a|+|b| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

for all $N > N_0$, so $\sum_1^\infty c_n$ converges to $A+iB$.
Second, suppose Σc_n converges, say to $A+iB$. Then, to each $\epsilon > 0$ there corresponds an M such that

$$|\sum_1^N (a_n + ib_n) - (A+iB)| < \epsilon \quad \text{for all } N > M$$

or, $|(\sum_1^N a_n - A) + i(\sum_1^N b_n - B)| < \epsilon$ for all $N > M$. Surely it follows from the latter that

$$|\sum_1^N a_n - A| < \epsilon \quad \text{for all } N > M$$
$$\text{and} \quad |\sum_1^N b_n - B| < \epsilon \quad \text{for all } N > M,$$

so Σa_n converges to A and Σb_n converges to B, which completes the proof.

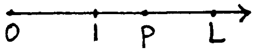
2. The triangle inequality states that $|z_1 + z_2| \leq |z_1| + |z_2|$. It follows that

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \leq |z_1| + |z_2 + z_3| \leq |z_1| + |z_2| + |z_3|.$$

Similarly, $|z_1 + z_2 + z_3 + z_4| = |z_1 + (z_2 + z_3 + z_4)| \leq |z_1| + |z_2 + z_3 + z_4|$
$\leq |z_1| + |z_2| + |z_3| + |z_4|$,

and so on.

3. Choose any number p such that $1 < p < L$.



Then by the definition of the convergence of $|c_{n+1}/c_n|$ to L, it follows that given p there must exist an N such that $|c_{n+1}/c_n| > p$ for all $n > N$. Hence, $|c_{n+1}| > |c_n|$ for all $n > N$. However, Theorem 24.2.2 says that $c_n \to 0$ as $n \to \infty$ is necessary for convergence, so Σc_n must be divergent.

4. Applying the ratio test (Thm 24.2.4) to $\sum_0^\infty a_n (z-a)^n$,

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) |z-a| = L|z-a| \begin{array}{l} < 1 \text{ for convergence,} \\ > 1 \text{ for divergence.} \end{array}$$

so (for $L \neq 0, \infty$) $|z-a| < 1/L$ gives convergence and $|z-a| > 1/L$ gives divergence. If $L = 0$ then

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|z-a| = 0\,|z-a| \text{ is } < 1 \text{ for all } z, \text{ so}$$

we have convergence for all z; i.e., for $|z-a| < \infty$. If $L = \infty$ then the latter gives $\infty|z-a|$ which is >1 for all $z \neq a$; hence we have divergence for all $z \neq a$. At $z = a$ we have convergence, of course, because the series is $a_0 + 0 + 0 + 0 + \cdots$ which converges to a_0.

5. (a) $\lim\limits_{n\to\infty}\left|\frac{(n+1)/(2+i)^{n+1}}{n/(2+i)^n}\right| = \lim\limits_{n\to\infty}\left|\frac{n+1}{n}\frac{1}{2+i}\right| = \frac{1}{\sqrt{5}} < 1$, hence convergent by the ratio test.

(b) $\lim\limits_{n\to\infty}\frac{(n+1)^{50}/3^{n+1}}{n^5/3^n} = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{50}\frac{1}{3} = \frac{1}{3} < 1$, hence conv. by ratio test.

(c) Let $M_n = 1/2^n$. Then $|c_n| \leq M_n$ for each $n \geq 2$. Since $\sum\limits^{\infty} M_n = \sum\limits^{\infty} (1/2)^n$ is a convergent geometric series (conv. because $1/2 < 1$), $\sum c_n$ converges by the comparison test.

(d) $c_n \to 1$ as $n \to \infty$; hence divergent by Theorem 24.2.2.

(e) $\lim\limits_{n\to\infty}\left|\frac{(1+3i)^{n+1}/(n+1)^{100}}{(1+3i)^n/n^{100}}\right| = \lim\limits_{n\to\infty}\left(\frac{\sqrt{10}^{\,n+1}}{\sqrt{10}^{\,n}}\left(\frac{n}{n+1}\right)^{100}\right) = \sqrt{10} > 1$, so div. by ratio test.

(f) $\lim\limits_{n\to\infty}\left|\frac{(n+1)^4 e^{-(5-i)(n+1)}}{n^4 e^{-(5-i)n}}\right| = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^4|e^{-5+i}| = |e^{-5}e^{i}| = e^{-5}|e^{i}| = e^{-5} < 1$, so conv. by ratio test.

(g) $|e^{-in}| = 1$ for all n so $c_n = e^{-in}$ does not $\to 0$ as $n \to \infty$. Hence, div. by Theorem 24.2.2.

(h) $|c_n| = \left|\sin\left(\frac{1+i}{2-i}\right)^n\right| \leq \left|\left(\frac{1+i}{2-i}\right)^n\right| = \sqrt{\frac{2}{5}}^{\,n}$. Since $\sum\limits^{\infty} M_n = \sum\limits^{\infty}\left(\sqrt{\frac{2}{5}}\right)^n$ is a conv. geometric series, it follows from the comparison test that $\sum c_n$ is convergent.

6. (a) $\sum\limits^{\infty} z^{2n} = \sum\limits^{\infty} (z^2)^n$ is a geometric series, which conv. if $|z^2| < 1$ (i.e., if $|z| < 1$) and div. if $|z^2| > 1$ (i.e., if $|z| > 1$).

(b) Use Thm. 24.2.5. $L = \lim|(n+1)^2/n^2| = 1$, so conv. in $|z-3| < 1$ and div. in $|z-3| > 1$.

(c) Again use Thm. 24.2.5. $L = \lim (n+1)!/n! = \lim(n+1) = \infty$ so conv. only at $z = -5$.

(d) $L = \lim(e^{n+1}/e^n) = e$ so conv. in $|z+i| < 1/e$, div. in $|z+i| > 1/e$.

(e) $L = \lim(e^{-(n+1)}/e^{-n}) = e^{-1}$ so conv. in $|z| < e$, div. in $|z| > e$.

(f) $L = \lim \left(\frac{n+1}{n}\right)^{100} \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$ so conv. for all z

(g) $|e^{in}| = 1$, so $L = 1$, so conv. in $|z| < 1$, div. in $|z| > 1$

(h) $\lim \frac{a_{n+1}}{a_n} = \lim \left(\frac{\cos(n+1)}{\cos n} \frac{n^2+1}{(n+1)^2+1} \right) = \lim \frac{\cos(n+1)}{\cos n}$ does not exist, so the ratio test (Thm 24.2.5) does not apply. We can, at least, say that $|c_n| = \left| \frac{\cos n}{n^2+1} z^n \right| < |z|^n < r^n$ inside the disk $|z| < r$.

Now, if $r < 1$ then $\sum^{\infty} r^n$ is a convergent geometric series, so we can at least say that the given series converges in $|z| < r$ for each $r < 1$ i.e., the series converges in $|z| < 1$. (No information for $|z| \geq 1$.)

(i) It's a geometric series: conv. in $|(2-i)z| < 1$, i.e., in $|z| < 1/\sqrt{5}$, and div. in $|z| > 1/\sqrt{5}$.

(j) $L = \lim \frac{e^{(n+1)^2}}{(n+1)!} \frac{n!}{e^{n^2}} = \lim \frac{e^{2n+1}}{n+1} \overset{\text{l'Hôpital}}{=} \lim \frac{2e^{2n+1}}{1} = \infty$ so, by Theorem 24.2.5, the series converges <u>only</u> at $z = 0$.

7. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$, $f'(x) = 2e^{-1/x^2}/x^3$ for $x \neq 0$.

$$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x} = \lim_{x \to 0} \frac{1}{x(1 + \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x^4} + \cdots)} = 0$$

$$f''(0) = \lim_{x \to 0} \frac{2e^{-1/x^2}/x^3 - 0}{x} = 2 \lim_{x \to 0} \frac{e^{-1/x^2}}{x^4} = 2 \lim_{x \to 0} \frac{1}{x^4(1 - \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x^4} - \cdots)} = 0$$

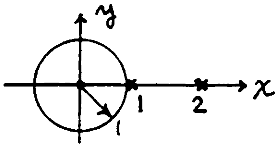
and similarly for $f'''(0), \ldots$.

8.(a) $L = \lim \left| \frac{(-1)^{n+2}(n+1)}{(-1)^{n+1} n} \right| = 1$, so the power series converges in $|z-1| < 1$ and diverges in $|z-1| > 1$. It is the Taylor series of its sum function in $|z-1| < 1$.

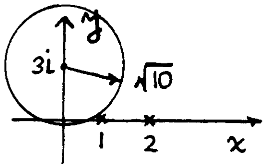
(b) $L = \lim \frac{4^{n+1}}{4^{n+2}} = \frac{1}{4}$, so the power series converges in $|z| < 1/4$ and diverges in $|z| > 1/4$. It is the Taylor series of its sum function in $|z| < 1/4$.

(c) It is a geometric series (missing the first several terms) $\sum^{\infty} \left[\left(\frac{z+i}{1+i} \right)^2 \right]^n$ so it converges in $\left| \frac{z+i}{1+i} \right|^2 < 1$, i.e., in $|z+i| < \sqrt{2}$, and diverges in $|z+i| > \sqrt{2}$. It is the Taylor series of its sum function in $|z+i| < \sqrt{2}$.

9. (a) $z^2-3z+2=0$ at $z=1,2$ so the TS about $z=0$ will converge in $|z|<1$.



(b)

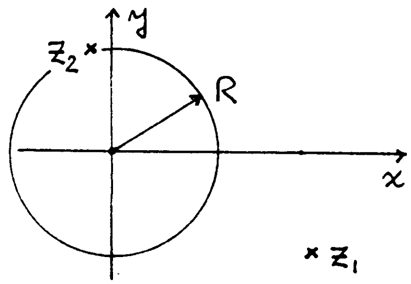


Conv. in $|z-3i|<\sqrt{10}$

(c) Conv. in $|z-(1-5i)|<5$

(d) Conv. in $|z-(5-i)|<\sqrt{10}$

10. (a) denominator $= z^2-2z+3i+1=0$ at $z=(1+\sqrt{\tfrac{3}{2}})-\sqrt{\tfrac{3}{2}}\,i \equiv z_1$ and $(1-\sqrt{\tfrac{3}{2}})+\sqrt{\tfrac{3}{2}}\,i \equiv z_2$.



The numerator does not vanish at either of these points so z_1, z_2 are indeed singular points of the given function.

$$R=|z_2|=\sqrt{(1-\sqrt{3/2})(1-\sqrt{3/2})+3/2}=\sqrt{4-\sqrt{6}}.$$

(b) $R=|10i-z_2|=\sqrt{(\sqrt{\tfrac{3}{2}}-1)^2+(10-\sqrt{\tfrac{3}{2}})^2}=\sqrt{104-22\sqrt{\tfrac{3}{2}}}$ since it is evident that z_2 is closer to $10i$ than z_1

(c) $R=|2-5i-z_1|$ since it is evident that z_1 is closer to $2-5i$ than z_1
$=\sqrt{(1-\sqrt{3/2})^2+(5-\sqrt{3/2})^2}=\sqrt{29-6\sqrt{6}}$

(d) $R=|20-z_1|=\sqrt{(19-\sqrt{3/2})^2+(\sqrt{3/2})^2}=\sqrt{364-19\sqrt{6}}$

11. (a) $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$, $R=\infty$.

(b) $\sin z = \sin a+(\cos a)(z-a)-\frac{\sin a}{2!}(z-a)^2-\frac{\cos a}{3!}(z-a)^3+\frac{\sin a}{4!}(z-a)^4+\cdots$, $R=\infty$, where $\sin a=\sin(2-i)=\sin 2\cos i-\sin i\cos 2=\sin 2\cosh 1-i\sinh 1\cos 2$ and $\cos a=\cos(2-i)=\cos 2\cos i+\sin 2\sin i=\cos 2\cosh 1+i\sin 2\sinh 1$.

(c) $\cos 2z=\cosh 6-2i(\sinh 6)(z-3i)-2(\cosh 6)(z-3i)^2+\frac{4}{3}i(\sinh 6)(z-3i)^3+\cdots$, $R=\infty$.

(d) Can reduce the labor by letting $z^6\equiv w$, say. Then, expanding e^w in powers of w will give the desired series in powers of z:

$$e^{z^6}=e^w=1+w+w^2/2!+\cdots=1+z^6+\frac{1}{2!}z^{12}+\frac{1}{3!}z^{18}+\cdots,\quad R=\infty.$$

(e) To use the known geometric series re-express as

$$\frac{1}{i+z}=\frac{1}{i(1-iz)}=-i\frac{1}{1-iz}=-i\left[1+iz+(iz)^2+(iz)^3+\cdots\right]$$
$$=-i+z+iz^2-z^3-\cdots$$ in $|iz|<1$, i.e., in $|z|<1$, so $R=1$.

(f) Get in geometric series form:

$$\frac{z^3}{2-iz} = \frac{z^3}{2}\frac{1}{1-\frac{iz}{2}} = \frac{z^3}{2}\left(1+\frac{iz}{2}+\left(\frac{iz}{2}\right)^2+\left(\frac{iz}{2}\right)^3+\cdots\right) = \frac{1}{2}z^3+\frac{i}{4}z^4-\frac{1}{8}z^5-\frac{i}{16}z^6-\cdots$$

or, in summation form, $= \sum_0^\infty \frac{i^n}{2^{n+1}} z^{n+3}$; $R=2$ since we need $|\frac{iz}{2}|<1$

(g) Let $z^8 \equiv w$, say. Then $\sin z^8 = \sin w = w - \frac{1}{3!}w^3 + \frac{1}{5!}w^5 - \cdots$

$$= z^8 - \frac{1}{3!}z^{24} + \frac{1}{5!}z^{40} - \cdots$$

or, in summation form,

$$= \sum_1^\infty (-1)^{n+1} \frac{z^{16n-8}}{(2n-1)!}, \quad R=\infty.$$

The $z^8=w$ idea was important so we don't need to waste our time working out all the in-between terms, the coefficients of which are 0.

(h) $z^3 = (-2i)^3 + 3(-2i)^2(z+2i) + \frac{6(-2i)}{2!}(z+2i)^2 + \frac{6}{3!}(z+2i)^3$

$= 8i - 12(z+2i) - 6i(z+2i)^2 + (z+2i)^3$; $R=\infty$.

The series <u>terminates</u>. NOTE: If you want a Taylor series about $-2i$, do not expand the powers on the right-hand side and simplify, which would merely give z^3 !

(i) $1/(1+2z^{35}) = 1 - 2z^{35} + 4z^{70} - 8z^{105} + \cdots$, or, $= \sum_0^\infty (-2z^{35})^n = \sum_0^\infty (-2)^n z^{35n}$;
need $|2z^{35}|<1$ or $|z| < 1/2^{1/35}$; $R = 1/2^{1/35}$.

(j) $z^2 - iz = (-4+2) + 3i(z-2i) + \frac{2}{2!}(z-2i)^2 = -2 + 3i(z-2i) + (z-2i)^2$; $R=\infty$.

12. (b) $\frac{1}{(3-z)^2} = \frac{1}{(3-i)^2\left[1-\left(\frac{z-i}{3-i}\right)\right]^2}$ so "z" is $\frac{z-i}{3-i}$ and "m" is 2

$$= \frac{1}{(3-i)^2}\sum_0^\infty \frac{(2+n-1)!}{(2-1)!\,n!}\left(\frac{z-i}{3-i}\right)^n = \sum_0^\infty \frac{(n+1)!}{n!}\frac{(z-i)^n}{(3-i)^{n+2}} = \sum_0^\infty \frac{n+1}{(3-i)^{n+2}}(z-i)^n$$

in $|\text{"}z\text{"}| = \left|\frac{z-i}{3-i}\right| = \frac{|z-i|}{\sqrt{10}} < 1$, i.e., in $|z-i| < \sqrt{10}$.

13. (a) $\frac{1}{(2z+1)^3} = \frac{1}{[1-(-2z)]^3} = \sum_0^\infty \frac{(3+n-1)!}{(3-1)!\,n!}(-2z)^n = \sum_0^\infty \frac{(n+2)!}{2n!}(-2z)^n$

$$= \frac{1}{2}\sum_0^\infty (-1)^n (n+2)(n+1)2^n z^n$$

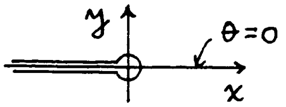
in $|-2z| = 2|z| < 1$, i.e., in $|z| < 1/2$.

(b) $\frac{1}{(2z+1)^3} = \frac{1}{[2(z-2)+5]^3} = \frac{1}{125}\,\frac{1}{[1-(-\frac{2(z-2)}{5})]^3}$ so "z" is $-\frac{2(z-2)}{5}$, "m" is 3

$$= \frac{1}{125}\sum_0^\infty \frac{(3+n-1)!}{2!\,n!}\left[-\tfrac{2}{5}(z-2)\right]^n = \frac{1}{250}\sum_0^\infty (-1)^n(n+1)(n+2)\left(\tfrac{2}{5}\right)^n(z-2)^n$$

in $\left|-\frac{2}{5}(z-2)\right| < 1$ or, $|z-2| < 5/2$.

14. (a) $f(z)=\sqrt{z}$, $f' = \frac{1}{2}z^{-1/2}$, $f'' = -\frac{1}{4}z^{-3/2}$, ...

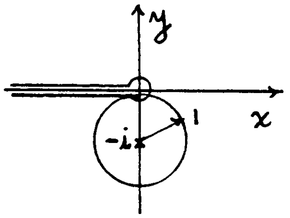


$a=1$: $f(z) = \underset{\uparrow}{\sqrt{1}} + \frac{1}{2}\underset{\uparrow}{1}^{-1/2}(z-1) - \frac{1}{4\cdot 2!}\underset{\uparrow}{1}^{-3/2}(z-1)^2 + \cdots$ and we need to evaluate these coefficients according to the branch cut chosen: $1^{1/2} = (1e^{i0})^{1/2} = 1$, $1^{-1/2} = (1e^{i0})^{-1/2} = 1$, $1^{-3/2} = (1e^{i0})^{-3/2} = 1$, so

$\sqrt{z} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots$ in $|z-1| < 1$, since if we make the circle any larger it will contain part of the branch cut so f will not be analytic throughout that disk.

(b) $a=-i$: $f(z) = (-i)^{1/2} + \frac{1}{2}(-i)^{-1/2}(z+i) - \frac{1}{4\cdot 2!}(-i)^{-3/2}(z+i)^2 - \cdots$
where

$(-i)^{1/2} = (1e^{-\pi i/2})^{1/2} = e^{-\pi i/4} = (1-i)/\sqrt{2}$,
$(-i)^{-1/2} = (\ \text{"}\ )^{-1/2} = e^{\pi i/4} = (1+i)/\sqrt{2}$,
$(-i)^{-3/2} = (\ \text{"}\ )^{-3/2} = e^{3\pi i/4} = (-1+i)/\sqrt{2}$

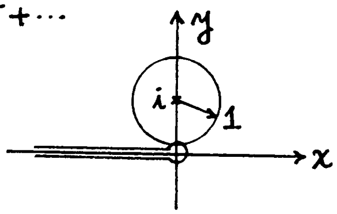


and so on. Thus,

$\sqrt{z} = \frac{1-i}{\sqrt{2}} + \frac{1}{2}\,\frac{1+i}{\sqrt{2}}(z+i) - \frac{1}{8}\,\frac{(-1+i)}{\sqrt{2}}(z+i)^2 + \cdots$ in $|z+i| < 1$.

(c) $a=i$: $f(z) = i^{1/2} + \frac{1}{2}i^{-1/2}(z-i) + (-\frac{1}{4\cdot 2!})i^{-3/2}(z-i)^2 + \cdots$
where

$(i)^{1/2} = (e^{\pi i/2})^{1/2} = e^{\pi i/4} = (1+i)/\sqrt{2}$
$(i)^{-1/2} = (\ \text{"}\ )^{-1/2} = e^{-\pi i/4} = (1-i)/\sqrt{2}$
$(i)^{-3/2} = (\ \text{"}\ )^{-3/2} = e^{-3\pi i/4} = (-1-i)/\sqrt{2}$



and so on. Thus,

$\sqrt{z} = \frac{1+i}{\sqrt{2}} + \frac{1}{2}\,\frac{1-i}{\sqrt{2}}(z-i) - \frac{1}{4\cdot 2!}\,\frac{-1-i}{\sqrt{2}}(z-i)^2 + \cdots$ in $|z-i| < 1$

15. $1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots = (a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + \cdots)$
$\times\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right)$

z^4: $1/24 = a_0/24 - a_2/2 + a_4$ gives $a_4 = \frac{1}{24} - \frac{1}{24} + \frac{1}{2} = \frac{1}{2}$
z^5: $1/120 = a_1/24 - a_3/2 + a_5$ gives $a_5 = \frac{1}{120} - \frac{1}{24} + \frac{1}{3} = \frac{3}{10}$
so $\frac{e^z}{\cos z} = 1 + z + z^2 + \frac{2}{3}z^3 + \frac{1}{2}z^4 + \frac{3}{10}z^5 + \cdots$

16.(a) $\tan z = \frac{\sin z}{\cos z} = \frac{z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots}{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots} = a_0 + a_1 z + a_2 z^2 + \cdots$

$$z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots = (1 - \tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 - \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$$

z^0: $0 = a_0$

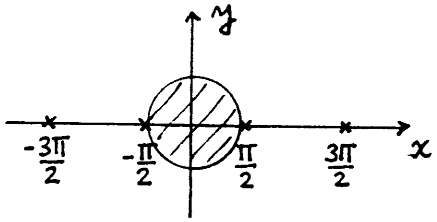
z^1: $1 = a_1$

z^2: $0 = -\frac{1}{2}a_0 + a_2 \rightarrow a_2 = 0$

z^3: $-1/6 = a_3 - \frac{1}{2}a_1 \rightarrow a_3 = 1/3$

z^4: $0 = a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 \rightarrow a_4 = 0$

z^5: $1/120 = a_5 - \frac{1}{2}a_3 + \frac{1}{24}a_1 \rightarrow \quad a_5 = 2/15$



and so on, so

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

in $|z| < \pi/2$ since $\tan z$ is singular at the zeros of $\cos z$, namely, at $\pm\pi/2, \pm 3\pi/2, \ldots$. The distance from $z = 0$ to the closest of these is $\pi/2$.

(b) $\sec z = 1/\cos z$ so $1 = (1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$

z^0: $1 = a_0$

z^1: $0 = a_1$

z^2: $0 = a_2 - \frac{1}{2}a_0 \rightarrow a_2 = 1/2$

z^3: $0 = a_3 - \frac{1}{2}a_1 \rightarrow a_3 = 0$

z^4: $0 = a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 \rightarrow a_4 = 5/24$

and so on, so

$$\sec z = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots \quad \text{in } |z| < \pi/2, \text{ as in part (a).}$$

(c) $\csc z = 1/\sin z$ does not admit a Taylor (i.e., Maclaurin) series about $z = 0$ because it is singular at $z = 0$ (since $\sin 0 = 0$).

(d) $1 + z = (1 + 2z + 3z^2)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$

z^0: $1 = a_0$

z^1: $1 = a_1 + 2a_0 \rightarrow a_1 = -1$

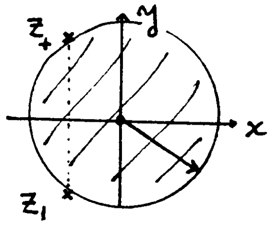
z^2: $0 = a_2 + 2a_1 + 3a_0 \rightarrow a_2 = -1$

z^3: $0 = a_3 + 2a_2 + 3a_1 \rightarrow a_3 = 5$

z^4: $0 = a_4 + 2a_3 + 3a_2 \rightarrow a_4 = -7$

and so on, so

$$\frac{1+z}{1+2z+3z^2} = 1 - z - z^2 + 5z^3 - 7z^4 + \cdots .$$



$3z^2 + 2z + 1 = 0$ gives $z = (-2 \pm \sqrt{-8})/6 = (-1 \pm i\sqrt{2})/3 \equiv z_\pm$

so convergence is in $|z| < 1/\sqrt{3}$

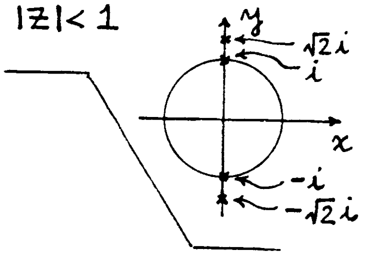
(e) Let us merely use Maple:

```
taylor((3-z)/(2+3*z^2+z^4), z=0, 8);
```

gives

$$\frac{3-z}{2+3z^2+z^4} = \frac{3}{2} - \frac{1}{2}z - \frac{9}{4}z^2 + \frac{3}{4}z^3 + \frac{21}{8}z^4 - \frac{7}{8}z^5 - \frac{45}{16}z^6 + \frac{15}{16}z^7 + \cdots$$

In what disk? $z^4 + 3z^2 + 2 = 0$ gives $z^2 = (-3 \pm \sqrt{1})/2 = -1, -2$, so $z = \pm i, \pm\sqrt{2}\,i$. Thus, the series converges in $|z| < 1$



(f) $e^z/\sin 2z$ does not admit a Taylor series about $z=0$ because it is not analytic there.

(h) $1 = (4 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$

z^0: $1 = 4a_0$

z^1: $0 = 4a_1 \rightarrow a_1 = 0$

z^2: $0 = 4a_2 - \frac{1}{2}a_0 \rightarrow a_2 = 1/32$

z^3: $0 = 4a_3 - \frac{1}{2}a_1 \rightarrow a_3 = 0$

z^4: $0 = 4a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0 \rightarrow a_4 = 1/768$

NOTE: actually, we could have omitted the $a_1 z, a_3 z^3, \ldots$ terms since $1/(3+\cos z)$ is an even function of z.

and so on, so

$$\frac{1}{3+\cos z} = \frac{1}{4} + \frac{1}{32}z^2 + \frac{1}{768}z^4 + \cdots$$

In what disk? Set $3 + \cos z = 0$. $3 + (e^{iz} + e^{-iz})/2 = 0$. Let e^{iz} be t.
$t^2 + 6t + 1 = 0$, $t = (-6 \pm \sqrt{32})/2 = -3 \pm 2\sqrt{2}$ (both negative)

so $iz = \log(-3 \pm 2\sqrt{2}) = \ln(3 \mp 2\sqrt{2}) + i(\pi + 2n\pi)$

$z = (\pi + 2n\pi) - i\ln(3 \mp 2\sqrt{2})$, $n = 0, \pm 1, \pm 2, \ldots$

of which the smallest one (i.e., the one closest to the point of expansion, which is the origin) is $z = \pi - i\ln(3+2\sqrt{2})$ (actually, $\ln(3-2\sqrt{2})$ is $= -\ln(3+2\sqrt{2})$, so either one will do), so

$$R = \sqrt{\pi^2 + [\ln(3+2\sqrt{2})]^2}.$$

Section 24.3

1. $t = (t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \cdots)(a_0 + a_2 t^2 + a_4 t^4 + a_6 t^6$

t: $1 = a_0$

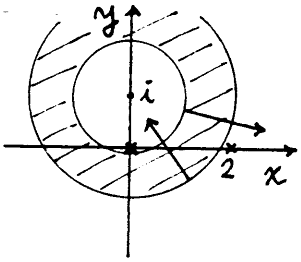
t^3: $0 = a_2 - \frac{1}{6}a_0 \rightarrow a_2 = 1/6$

t^5: $0 = a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0 \rightarrow a_4 = 7/360$

t^7: $0 = a_6 - \frac{1}{6}a_4 + \frac{1}{120}a_2 - \frac{1}{5040}a_0 \rightarrow a_6 = 31/15120$

so $t/\sin t = 1 + (1/6)/t^2 + (7/360)t^4 + (31/15120)t^6 + \cdots$

2. $f(z)=\frac{1}{z(z-2)}=-\frac{1}{2}\frac{1}{z}+\frac{1}{2}\frac{1}{z-2}$



Expand the $1/(z-2)$ in $|z-i|<\sqrt{5}$ in a TS (Taylor series) and expand the $1/z$ in $|z-i|>1$ in a LS (Laurent series), as indicated by the arrows at the right.

$$f(z)=-\frac{1}{2}\frac{1}{i+(z-i)}+\frac{1}{2}\frac{1}{-2+i+(z-i)}=-\frac{1}{2}\frac{1}{i+t}+\frac{1}{2}\frac{1}{-2+i+t}$$

$$=-\frac{1}{2t}\frac{1}{1+\frac{i}{t}}+\frac{1}{2}\frac{1}{-2+i}\frac{1}{1+\frac{t}{-2+i}}=-\frac{1}{2t}\sum_0^\infty\left(-\frac{i}{t}\right)^n-\frac{2+i}{10}\frac{1}{1-\frac{2+i}{5}t}$$

$$=-\frac{1}{2}\sum_0^\infty(-i)^n t^{-n-1}-\frac{2+i}{10}\sum_0^\infty\left(\frac{2+i}{5}\right)^n t^n$$

$$=\underbrace{\underbrace{-\frac{1}{2}\sum_0^\infty(-i)^n(z-i)^{-n-1}}_{\text{Conv. in } |z-i|>1}\;\underbrace{-\frac{1}{2}\sum_0^\infty\left(\frac{2+i}{5}\right)^{n+1}(z-i)^n}_{\text{Conv. in } |z-i|<\sqrt{5}}}_{\text{Valid in the overlap } 1<|z-i|<\sqrt{5}.}$$

3. $f(z)=\frac{1}{z(z-2)}=\frac{1}{i+(z-i)}\frac{1}{i-2+(z-i)}=\frac{1}{(z-i)^2}\frac{1}{1+\frac{i}{z-i}}\frac{1}{1+\frac{i-2}{z-i}}$ (let $z-i$ be t)

$$=\frac{1}{t^2}\sum_{n=0}^\infty\left(-\frac{i}{t}\right)^n\sum_{m=0}^\infty\left(\frac{2-i}{t}\right)^m=\frac{1}{t^2}\sum_{n=0}^\infty\sum_{m=0}^\infty(-i)^n(2-i)^m t^{-(m+n)}$$

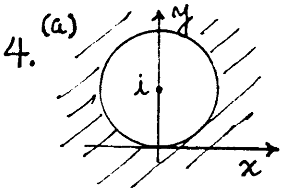
Now let $p=m+n$, $q=m$ (or n; either way is fine)
or, $n=p-q$
$m=q$
so the boundaries $m=0, n=0$ of the m,n quarter plane map into $0=p-q$ (ie, $p=q$) and $q=0$, hence the image is the wedge from $\pi/4$ to $\pi/2$, as shown in the figure in the text. It will be best, in writing the iterated sum on p and q, to sum on q first because the q limits will then be finite.

$$=\sum_{p=0}^\infty\left(\sum_{q=0}^p(-i)^{p-q}(2-i)^q\right)t^{-(p+2)}$$ or, writing out through $p=3$, say,

$$=\left((-i)^{0-0}(2-i)^0\right)t^{-(0+2)}+\left((-i)^{1-0}(2-i)^0+(-i)^{1-1}(2-i)^1\right)t^{-(1+2)}$$
$$+\left((-i)^{2-0}(2-i)^0+(-i)^{2-1}(2-i)^1+(-i)^{2-2}(2-i)^2\right)t^{-(2+2)}+\cdots$$

$$=t^{-2}+(2-2i)t^{-3}+(1-6i)t^{-4}+\cdots$$, which does agree with (33).

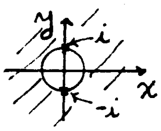
4. (a)



$$\frac{1}{z}=\frac{1}{i+(z-i)}=\frac{1}{t}\frac{1}{1+\frac{i}{t}}=\frac{1}{t}\left(1-\frac{i}{t}+\frac{i^2}{t^2}-\frac{i^3}{t^3}+\cdots\right)=\sum_0^\infty\frac{(-i)^n}{(z-i)^{n+1}}$$

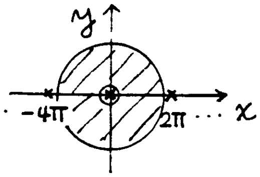
$$=\frac{1}{z-i}-i\frac{1}{(z-i)^2}-\frac{1}{(z-i)^3}+\frac{i}{(z-i)^4}-\cdots \quad \text{in } 1<|z-i|<\infty$$

(b) $\frac{1}{z^2+1} = \frac{1}{z^2(1+\frac{1}{z^2})} = \frac{1}{z^2}\sum_0^\infty \left(-\frac{1}{z^2}\right)^n = \sum_0^\infty (-1)^n \frac{1}{z^{2(n+1)}}$ in $1<|z|<\infty$



(c) $\frac{z^2+3}{z} = \frac{3}{z}+z$ in $0<|z|<\infty$

(d) $\frac{1}{e^z-1} = ?$ Singularities at the roots of $e^z=1$, namely, at $z=\log 1 = 2n\pi i$ $(n=0,\pm1,\pm2,\dots)$



$e^z-1 = 1+z+\frac{z^2}{2!}+\cdots-1 = z+\cdots$, hence there is a first order pole at $z=0$. Thus, write

$\frac{1}{e^z-1} = \frac{1}{z}\boxed{\frac{z}{e^z-1}} \leftarrow$ analytic at $z=0$ and in $|z|<2\pi$, so set

$\frac{z}{e^z-1} = a_0+a_1z+\cdots$ or, $z = (z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\cdots)(a_0+a_1z+a_2z^2+\cdots)$

z: $1=a_0$,

z^2: $0=a_1+\frac{1}{2}a_0 \to a_1=-1/2$,

z^3: $0=a_2+\frac{1}{2}a_1+\frac{1}{6}a_0 \to a_2= 1/12$,

and so on, so

$$\frac{1}{e^z-1} = \frac{1}{z}\left(1-\frac{1}{2}z+\frac{1}{12}z^2+\cdots\right) = \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z+\cdots \quad \text{in } 0<|z|<2\pi$$

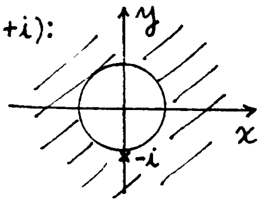
(e) $\frac{1}{z(z^3+2)} = \frac{1}{2z}\frac{1}{1+z^3/2} = \underbrace{\frac{1}{2z}}_{\text{one-term LS in } 0<|z|<\infty}\underbrace{\left(1-\frac{z^3}{2}+\frac{z^6}{4}-\cdots\right)}_{\text{TS in } 0\le|z|<\sqrt[3]{2}}$

$= \frac{1}{2z}-\frac{1}{4}z^2+\frac{1}{8}z^5-\frac{1}{16}z^8+\cdots$ in $0<|z|<\sqrt[3]{2}$

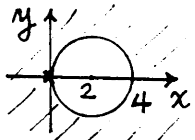
(f) $\frac{1}{z}+\frac{z}{z+i} = \frac{1}{z}+\frac{1}{1+\frac{i}{z}} = \underset{\substack{\uparrow \\ \text{LS in } 0<|z|<\infty}}{\frac{1}{z}} + \underset{\substack{\uparrow \\ \text{LS in } 1<|z|<\infty}}{\left(1-\frac{i}{z}+\frac{i^2}{z^2}-\frac{i^3}{z^3}+\cdots\right)}$

$= 1+(1-i)\frac{1}{z}-\frac{1}{z^2}+i\frac{1}{z^3}-\cdots$ in $1<|z|<\infty$

$z/(z+i)$:

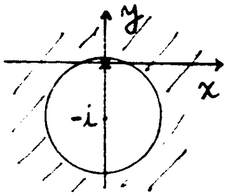


(g) $\frac{1}{z^3} = \frac{1}{[2+(z-2)]^3} = \frac{1}{t^3(1+\frac{2}{t})^3} = \frac{1}{t^3}\underbrace{\left(1+\frac{2}{t}\right)^{-3}}_{\text{Do a T.S. in } \varphi=2/t \text{ about } \varphi=0}$ $(t=z-2)$



$= \frac{1}{t^3}\left(1-3\left(\frac{2}{t}\right)+6\left(\frac{2}{t}\right)^2-10\left(\frac{2}{t}\right)^3+\cdots\right) = \frac{1}{(z-2)^3}-6\frac{1}{(z-2)^4}+24\frac{1}{(z-2)^5}-80\frac{1}{(z-2)^6}+\cdots$

in $2<|z-2|<\infty$.

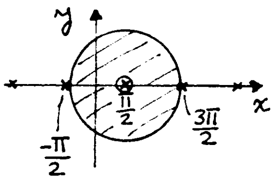


(h) $\frac{1}{z^2} = \frac{1}{[(z+i)-i]^2} = \frac{1}{t^2}\frac{1}{(1-\frac{i}{t})^2} \quad (t = z+i)$

$= \frac{1}{t^2}\left(1 + \frac{2i}{t} + 3\left(\frac{i}{t}\right)^2 + 4\left(\frac{i}{t}\right)^3 + \cdots\right)$ ← TS in $\varphi = i/t$, in $|\varphi| < 1$ (i.e., in $|t| > 1$)

$= \frac{1}{(z+i)^2} + 2i\frac{1}{(z+i)^3} + 3i^2\frac{1}{(z+i)^4} + 4i^3\frac{1}{(z+i)^5} + \cdots$ in $1 < |z+i| < \infty$

$= \sum_0^\infty \frac{(n+1)i^n}{(z+i)^{n+2}}$



(i) $\cos z = 0$ at $z = \pm\pi/2, \pm 3\pi/2, \ldots$

$1/\cos z = 1/\cos[\pi/2 + (z - \pi/2)] = 1/[\overset{\to 0}{\cos\frac{\pi}{2}}\cos(z-\frac{\pi}{2}) - \sin\frac{\pi}{2}\sin(z-\frac{\pi}{2})]$

$= -\frac{1}{\sin(z-\frac{\pi}{2})} = -\frac{1}{\sin t} = -\frac{1}{t}\boxed{\frac{t}{\sin t}}$ ← analytic in $0 \le |t| < \pi$ so Taylor expand it in that disk

$\frac{t}{\sin t} = a_0 + a_1 t + a_2 t^2 + \cdots$ (Can omit $a_1 t, a_3 t^3, \ldots$ since $t/\sin t$ is even in t.)

$t = (a_0 + a_2 t^2 + a_4 t^4 + \cdots)(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \cdots)$

t: $1 = a_0$

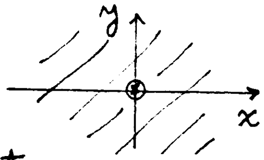
t^3: $0 = a_2 - \frac{1}{6}a_0 \to a_2 = 1/6$

t^5: $0 = a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0 \to a_4 = 7/360$

and so on, so

$$\frac{1}{\cos z} = -\frac{1}{t}\left(1 + \frac{1}{6}t^2 + \frac{7}{360}t^4 + \cdots\right) = -\frac{1}{z-\frac{\pi}{2}} - \frac{1}{6}\frac{1}{(z-\frac{\pi}{2})^3} - \frac{7}{360}\frac{1}{(z-\frac{\pi}{2})^5} - \cdots$$

in $0 < |z - \frac{\pi}{2}| < \pi$



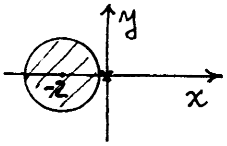
5. (a) $\sin 1/z$ is singular only at the point of expansion, $a=0$, so there is only one expansion possible, a LS in $0 < |z| < \infty$. To obtain it, write

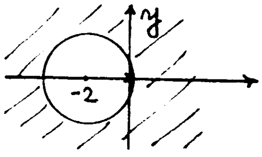
$\sin\frac{1}{z} = \sin t \quad (t = 1/z)$

and do a TS of $\sin t$ in $0 \le |t| < \infty$, which is equivalent to $0 < |z| < \infty$:

$$\sin\frac{1}{z} = \sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots \text{ in } 0 < |z| < \infty$$

(b) Two possible expansions: In $0 \le |z+2| < 2$: In $2 < |z+2| < \infty$:





In $0 \le |z+2| < 2$:

$$\frac{1}{z} = \frac{1}{-2+(z+2)} = -\frac{1}{2}\frac{1}{1-\frac{z+2}{2}} = -\frac{1}{2}\left(1 + \frac{z+2}{2} + \left(\frac{z+2}{2}\right)^2 + \cdots\right) = -\frac{1}{2} - \frac{1}{4}(z+2) - \frac{1}{8}(z+2)^2 - \cdots$$

In $2 < |z+2| < \infty$:

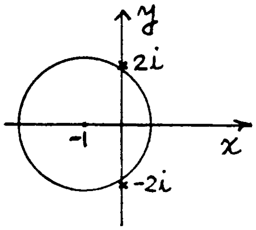
$$\frac{1}{z} = \frac{1}{-2+(z+2)} = \frac{1}{z+2}\frac{1}{1-\frac{2}{z+2}} = \frac{1}{z+2}\left(1 + \frac{2}{z+2} + \left(\frac{2}{z+2}\right)^2 + \left(\frac{2}{z+2}\right)^3 + \cdots\right) = \frac{1}{z+2} + \frac{2}{(z+2)^2} + \cdots$$

(c) Singular only at $z=0$ so the only expansion possible, about $z=0$, is in $0<|z|<\infty$:

$$e^{-1/z^3} = e^{-t} = 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \cdots \quad \text{in } 0 \le |t| < \infty \quad (t = 1/z^3)$$
$$= 1 - \frac{1}{z^3} + \frac{1}{2!}\frac{1}{z^6} - \frac{1}{3!}\frac{1}{z^9} + \cdots \quad \text{in } 0<|z|<\infty .$$

(d) In $0 \le |z+1| < \sqrt{5}$: TS gives

$$\frac{z^2+5}{z^2+4} = \frac{6}{5} + \frac{2}{25}(z+1) - \frac{1}{125}(z+1)^2 - \frac{12}{625}(z+1)^3 - \frac{19}{3125}(z+1)^4 - \cdots$$



In $\sqrt{5} < |z+1| < \infty$: LS

$$\frac{z^2+5}{z^2+4} = \frac{z^2+5}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right).$$ Then, with $t = z+1$,

$$\frac{1}{z-2i} = \frac{1}{(z+1)-(2i+1)} = \frac{1}{z+1}\,\frac{1}{1-\frac{2i+1}{z+1}} = \frac{1}{t}\left[1 + \frac{2i+1}{t} + \frac{4i-3}{t^2} + \cdots\right] \qquad ①$$

and, merely changing $i \to -i$,

$$\frac{1}{z+2i} = \frac{1}{t}\left[1 + \frac{-2i+1}{t} + \frac{-4i-3}{t^2} + \cdots\right] \qquad ②$$

Also, $z^2+5 = (t-1)^2+5 = t^2-2t+6$, so

$$\frac{z^2+5}{z^2+4} = \frac{t^2-2t+6}{4it}\left[\cancel{1} + \frac{1+2i}{t} - \frac{3i-4}{t^2} + \cdots - \cancel{1} - \frac{1-2i}{t} - \frac{-3-4i}{t^2} - \cdots\right] \qquad ③$$

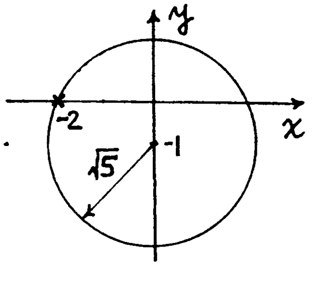
$$= \frac{1}{4i}\left(t-2+\frac{6}{t}\right)\left(\frac{4i}{t} + \frac{7+i}{t^2} + \cdots\right) = \frac{1}{4i}\left(4i + \frac{7+i}{t} + \cdots - \frac{8i}{t} - \cdots + \cdots\right)$$

$$= \frac{1}{4i}\left(4i + \frac{7-7i}{t} + \cdots\right) = 1 - \frac{7}{4}(1+i)\frac{1}{z+1} + \cdots . \qquad ④$$

I thought I would obtain the first 3 terms, in ④, by carrying ① and ② through 3 terms, but the cancelling 1's in ③ reduced ④ to only 2 terms. Thus, we need to include at least one more term in ① and in ②.

(e) In $0 \le |z+i| < \sqrt{5}$: TS gives

$$\frac{1+z}{2+z} = \left(\frac{3-i}{5}\right) + \left(\frac{3+4i}{25}\right)(z+i) - \left(\frac{2+11i}{125}\right)(z+i)^2 - \left(\frac{7-24i}{625}\right)(z+i)^3 + \left(\frac{38-41i}{3125}\right)(z+i)^4 + \cdots$$



In $\sqrt{5} < |z+i| < \infty$: LS gives

$$\frac{1+z}{2+z} = \frac{1-i+(z+i)}{2-i+(z+i)} = \frac{1}{z+i}\left[(1-i)+(z+i)\right]\frac{1}{1+\frac{2-i}{z+i}}$$

$$= \left(1 + \frac{1-i}{z+i}\right)\left[1 - \left(\frac{2-i}{z+i}\right) + \left(\frac{2-i}{z+i}\right)^2 - \cdots\right]$$

$$= 1 - \frac{1}{z+i} + (2-i)\frac{1}{(z+i)^2} - \cdots$$

(f) Singular only at $z=0$ so the only expansion is the LS in $0<|z|<\infty$.

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots\right) = \frac{1}{z^3}-\frac{1}{6}\frac{1}{z}+\frac{1}{120}z-\cdots$$

(g) Singular only at $z=-i$ so the only expansion (about $z=-i$, that is) is the LS in $0<|z+i|<\infty$. The $1/(z+i)^2$ is already in powers of $(z+i)$ so leave it alone, and Taylor expand $\cos 2z$ about $-i$:

$$\frac{\cos 2z}{(z+i)^2} = \frac{1}{(z+i)^2}\left[\cosh 2+2i\sinh 2\,(z+i)-2\cosh 2\,(z+i)^2-\frac{4}{3}i\sinh 2\,(z+i)^3-\cdots\right]$$

$$= \cosh 2\,\frac{1}{(z+i)^2}+2i\sinh 2\,\frac{1}{z+i}-2\cosh 2-\frac{4}{3}i\sinh 2\,(z+i)-\cdots$$

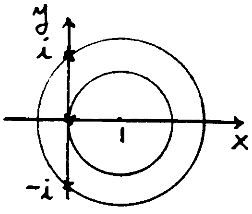
(h) Analytic everywhere, so we have only the TS

$$e^{-z^2} = 1-z^2+\frac{1}{2!}z^4-\frac{1}{3!}z^6+\cdots \quad \text{in } 0\le|z|<\infty$$

(i) We have only the LS

$$e^{-1/z} = 1-\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}-\frac{1}{3!}\frac{1}{z^3}+\cdots \quad \text{in } 0<|z|<\infty$$

(j) TS in $0\le|z-1|<1$:



$$\frac{1}{z(z^2+1)} = \frac{1}{2}-(z-1)+\frac{5}{4}(z-1)^2-\frac{5}{4}(z-1)^3+\frac{9}{8}(z-1)^4-\cdots$$

LS in $1<|z-1|<\sqrt{2}$:

$$\frac{1}{z(z^2+4)} = \frac{1}{1+(z-1)}\,\frac{1}{[(z-1)+1+2i][(z-1)+1-2i]} = \frac{1}{(1+t)(t+(1+2i))(t+(1-2i))}$$

$$= \frac{1}{4}\frac{1}{t+1}-\frac{1}{8}\frac{1}{t+(1+2i)}-\frac{1}{8}\frac{1}{t+(1-2i)}$$

Key step:

$$= \frac{1}{4t}\frac{1}{1+\frac{1}{t}}-\frac{1}{8(1+2i)}\frac{1}{1+\frac{t}{1+2i}}-\frac{1}{8(1-2i)}\frac{1}{1+\frac{t}{1-2i}}$$

$$= \frac{1}{4t}\left(1-\frac{1}{t}+\frac{1}{t^2}+\cdots\right)-\frac{1}{8(1+2i)}\left[1-\frac{t}{1+2i}+\frac{t^2}{(1+2i)^2}-\cdots\right]$$

$$-\frac{1}{8(1-2i)}\left[1-\frac{t}{1-2i}+\frac{t^2}{(1-2i)^2}-\cdots\right]$$

$$= \cdots-\frac{1}{4}\frac{1}{t^2}+\frac{1}{4}\frac{1}{t}-\frac{1}{20}-\frac{3}{100}t+\frac{11}{500}t^2-\cdots, \quad \text{where } t=z-1.$$

6. $\frac{1}{z^2}+\frac{1}{z^3}+\cdots = \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$ in $1<|z|<\infty$

so $f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z(z-1)}$ everywhere (except at $z=0,1$)

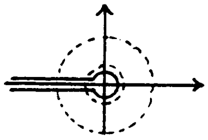
so $f(2) = (2i) = (-2+i)/10$, $f(i/3) = (-9+27i)/10$

7. $f(z) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$ in $1 < |z| < \infty$

$$= -\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) + 1 = 1 - \frac{1}{1 + 1/z} = \frac{1}{z+1}$$ everywhere (except at $z=-1$)

so $f(2) = 1/3$, $f(1/3) = 3/4$.

8. No, there is no annulus of analyticity about $z=0$ due to the intrusion of the cut; i.e., $z=0$ is not an isolated singular point of $\log z$.



9. (a) $e^{\frac{x}{2}(z - \frac{1}{z})} = e^{\frac{x}{2}z} e^{-\frac{x}{2}\frac{1}{z}}$. The first factor is analytic everywhere and the second is analytic everywhere except at $z=0$ where it has an essential singularity. Thus, the LS on the RHS must be valid in $0 < |z| < \infty$.

10. "c_n" is $J_n(x)$, so $J_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{x}{2}(\zeta - 1/\zeta)}}{(\zeta - 0)^{n+1}} d\zeta$ but $\zeta = e^{i\theta}$ on C, so

$$J_n(x) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}}{(e^{i\theta})^{n+1}} i e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\underbrace{\cos(x\sin\theta - n\theta)}_{\text{even}} + i \cancelto{0}{\underbrace{\sin(x\sin\theta - n\theta)}_{\text{odd}}}] d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta) d\theta$$

Section 24.4

2.(a) $z^2 - z = 0$ at $z=0$ and $z=1$.

TS about 0 is $= -z + z^2 \sim -z$, so first-order zero at 0

TS about 1 is $= (z-1) + (z-1)^2 \sim (z-1)$ so " " " " 1

(b) $e^z - 1 = 0$ at $z = \log 1 = \ln 1 + 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$
$= 2n\pi i$

TS about $2n\pi i$ is $= (z - 2n\pi i) + \frac{1}{2!}(z - 2n\pi i)^2 + \cdots \sim (z - 2n\pi i)$ so $e^z - 1$ has first-order zeros at $2n\pi i$ for $n = 0, \pm 1, \ldots$.

(c) $z\sin z = 0$ at $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$.

TS about $z=0$ is $= z^2 - \frac{1}{3!}z^4 + \frac{1}{5!}z^6 - \cdots$, so 2nd-order zero at $z=0$

TS about $z = n\pi$ $(n \neq 0)$ is $= [n\pi + (z - n\pi)][\cos n\pi\,(z - n\pi) - \frac{1}{2!}(z - n\pi)^2 + \cdots]$
$\sim n\pi\cos n\pi\,(z - n\pi)$ so 1st order zeros at $n\pi$ $(n \neq 0)$.

(d) It's easiest to factor f as $(z)(\cos z)(\cos z)$.

TS about $z=0$ is $= (z)(1 + \cdots)(1 + \cdots) = z + \cdots \sim z$ so 1st order zero at 0.

TS about $z = n\pi/2$ (n an odd integer) is $= [\frac{n\pi}{2} + (z - \frac{n\pi}{2})][-(\sin\frac{n\pi}{2})(z - \frac{n\pi}{2}) + \cdots]^2$

$\sim \frac{n\pi}{2}\left(\sin\frac{n\pi}{2}\right)^2\left(z-\frac{n\pi}{2}\right)^2$ so 2nd order zeros at $n\pi/2$ (n odd)

(e) $(z^2+1)^3 = (z+i)^3(z-i)^3$. First and second factors have 0th and 3rd order zeros at i so f has a $0+3 = $ 3rd order zero at i. First and second factors have 3rd and 0th order zeros at $-i$ so f has a $3+0 = $ 3rd order zero at $-i$.

(f) zeros at $z = \log(-2) = \ln 2 + (2n+1)\pi i$ for $n = 0, \pm 1, \pm 2, \dots$. TS about that point is $= -2[z-(\ln 2 + (2n+1)\pi i)] + \cdots$, so f has 1st order zeros at those points

(g) 1st order zeros at $z = (-1+\sqrt{3}i)/2$ and at $z = (-1-\sqrt{3}i)/2$.

(h) $1-z^4 = 0$ at $z = 1^{1/4} = 1, i, -1, -i$, at each of which f has a 1st order zero

3. (a) Singular only at $z=0$; 2nd order pole

(b) Singular at $z = 2n\pi i$ ($n = \pm 1, \pm 2, \dots$ but not $n=0$ because at $z=0$ the 2nd order zero in the numerator overpowers the 1st order zero in the denominator); 1st order poles

(c) 1st order poles at each of the 3 one-third roots of 1, namely, at $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(d) 1st order pole at -2

(e) $\sinh z$ has 1st order zeros at $z = n\pi i$ ($n = 0, \pm 1, \pm 2, \dots$) so $1/\sinh z$ has first order poles there

(f) $\cosh z = \cos iz$ has 1st order zeros at $iz = n\pi/2$ ($n = \pm 1, \pm 2, \dots$), that is, at $z = m\pi i/2$ ($m = \pm 1, \pm 2, \dots$) so $1/\cosh z$ has 1st order poles at those points

(g) $\sin z$ has 1st order zeros at $n\pi$ ($n = 0, \pm 1, \pm 2, \dots$) so $\sin^3 z$ has 3rd order zeros at those points. Thus, $z/\sin^3 z$ has 3rd order poles at $z = n\pi$ ($n = \pm 1, \pm 2, \dots$) but a 2nd order pole at $z=0$ (since the numerator has a first order pole there).

(h) Singular only at $z=0$, where it has an essential singularity

(i)-(m) Same as for (h)

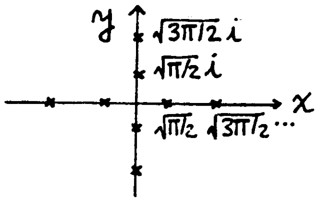
(n) 2nd order pole at $z=1$

(o) e^{-z} is analytic for all z. (e^z has no zeros)

(p) $\tan z^2 = \frac{\sin z^2}{\cos z^2}$. Since $\sin^2 z^2 + \cos^2 z^2 = 1$ for all z it follows that $\cos z^2$ and $\sin z^2$ cannot vanish at the same point so we need merely attend to the zeros of the denominator, $\cos z^2$, namely, $z^2 = n\pi/2$ ($n = \pm 1, \pm 3, \dots$) or $z = \begin{cases} \pm\sqrt{n\pi/2} & \text{for } n = 1, 3, \dots \\ \pm i\sqrt{|n|\pi/2} & \text{for } n = -1, -3, \dots \end{cases}$

At any of those points, say z_n, $\cos z^2 = 0 - \left.(\underbrace{2z\sin z^2}_{\neq 0})\right|_{z_n}(z-z_n) + \cdots$

so $\cos z^2$ has a 1st order zero and $\tan z^2$ has a 1st order pole there.



(q) Same idea as in (p): 1st order poles where $1/z^2 = n\pi/2$ $(n=\pm1, \pm3, \ldots)$, namely, at the points $z = \pm 1/\sqrt{n\pi/2}$ for $n=1,3,\ldots$ and $\pm i/\sqrt{|n|\pi/2}$ for $n=-1,-3,\ldots$

(r) No singular points since e^z (and hence e^{z^2}) $\neq 0$ for all z.

(s) 1st order poles where $z-2=n\pi$, namely, at $z=2+n\pi$ for $n=0,\pm1,\pm2,\ldots$

(t) 1st order poles where $1/z=n\pi$, namely, at $z=1/n\pi$ for $n=\pm1,\pm2,\ldots$ (I'll omit $n=0$ since that would give $z=0$, whereas we are considering here only the finite z plane.)

4. (a) With $t=1/z$ (so $z=\infty \to t=0$), $\dfrac{e^z-1}{z^3} = (e^{1/t}-1)t^3$ which has an essential singularity at $t=0$ and hence at $z=\infty$.

(b) $\dfrac{z^2}{e^z-1} = \dfrac{1}{t^2(e^{1/t}-1)}$ has an essential singularity at $t=0$, hence at $z=\infty$. Why? CRUDELY put, for small t the -1 is inconsequential so the $1/(e^{1/t}-1)$ is "like" $1/e^{1/t} = e^{-1/t}$, which has an essential singularity at $t=0$. More convincingly, let us seek the Laurent expansion of $1/(e^{1/t}-1)$ about $t=0$. (We can ignore the $1/t^2$ because if $1/(e^t-1)$ has an essential sing. at $t=0$ then so does $1/t^2$ times it.) The form

$$\frac{1}{e^{1/t}-1} = t + a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots$$

will work since $1 = \left(\frac{1}{t} + \frac{1}{2!}\frac{1}{t^2} + \frac{1}{3!}\frac{1}{t^3} + \cdots\right)\left(t + a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots\right)$ gives:

1: $1 = 1$ ✓

t^{-1}: $0 = a_0 + \frac{1}{2!} \to a_0 = -1/2!$

t^{-2}: $0 = a_1 + \frac{a_0}{2!} + \frac{1}{3!} \to a_1 =$ etc

t^{-3}: $0 = a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} + \frac{1}{4!} \to a_2 =$ etc.,

and so on.

(c) $\dfrac{1}{z^3-1} = \dfrac{t^3}{1-t^3}$ is analytic at $t=0$ and hence at $z=\infty$.

(d) $\dfrac{1}{1+\frac{1}{1+z}} = \dfrac{z+1}{z+2} = \dfrac{1+t}{1+2t}$ $(t=1/z)$ is analytic at $t=0$ and hence at $z=\infty$.

5. (a) $f(z) = \frac{1}{z^2}(1+\frac{1}{z}+\frac{1}{z^2}+\cdots) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z(z-1)}$, which has a 1st-order pole at $z=0$.

(b) $f(z) = -[1-\frac{1}{2z}+(\frac{1}{2z})^2-(\frac{1}{2z})^3+\cdots]+1 = 1-\frac{1}{1+\frac{1}{2z}} = 1-\frac{2z}{2z-1} = \frac{1}{1-2z}$,

which is analytic at $z=0$.

(c) f has a 4th-order pole at $z=0$.

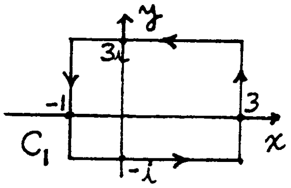
(d) $f(z) = [1+\frac{1}{1!}(\frac{1}{z})+\frac{1}{2!}(\frac{1}{z})^2+\cdots]-1-\frac{1}{z} = e^{1/z}-1-\frac{1}{z}$ has an essential singularity at $z=0$

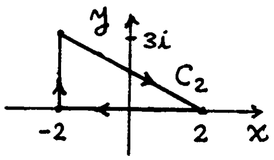
(e) $f(z) = \frac{1}{z^5}\frac{1}{1+\frac{2}{z^3}} = \frac{1}{z^2(z^3+2)}$ has a 2nd order pole at $z=0$.

6. "has an infinite number of negative powers of z" is incorrect; it has an infinite number of positive powers of z in the denominator, which is not the same thing. Indeed, $1/e^z = e^{-z}$ is analytic for all z and has the TS $e^{-z} = 1-\frac{1}{1!}z+\frac{1}{2!}z^2-\cdots$ in $0 \le |z| < \infty$.

Section 24.5

1.





(a) $\sin 2z = 0$ at $z = n\pi/2 = 0, \pm\pi/2, \pm\pi, \ldots$, of which 0 and $\pi/2$ are within C_1. Thus,

$$\int_{C_1} \frac{dz}{\sin 2z} = 2\pi i(\text{Res}@0 + \text{Res}@\tfrac{\pi}{2})$$
$$= 2\pi i\left(\lim_{z\to 0}\frac{z}{\sin 2z} + \lim_{z\to\pi/2}\frac{z-\pi/2}{\sin 2z}\right)$$
$$= 2\pi i\left(\tfrac{1}{2}-\tfrac{1}{2}\right) \text{ by l'Hôpital's rule}$$
$$= 0$$

(b) Second order pole @ 0 where Res $= -1$ since $\frac{1}{z^2e^z} = \frac{1}{z^2}e^{-z} = \frac{1}{z^2}(1-z+\cdots)$

Thus, $\int_{C_1}\frac{dz}{z^2e^z} = 2\pi i(-1) = -2\pi i$

(c) $\sinh 2z = -i\sin(i2z) = 0$ at $z = 0, \pm\pi i/2, \pm\pi i, \pm 3\pi i/2, \pm 4\pi i/2, \ldots$, of which $0, \pi i/2$ are within C_1. $z^2/\sinh 2z$ is analytic at 0, however, and has 1st-order pole @ $\pi i/2$, with Res $= \lim_{z\to\pi i/2}(\frac{z-\pi i/2}{\sinh 2z}z^2)$ $= (\lim_{z\to\pi i/2}\frac{z-\frac{\pi i}{2}}{\sinh 2z})(\frac{\pi i}{2})^2 = \frac{1}{2\cosh\pi i}(-\frac{\pi^2}{4}) = \frac{\pi^2}{8}$, so $I = 2\pi i(\pi^2/8) = \pi^3 i/4$

(d) Integrand has 3rd-order pole at 1 and (with $z-1=t$)

$$\left(\frac{z+1}{z-1}\right)^3 = \frac{(2+t)^3}{t^3} = \frac{8}{t^3} + \frac{12}{t^2} + \frac{6}{t} + 1 \quad \text{so Res@1} = 6 \quad \text{and } I = 2\pi i(6) = 12\pi i$$

(e) $z^2 - 2iz - 2 = [z-(1+i)][z-(-1+i)]$ so the integrand has 1st order poles at $1+i$ and $-1+i$, of which $1+i$ is outside of C_2 and $-1+i$ is inside, so

$$I = -2\pi i\,\text{Res@}(-1+i) = -2\pi i \lim_{z\to -1+i} [z-\cancel{(-1+i)}]\frac{1}{[z-(1+i)][z-\cancel{(-1+i)}]} = \pi i$$

(f) $\cosh(\pi z/2) = \cos(i\pi z/2) = 0$ at $i\pi z/2 = \pm\pi/2, \pm 3\pi/2, \dots$
or, $z = \pm i, \pm 3i, \dots,$
of which only $+i$ is within C_2. at i, $1/\cosh^2(\pi z/2)$ has a 2nd-order pole, so

$$\text{Res@}i = \lim_{z\to i}\frac{d}{dz}[(z-i)^2/\cosh^2(\pi z/2)]$$

$$= \lim_{z\to i}\left\{\frac{2(z-i)}{\cosh^2\frac{\pi z}{2}} + \frac{(z-i)^2(-2)(\pi/2)\sinh\frac{\pi z}{2}}{\cosh^3\frac{\pi z}{2}}\right\}$$

$$= \lim_{z\to i}\frac{2(z-i)\cosh\frac{\pi z}{2} - \pi(z-i)^2\sinh\frac{\pi z}{2}}{\cosh^3\frac{\pi z}{2}}$$

$$= \lim_{z\to i}\frac{2\cosh\frac{\pi z}{2} + \pi(z-i)\sinh\frac{\pi z}{2} - 2\pi(z-i)\sinh\frac{\pi z}{2} - \frac{\pi^2}{2}(z-i)^2\cosh\frac{\pi z}{2}}{\frac{3\pi}{2}\cosh^2\frac{\pi z}{2}\sinh\frac{\pi z}{2}}$$

by l'Hôpital, but we still need to apply l'Hôpital again – twice in fact, and it is looking tedious, so let's try evaluating the Res more directly, by developing the LS of the integrand about $z=i$: With $z-i=t$,

$$\cosh\frac{\pi z}{2} = \cosh\frac{\pi}{2}(t+i) = \cosh\frac{\pi t}{2}\overset{\cos\pi/2=0}{\cosh\frac{\pi i}{2}} + \sinh\frac{\pi t}{2}\sinh\frac{\pi i}{2}$$

$$= i\sinh\frac{\pi t}{2} = i\left(\frac{\pi t}{2} + \frac{\pi^3 t^3}{6} + \cdots\right)$$

* so $$\frac{1}{\cosh^2\frac{\pi z}{2}} = \frac{1}{(i\frac{\pi t}{2})^2}\frac{1}{(1+\frac{\pi^2 t^2}{3}+\cdots)^2}$$

$$= -\frac{4}{\pi^2 t^2}\Big[1 + \underbrace{\left(\frac{\pi^2 t^2}{3}+\cdots\right)}\Big]^{-2}$$

calling this μ, say,

$$(1+\mu)^{-2} = 1 - 2\mu + 3\mu^2 - \cdots$$

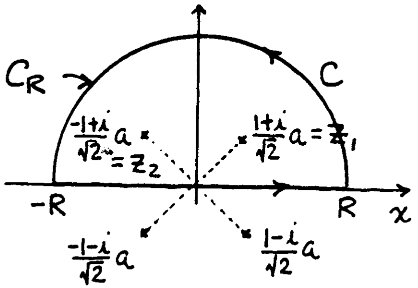
$$= 1 - 2\left(\frac{\pi^2 t^2}{3}+\cdots\right) + 3\left(\frac{\pi^2 t^2}{3}+\cdots\right)^2 - \cdots$$

$$= 1 - \frac{2\pi^2}{3}t^2 + (\text{etc})t^4 + (\text{etc})t^6 + \cdots$$

so the residue (i.e., the coeff. of $1/t$) is seen to be 0. Hence, $I = 0$.

NOTE: The * method is useful and might be worth discussing in class.

2.(a) $\mathscr{I} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = ?$ Consider



$$\mathscr{J} = \oint_C \frac{dz}{z^4+a^4} = 2\pi i\,(\text{Res} @ z_1 + \text{Res} @ z_2)$$

$$= 2\pi i\left(\lim_{z\to z_1} \frac{z-z_1}{z^4+a^4} + \lim_{z\to z_2} \frac{z-z_2}{z^4+a^4}\right)$$

$= 2\pi i\left(\frac{1}{4z_1^3} + \frac{1}{4z_2^3}\right)$ by l'Hôpital. Best to express z_1, z_2 in polar form now: $z_1 = ae^{\pi i/4}$, $z_2 = ae^{3\pi i/4}$ or, better yet, $z_2 = -ae^{-\pi i/4}$

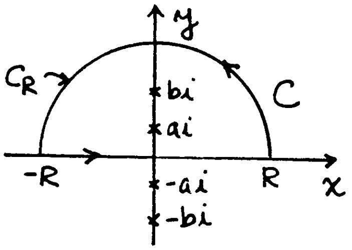
$$= \frac{2\pi i}{4a^3}\left(\frac{e^{-3\pi i/4} - e^{+3\pi i/4}}{2i}\right)2i = \frac{2\pi i}{4a^3}\left(\sin\frac{3\pi}{4}\right)(2i) = -\frac{\pi}{a^4}\left(-\frac{1}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}a^4}$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+a^4} + \lim_{R\to\infty} \int_{C_R}$$

$$\left|\int_{C_R}\right| \le \max\left|\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\right| \pi R \le \frac{\pi R}{(R-a)^4} \to 0 \text{ as } R\to\infty$$

$= \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} + 0 = 2\mathscr{I}$, so $\mathscr{I} = \frac{\pi}{2\sqrt{2}a^4}$

(b) $\mathscr{I} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = ?$ Consider



$$\mathscr{J} = \oint_C \frac{dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i\,(\text{Res} @ ai + \text{Res} @ bi)$$

$$= 2\pi i\left(\lim_{z\to ai} \frac{z-ai}{(z^2+a^2)(z^2+b^2)} + \lim_{z\to bi} \frac{z-bi}{(z^2+a^2)(z^2+b^2)}\right)$$

$$= 2\pi i\left(\frac{1}{2ai(b^2-a^2)} + \frac{1}{(a^2-b^2)2bi}\right) = \frac{\pi}{ab(a+b)}$$

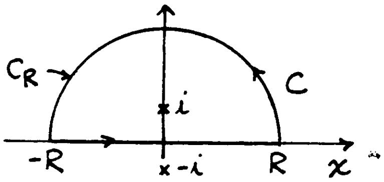
$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+a^2)(x^2+b^2)} + \lim_{R\to\infty} \int_{C_R}$$

Let $\max\{a,b\} \equiv \alpha$. Then $\left|\int_{C_R}\right| \le \frac{1}{(R-\alpha)^4}\pi R \to 0$ as $R\to\infty$

$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} + 0 = 2\mathscr{I}$, so $\mathscr{I} = \frac{\pi}{2ab(a+b)}$

(c) $\int_0^{\infty} \frac{x^2}{x^4+1}\,dx = \pi\sqrt{2}/4$ (by maple)

(d) $\mathcal{I} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = ?$ Consider



$$\mathcal{J} = \oint_C \frac{dz}{(z^2+1)^2} = 2\pi i \operatorname{Res}@\, i = 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2}{(z^2+1)^2}$$

$$= 2\pi i \lim_{z\to i} \frac{d}{dz}(z+i)^{-2} = \pi/2$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \lim_{R\to\infty} \int_{C_R}$$

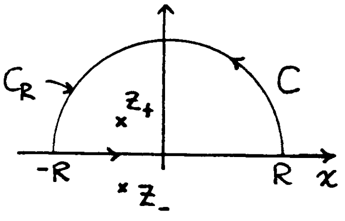
$$\left|\int_{C_R}\right| \leq \frac{1}{(R-1)^2}\pi R \sim \frac{\pi}{R} \to 0 \text{ as } R\to\infty$$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} + 0 = 2\mathcal{I}, \text{ so } \mathcal{I} = \pi/4$$

(e) $\mathcal{I} = \int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1} = ?$ Consider

$$\mathcal{J} = \oint_C \frac{dz}{4z^2+2z+1} = \frac{1}{\underline{\underline{\underline{4}}}}\oint_C \frac{dz}{(z-z_+)(z-z_-)}$$ where $z_+ = (-1+i\sqrt{3})/4$ is in C and $z_- = (-1-i\sqrt{3})/4$ is not

$$= 2\pi i \frac{1}{4} \operatorname{Res}@\, z_+ = \frac{2\pi i}{4}\frac{1}{z_+ - z_-} = \frac{\pi}{\sqrt{3}}$$



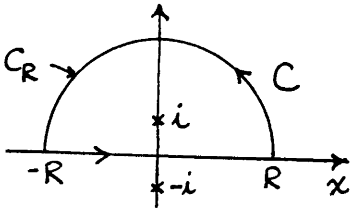
$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{4x^2+2x+1} + \lim_{R\to\infty} \int_{C_R}$$

$$\left|\int_{C_R}\right| \leq \frac{1}{4(R-|z_+|)^2}\pi R \sim \frac{\pi}{4R} \to 0$$

$$= \int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1} + 0 = \mathcal{I}, \text{ so } \mathcal{I} = \pi/\sqrt{3}.$$

(f) Maple gives $\mathcal{I} = \pi/6$, namely, $2\pi i$ times the sum of the residues at the first order poles in the upper half plane, at $\pm\frac{\sqrt{3}}{2} + \frac{i}{2}$.

(g) $\mathcal{I} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\,dx = ?$ Consider



$$\mathcal{J} = \oint_C \frac{e^{i2z}}{(z^2+1)^2}\,dz = 2\pi i \operatorname{Res}@\, i$$

$$= 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{\cancel{(z-i)^2}\, e^{i2z}}{(z+i)^2\cancel{(z-i)^2}} = \frac{3\pi}{2e^2}$$

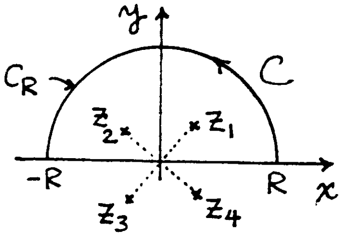
$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x + i\sin 2x}{(x^2+1)^2}\,dx + \lim_{R\to\infty} \int_{C_R} \frac{e^{i2z}}{(z^2+1)^2}\,dz$$

On C_R, $|e^{i2z}| = |e^{i2(x+iy)}| = e^{-2y} \leq 1$ so

$$\left|\int_{C_R}\right| \leq \frac{1}{(R-1)^4}\pi R \sim \frac{\pi}{R^3} \to 0 \text{ as } R\to\infty$$

$$= \int_{-\infty}^{\infty} \frac{\cos 2x + \overset{0 \text{ by odd integrand}}{\cancel{i \sin 2x}}}{(x^2+1)^2}\,dx + 0 = 2I, \text{ so } I = 3\pi/(4e^2)$$

(h) $x\sin x$ is even, as is x^4+16, so $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+16}\,dx = ?$ Consider



$z_1 = 2e^{\pi i/4}$, $z_2 = 2e^{3\pi i/4}$
$= -2e^{-\pi i/4}$

$$J = \oint_C \frac{z e^{iz}}{z^4+16}\,dz = 2\pi i\,(\text{Res}@z_1 + \text{Res}@z_2)$$

$$= 2\pi i\left(\lim_{z\to z_1} \frac{(z-z_1)\,z e^{iz}}{z^4+16} + \lim_{z\to z_2} \frac{(z-z_2)\,z e^{iz}}{z^4+16}\right)$$

$$= 2\pi i\left(z_1 e^{iz_1} \lim_{z\to z_1} \frac{z-z_1}{z^4+16} + z_2 e^{iz_2} \lim_{z\to z_2} \frac{z-z_2}{z^4+16}\right)$$

$$= 2\pi i\left(z_1 e^{iz_1} \frac{1}{4z_1^3} + z_2 e^{iz_2}\frac{1}{4z_2^3}\right) = \frac{2\pi i}{4}\left(\frac{e^{iz_1}}{z_1^2} + \frac{e^{iz_2}}{z_2^2}\right)$$

$$= \frac{\pi i}{2}\left[\frac{e^{i(2)\left(\frac{1+i}{\sqrt{2}}\right)}}{4i} + \frac{e^{i(2)\left(\frac{-1+i}{\sqrt{2}}\right)}}{-4i}\right] = \frac{\pi}{8} e^{-\sqrt{2}}\left(\frac{e^{i\sqrt{2}} - e^{-i\sqrt{2}}}{2i}\right)2i = \frac{\pi i}{4} e^{-\sqrt{2}} \sin\sqrt{2}$$

$$= \lim_{R\to\infty}\int_{-R}^{R} \frac{x e^{ix}}{x^4+16}\,dx + \lim_{R\to\infty}\int_{C_R} \frac{z e^{iz}}{z^4+16}\,dz$$

On C_R, $|e^{iz}| = |e^{i(x+iy)}| = e^{-y} \le 1$

so $\left|\int_{C_R}\right| \le \frac{R}{(R-2)^4}\pi R \sim \frac{\pi}{R^2} \to 0$ as $R \to \infty$

$$= \int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{x^4+16}\,dx + 0 = 0 + i2I$$

$\uparrow$ since $x\cos x/(x^4+16)$ is odd

so $2iI = \frac{\pi i}{4} e^{-\sqrt{2}} \sin\sqrt{2}$, $I = \frac{\pi e^{-\sqrt{2}}}{8} \sin\sqrt{2}$

NOTE: I tried checking this result by the maple command

`int( x* sin(x)/(x^4+16), x=0.. infinity );`

but it didn't give an answer. Using numerical integration, however,

`evalf( int (x*sin(x)/(x^4+16), x=0.. infinity ));`

gives $I = 0.094303712$, which does agree with our analytical result.

(i) $I = \int_{-\infty}^{\infty} \frac{\cos x\,dx}{8x^2+12x+5} = ?$ Consider $J = \oint_C \frac{e^{iz}\,dz}{8z^2+12z+5}$ where C is the "usual" contour, as above. Then $8z^2+12z+5 = 8(z-z_1)(z-z_2)$ where $z_1 = (-3+i)/4$ is in C and $z_2 = (-3-i)/4$ is not, so

$$J = 2\pi i\,\text{Res}@z_1 = 2\pi i\,\frac{e^{iz_1}}{8(z_1-z_2)} = \frac{\pi}{2} e^{-1/4}\left(\cos\frac{3}{4} - i \sin\frac{3}{4}\right)$$

and (omitting the $\lim_{R\to\infty}$ steps for brevity) this $= \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{8x^2+12x+5}\,dx$

so equating real parts gives $I = \int_{-\infty}^{\infty}\frac{\cos x\,dx}{8x^2+12x+5} = \frac{\pi}{2}e^{-1/4}\cos\frac{3}{4}$

which is verified using the Maple evalf(int( )) command.

3. (a) $I = \frac{1}{2}\int_{-\pi}^{\pi}\sin^2 x\,dx = \frac{1}{2}\oint_C\left(\frac{z^2-1}{2iz}\right)^2\frac{dz}{iz} = \frac{1}{-8i}\oint_C\frac{z^4-2z^2+1}{z^3}\,dz$

$= -\frac{1}{8i}2\pi i(0-2+0) = \pi/2$.

(b) $I = \frac{1}{2}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{2}\oint_C\left(\frac{z^2+1}{2z}\right)^2\frac{dz}{iz} = \frac{1}{8i}\oint_C\frac{z^4+2z^2+1}{z^3}\,dz = \frac{2}{8i}2\pi i = \frac{\pi}{2}$

(c) Sketching the graph of $\sin^2 x$ it is evident that $\int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\sin^2 x\,dx$. Then proceed as in (a). We obtain $I = \pi/4$.

(d) Sketching the graph of $\cos^2 x$ it is evident that $\int_{\pi/2}^{\pi}\cos^2 x\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\cos^2 x\,dx$ Then, proceeding as in (b), we obtain $I = \pi/4$.

(e) $I = \frac{1}{2}\int_{-\pi}^{\pi}\sin^4 x\,dx = \frac{1}{2}\oint_C\left(\frac{z^2-1}{2iz}\right)^4\frac{dz}{iz} = \frac{1}{32i}\oint_C\frac{(z^4-2z^2+1)(z^4-2z^2+1)}{z^5}\,dz$

To evaluate the residue merely pick out the coefficient of the z^4 term in the numerator, namely, 6, so $I = \frac{1}{32i}2\pi i(6) = 3\pi/8$

(g) Proceeding as in (e),

$I = \frac{1}{2}\oint_C\left(\frac{z^2-1}{2iz}\right)^6\frac{dz}{iz} = -\frac{1}{128i}\oint_C\frac{(z^4-2z^2+1)(z^4-2z^2+1)(z^4-2z^2+1)}{z^7}\,dz$

$= -\frac{1}{128i}(2\pi i)(-20) = 5\pi/16$

(i) $I = \oint_C\frac{1}{7+\frac{z^2+1}{2z}}\frac{dz}{iz} = \frac{2}{i}\oint_C\frac{dz}{z^2+14z+1} = \frac{2}{i}\oint_C\frac{dz}{(z-z_1)(z-z_2)}$ $\left\{\begin{matrix} z_1 = -7+4\sqrt{3} \\ z_2 = -7-4\sqrt{3}\end{matrix}\right\}$

z_1 is inside C and z_2 is outside, so

$I = \frac{2}{i}2\pi i\,\text{Res}@z_1 = \frac{2}{i}2\pi i\frac{1}{z_1-z_2} = \pi/2\sqrt{3}$ or $\pi\sqrt{3}/6$

4. $I = \frac{1}{2}\int_0^{2\pi}\frac{\cos t\,dt}{1-2a\cos t+a^2} = \frac{1}{2}\oint_C\frac{\frac{z+1/z}{2}\,\frac{dz}{iz}}{1+a^2-a(z+\frac{1}{z})} = -\frac{1}{4ai}\oint_C\frac{(z^2+1)dz}{z\underbrace{[z^2-(\frac{1+a^2}{a})z+1]}_{(z-\frac{1}{a})(z-a),}}$

$= -\frac{1}{4ai}[\text{Res}@0+\text{Res}@a]2\pi i = -\frac{\pi}{2a}\left[1+\frac{a^2+1}{a(a-\frac{1}{a})}\right] = \frac{\pi a}{1-a^2}$

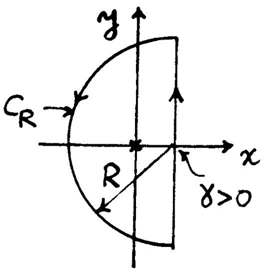
where $|a|<1$

5. (a) Consider $J = \oint_C \frac{e^{st}}{s^2} ds$. $J = 2\pi i \text{ Res@} s=0$
$= 2\pi i t$.

also, $J = \int_{\gamma - iR}^{\gamma + iR} \frac{e^{st}}{s^2} ds + \int_{C_R}$

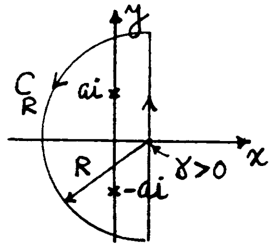
But $\left| \int_{C_R} \right| \leq \frac{\max |e^{(x+iy)t}|}{\min |s|^2} \pi R = \frac{e^{\gamma t}}{R^2} \pi R \to 0$ as $R \to \infty$

so $2\pi i t = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s^2} ds$, $L^{-1}\{\frac{1}{s^2}\} = \frac{1}{2\pi i} 2\pi i t = t$ (for $t>0$)



(b) Like (a), except Res@ $s=0$ is $= t^5/5!$. We obtain $L^{-1}\{\frac{1}{s^6}\} = \frac{t^5}{5!}$ (for $t>0$)

(c) Consider $J = \oint_C \frac{e^{st}}{s^2+a^2} ds = 2\pi i(\text{Res@} ai + \text{Res@} -ai)$
$= 2\pi i \left(\frac{e^{iat}}{2ai} + \frac{e^{-iat}}{-2ai} \right)$
$= \frac{2\pi i}{a} \sin at$

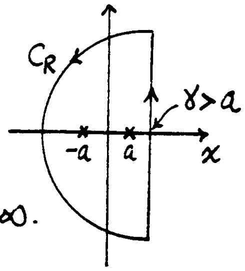


also, $J = \int_{\gamma - iR}^{\gamma + iR} \frac{e^{st}}{s^2+a^2} ds + \int_{C_R}$

But $\left| \int_{C_R} \right| \leq \frac{\max |e^{(x+iy)t}|}{\min |s-ai| \min |s+ai|} \pi R = \frac{e^{\gamma t}}{(R - \sqrt{a^2+\gamma^2})^2} \pi R \sim \frac{\pi e^{at}}{R} \to 0$ as $R \to \infty$

so, letting $R \to \infty$ in
$\frac{2\pi i}{a} \sin at = \int_{\gamma - iR}^{\gamma + iR} \frac{e^{st}}{s^2+a^2} ds + \int_{C_R}$
gives
$\frac{2\pi i}{a} \sin at = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s^2+a^2} ds + 0$
$= 2\pi i L^{-1}\{\frac{1}{s^2+a^2}\}$ so $L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$ (for $t>0$)

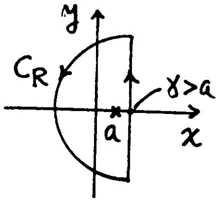
(d) Consider $J = \oint_C \frac{e^{st}}{s^2-a^2} ds = 2\pi i (\text{Res@} a + \text{Res@} -a)$
$= 2\pi i \left(\frac{e^{at}}{2a} + \frac{e^{-at}}{-2a} \right) = 2\pi i \frac{\sinh at}{a}$



But $\left| \int_{C_R} \right| \leq \frac{\max |e^{(x+iy)t}|}{\min |s-a| \min |s+a|} \pi R = \frac{e^{\gamma t}}{(R+a)(R-a)} \pi R \sim \frac{\pi e^{\gamma t}}{R} \to 0$ as $R \to \infty$.

So, letting $R \to \infty$ in
$2\pi i \frac{\sinh at}{a} = \int_{\gamma - iR}^{\gamma + iR} \frac{e^{st}}{s^2-a^2} ds + \int_{C_R}$
gives $2\pi i \frac{\sinh at}{a} = 2\pi i L^{-1}\{\frac{1}{s^2-a^2}\} + 0$ so $L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{\sinh at}{a}$ ($t>0$)

(e) Consider $J = \oint_C \frac{e^{st}}{(s-a)^4} ds = 2\pi i \text{ Res@} a$
$= 2\pi i \frac{1}{3!} \frac{d^3}{ds^3} \left(\frac{\cancel{(s-a)^4} e^{st}}{\cancel{(s-a)^4}} \right) \Big|_{s \to a}$
$= 2\pi i t^3 e^{at}/3!$



But $|\int_{C_R}| \le \frac{\max|e^{(x+iy)t}|}{\min|s-a|^4}\pi R = \frac{e^{\gamma t}}{(R-a)^4}\pi R \sim \frac{\pi e^{\gamma t}}{R^3} \to 0$ as $R\to\infty$

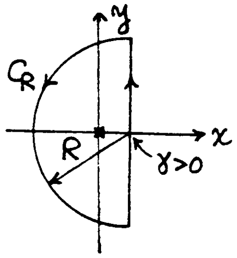
so, letting $R\to\infty$ in

$$2\pi i\, t^3 e^{at}/3! = \int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{(s-a)^4}\, ds + \int_{C_R}$$

gives $2\pi i\, t^3 e^{at}/3! = 2\pi i\, L^{-1}\left\{\frac{1}{(s-a)^4}\right\} + 0$ so $L^{-1}\left\{\frac{1}{(s-a)^4}\right\} = t^3 e^{at}/6 \quad (t>0)$

(k) Consider $J = \oint_C e^{st}\frac{e^{-as}}{s^3}\, ds = 2\pi i$ Res@$s=0$

$= 2\pi i \frac{(t-a)^2}{2!}$



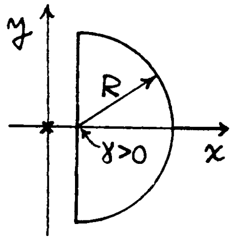
$|\int_{C_R}| \le \frac{\max|e^{(x+iy)(t-a)}|}{\min|s|^3}\pi R = \frac{e^{\gamma(t-a)}}{(R-\gamma)^3}$ if $\underline{\underline{t>a}}$

so, letting $R\to\infty$ in

$$2\pi i\frac{(t-a)^2}{2} = \int_{\gamma-iR}^{\gamma+iR}\frac{e^{s(t-a)}}{s^3}\,ds + \int_{C_R}$$

gives $2\pi i \frac{(t-a)^2}{2} = 2\pi i\, L^{-1}\left\{\frac{e^{-as}}{s^3}\right\} + 0$ so $L^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = \frac{(t-a)^2}{2}$ for $\underline{\underline{t>a}}$.

If $t<a$, close C on the right, as shown:
In that case the residue theorem (or Cauchy's theorem)
gives $J=0$. Also,



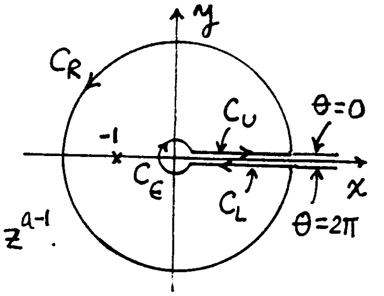
$|\int_{C_R}| \le \frac{\max|e^{-(x+iy)(a-t)}|}{\min|s|^3}\pi R = \frac{e^{-\gamma(a-t)}}{(R^2+\gamma^2)^{3/2}}\pi R \sim \frac{\pi e^{-\gamma(a-t)}}{R^2}$

$\to 0$ as $R\to\infty$

so $0 = \int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{s(t-a)}}{s^3}\,ds$. Thus, $L^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = \begin{cases}(t-a)^2/2, & t>a\\ 0 & , t<a\end{cases} = H(t-a)\frac{(t-a)^2}{2}$

6. (a) $I = \int_0^\infty \frac{x^{a-1}}{x+1}\,dx \quad (0<a<1)$

Consider $J = \oint_C \frac{z^{a-1}}{z+1}\,dz$ where C is as shown:



$J = 2\pi i$ Res@-1 $= 2\pi i(-1)^{a-1} = 2\pi i(1e^{\pi i})^{a-1} = 2\pi i\, e^{(a-1)\pi i}$
per the branch cut for z^{a-1}.

Also,

$$J = \int_{C_R} + \int_R^\epsilon \frac{(xe^{i2\pi})^{a-1}}{x+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R\frac{(xe^{i0})^{a-1}}{x+1}\,dx$$

so $2\pi i\, e^{(a-1)\pi i} = \int_{C_R}+\int_{C_\epsilon}+\int_\epsilon^R \frac{x^{a-1}}{x+1}\,dx + e^{2\pi(a-1)i}\int_R^\epsilon\frac{x^{a-1}}{x+1}\,dx.$

Now,

$$\left|\int_{C_R}\right| \leq \frac{R^{a-1}}{R-1}\pi R \sim \frac{\pi}{R^{1-a}} \to 0 \text{ as } R\to\infty \text{ , and}$$

$$\left|\int_{C_\epsilon}\right| \leq \frac{\epsilon^{a-1}}{1-\epsilon}2\pi\epsilon \sim 2\pi\epsilon^{a} \to 0 \text{ as } \epsilon\to 0 .$$

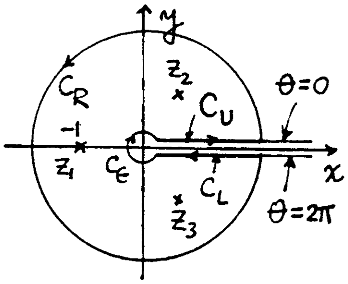
Thus, letting $R\to\infty$ and $\epsilon\to 0$ in ♀ gives

$$2\pi i e^{(a-1)\pi i} = (1-e^{2\pi(a-1)i})\int_0^\infty \frac{x^{a-1}}{x+1}dx$$

so

$$\pi = \frac{e^{-(a-1)\pi i}-e^{(a-1)\pi i}}{2i}\int_0^\infty \frac{x^{a-1}}{x+1}dx \text{, so } I = \frac{\pi}{\sin(1-a)\pi} = \frac{\pi}{\sin a\pi}.$$

(b) $I = \int_0^\infty \frac{\sqrt{x}}{x^3+1}dx$. Consider $J = \oint_C \frac{\sqrt{z}}{z^3+1}dz$ where C is:



The roots of $z^3+1=0$ are $z=-1\equiv z_1$, $z_2=e^{i\pi/3}$, and $z=e^{i5\pi/3}$ (not $e^{-i\pi/3}$, per the cut), so

$$J = 2\pi i(\text{Res} @ z_1 + \text{Res} @ z_2 + \text{Res} @ z_3)$$

$$= 2\pi i\left(\frac{(z-z_1)}{z^3+1}\sqrt{z}\bigg|_{z\to z_1} + \frac{(z-z_2)}{z^3+1}\sqrt{z}\bigg|_{z\to z_2} + \frac{(z-z_3)}{z^3+1}\sqrt{z}\bigg|_{z\to z_3}\right)$$

or, by l'Hôpital,

$$= 2\pi i\left(\frac{\sqrt{z_1}}{3z_1^2}+\frac{\sqrt{z_2}}{3z_2^2}+\frac{\sqrt{z_3}}{3z_3^2}\right)$$

$$= \frac{2\pi i}{3}\left(z_1^{-3/2}+z_2^{-3/2}+z_3^{-3/2}\right)$$ where, by the branch cut,

$$z_1^{-3/2} = (1e^{\pi i})^{-3/2} = e^{-3\pi i/2} = i$$

$$z_2^{-3/2} = (1e^{\pi i/3})^{-3/2} = e^{-\pi i/2} = -i$$

$$z_3^{-3/2} = (1e^{5\pi i/3})^{-3/2} = e^{-5\pi i/2} = -i$$

$$= \frac{2\pi i}{3}(i-i-i) = 2\pi/3.$$

Also, $$J = \int_{C_R} + \int_R^\epsilon \frac{(xe^{2\pi i})^{1/2}}{x^3+1}dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{(xe^{i0})^{1/2}}{x^3+1}dx \qquad *$$

Now, $$\left|\int_{C_R}\right| \leq \frac{\sqrt{R}}{(R-1)^3}2\pi R \sim 2\pi R^{-3/2} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{C_\epsilon}\right| \leq \frac{\sqrt{\epsilon}}{(1-\epsilon)^3}2\pi\epsilon \sim 2\pi\epsilon^{3/2} \to 0 \text{ as } \epsilon\to 0$$

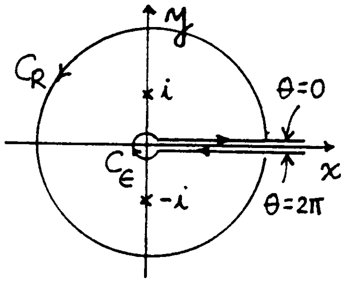
so, letting $R\to\infty$ and $\epsilon\to 0$ in $*$,

$$\frac{2\pi}{3} = 0 + \int_\infty^0 \frac{-\sqrt{x}}{x^3+1}dx + 0 + \int_0^\infty \frac{\sqrt{x}}{x^3+1}dx ,$$

so $$\int_0^\infty \frac{\sqrt{x}}{x^3+1}dx = \frac{\pi}{3}.$$

(c) $I = \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2}$. Consider $J = \oint_C \frac{dz}{\sqrt{z}(z^2+1)^2}$

$$= 2\pi i\,(\text{Res}@i + \text{Res}@-i)$$



Let's work out the residues.

@i: $$\frac{1}{\sqrt{z}\,(z^2+1)^2} = \frac{1}{\sqrt{i+(z-i)}\;[2i+(z-i)]^2[z-i]^2}$$

$$= \frac{1}{\sqrt{i}\,(2i)^2}(1+t)^{-1/2}\left(1+\frac{t}{2i}\right)^{-2}\frac{1}{t^2} \qquad (t=z-i)$$

$$= \frac{i^{-1/2}}{-4}\left(1-\tfrac{1}{2}t+\cdots\right)\left(1-2\frac{t}{2i}+\cdots\right)\frac{1}{t^2}$$

so $\text{Res}@i = -\frac{i^{-1/2}}{4}\left(-\frac{1}{2}-\frac{1}{i}\right) = \frac{(e^{\pi i/2})^{-1/2}}{4}\left(\frac{1}{2}+\frac{1}{i}\right) = \frac{-1-3i}{8\sqrt{2}}$

@-i: $$\frac{1}{\sqrt{z}\,(z^2+1)^2} = \frac{1}{\sqrt{-i+(z+i)}\;[z+i]^2[-2i+(z+i)]^2}$$

is same as above, with $i \to -i$, so

$\text{Res}@-i = -\frac{(-i)^{-1/2}}{4}\left(-\frac{1}{2}+\frac{1}{i}\right) = \frac{(e^{3\pi i/2})^{-1/2}}{4}\left(\frac{1}{2}-\frac{1}{i}\right) = \frac{1-3i}{8\sqrt{2}}$

so $J = 2\pi i\left(\frac{-1-3i}{8\sqrt{2}} + \frac{1-3i}{8\sqrt{2}}\right) = \frac{3\pi}{2\sqrt{2}}$

Also, $$J = \int_{C_R} + \int_R^\epsilon \frac{dx}{\sqrt{xe^{2\pi i}}\,(x^2+1)^2} + \int_{C_\epsilon} + \int_\epsilon^R \frac{dx}{\sqrt{xe^{i0}}\,(x^2+1)^2} \qquad \ddagger$$

Now,

$$\left|\int_{C_R}\right| \le \frac{1}{\sqrt{R}\,(R-1)^4}\,2\pi R \sim 2\pi R^{-7/2} \to 0 \text{ as } R\to\infty$$

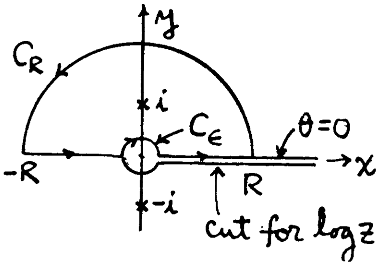
$$\left|\int_{C_\epsilon}\right| \le \frac{1}{\sqrt{\epsilon}\,(1-\epsilon)^4}\,2\pi\epsilon \sim 2\pi\sqrt{\epsilon} \to 0 \text{ as } \epsilon\to 0$$

so, letting $R\to\infty$ and $\epsilon\to 0$ in $\ddagger$ gives

$$\frac{3\pi}{2\sqrt{2}} = 0 + \int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)^2} + 0 + \int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)^2}$$

so $$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)^2} = \frac{3\pi}{4\sqrt{2}}$$

(d) $I = \int_0^\infty \frac{\ln x}{x^2+1}\,dx$. Consider $J = \oint_C \frac{\log z}{z^2+1}\,dz$ where C:



$J = 2\pi i\,\text{Res}@i = 2\pi i\,\frac{\log i}{2i} = \pi\left(\underbrace{\ln 1}_{0} + i\frac{\pi}{2}\right) = i\frac{\pi^2}{2}$

Also,

$$J = \int_{C_R} + \int_{-R}^{-\epsilon} \frac{\log(-xe^{i\pi})}{x^2+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\log(xe^{i0})}{x^2+1}\,dx$$

so $$i\frac{\pi^2}{2} = \int_{C_R} + \int_{-R}^{-\epsilon} \frac{\ln(-x)+i\pi}{x^2+1}\,dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\ln x + i0}{x^2+1}\,dx \quad ☆$$

Now, $$\left|\int_{C_R}\right| \le \frac{\max|\log(Re^{i\theta})|}{(R-1)^2}\pi R = \frac{\sqrt{(\ln R)^2+\pi^2}}{(R-1)^2}\pi R \sim \pi\frac{\ln R}{R} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{C_\epsilon}\right| \le \frac{\max|\log(\epsilon e^{i\theta})|}{(1-\epsilon)^2}\pi\epsilon = \frac{\sqrt{(\ln\epsilon)^2+\pi^2}}{(1-\epsilon)^2}\pi\epsilon \sim \pi\epsilon|\ln\epsilon| \to 0 \text{ as } \epsilon\to 0$$

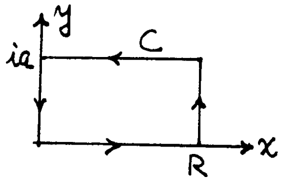
so letting R→∞ and ε→0 in ☆ gives

$$i\frac{\pi^2}{2} = 0 + \int_{-\infty}^0 \frac{\ln(-x)+i\pi}{x^2+1}\,dx + 0 + \int_0^\infty \frac{\ln x}{x^2+1}\,dx$$
$$= \int_\infty^0 \frac{\ln t + i\pi}{t^2+1}(-dt) + \int_0^\infty\frac{\ln x}{x^2+1}\,dx = 2\int_0^\infty \frac{\ln x}{x^2+1}\,dx + i\pi\int_0^\infty\frac{dx}{x^2+1}$$

and equating real and imaginary parts gives

$$\int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0 \text{ and (as a ``bonus'') } \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

7. Consider $J = \oint_C e^{-z^2}dz$. $J=0$ since e^{-z^2} is analytic everywhere. Thus,



$$0 = \int_0^R e^{-x^2}dx + \int_0^a e^{y^2-i2Ry-R^2}\,i\,dy + \int_R^0 e^{-x^2-i2ax+a^2}dx + \int_a^0 e^{y^2}\,i\,dy$$
$$\equiv K+L+M+N,$$

say. Now, $K\to\sqrt{\pi}/2$ as $R\to\infty$

$$|L| \le \max_{0\le y\le a}|e^{y^2-i2Ry-R^2}|\cdot a = \max_{0\le y\le a} a e^{y^2-R^2} = ae^{a^2-R^2} \to 0 \text{ as } R\to\infty$$

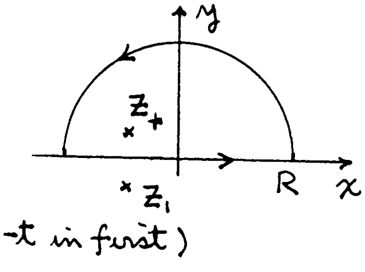
$$M = -\int_0^R e^{a^2}e^{-x^2}(\cos 2ax - i\sin 2ax)\,dx \to -e^{a^2}\int_0^\infty e^{-x^2}\cos 2ax\,dx$$ plus imaginary term

N = imaginary

so, equating real parts gives $0 = \frac{\sqrt{\pi}}{2} - e^{a^2}\int_0^\infty e^{-x^2}\cos 2ax\,dx$

or, $\int_0^\infty e^{-x^2}\cos 2ax\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2}$

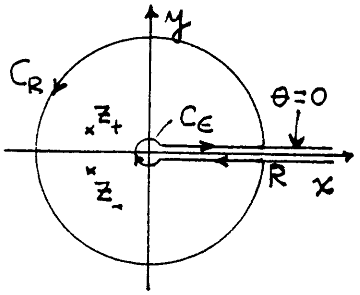
8. (a) $J = \oint_C \frac{dz}{z^2+z+1}$. $z^2+z+1=0 \to z = \frac{-1\pm\sqrt{3}i}{2} \equiv z_\pm$



$$2\pi i\,\text{Res}@z_+ = \int_{-\infty}^\infty \frac{dx}{x^2+x+1} \quad \text{(by letting } R\to\infty)$$
$$= \int_{-\infty}^0 \frac{dx}{x^2+x+1} + \int_0^\infty\frac{dx}{x^2+x+1} \quad (x=-t \text{ in first})$$
$$= \int_\infty^0 \frac{-dt}{t^2-t+1} + \text{"} = +\int_0^\infty \frac{dx}{x^2-x+1} + \int_0^\infty\frac{dx}{x^2+x+1},$$

which is one equation in the two unknown integrals.

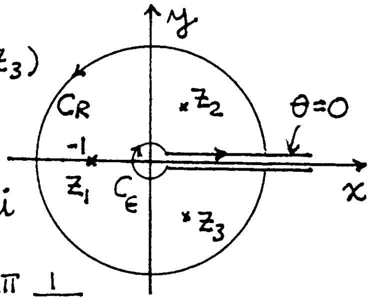
(b) $$J = \oint_C \frac{\log z}{z^2+z+1}\,dz = 2\pi i(\text{Res}@z_+ + \text{Res}@z_-)$$
$$= 2\pi i\left(\frac{\log z_+}{z_+-z_-} + \frac{\log z_-}{z_--z_+}\right)$$
$$= 2\pi i\left(\frac{\ln 1 + i2\pi/3}{\sqrt{3}i} + \frac{\ln 1 + i4\pi/3}{-\sqrt{3}i}\right)$$
$$= -\frac{4\pi^2}{3\sqrt{3}}i$$



$$\left|\int_{C_R}\right| \le \frac{\max|\ln R+i\theta|}{\min|z-z_+|\,\min|z-z_-|}2\pi R = \frac{\sqrt{(\ln R)^2+4\pi^2}}{(R-1)^2}2\pi R \sim 2\pi\frac{\ln R}{R} \to 0 \text{ as } R\to\infty$$
$$\left|\int_{C_\epsilon}\right| \le \frac{\max|\ln\epsilon+i\theta|}{\min|z-z_+|\,\min|z-z_-|}2\pi R = \frac{\sqrt{(\ln\epsilon)^2+4\pi^2}}{(1-\epsilon)^2}2\pi\epsilon \sim 2\pi\epsilon|\ln\epsilon| \to 0 \text{ as } \epsilon\to 0$$

so letting $R\to\infty$ and $\epsilon\to 0$ in
$$-\frac{4\pi^2}{3\sqrt{3}}i = \int_{C_R} + \int_R^\epsilon \frac{\cancel{\ln x}+i2\pi}{x^2+x+1}dx + \int_{C_\epsilon} + \int_\epsilon^R \frac{\cancel{\ln x}+i0}{x^2+x+1}dx$$
gives $-\frac{4\pi^2}{3\sqrt{3}}i = -i2\pi\int_0^\infty \frac{dx}{x^2+x+1}$ or, $\int_0^\infty \frac{dx}{x^2+x+1} = \frac{2\pi}{3\sqrt{3}}$

9. (a) Consider $J = \oint_C \frac{\log z}{z^3+1}dz = 2\pi i(\text{Res}@z_1+\text{Res}@z_2+\text{Res}@z_3)$



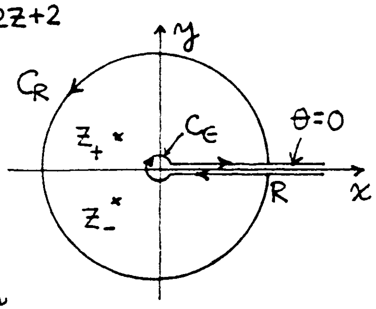
$z_1 = -1 = e^{\pi i}$, $z_2 = e^{\pi i/3}$, $z_3 = e^{5\pi i/3}$
$$\text{Res}@z_1 = \frac{\log z_1}{(z_1-z_2)(z_1-z_3)} = \frac{\ln 1+\pi i}{\left[-1-\left(\frac{1+\sqrt{3}i}{2}\right)\right]\left[-1-\left(\frac{1-\sqrt{3}i}{2}\right)\right]} = \frac{\pi}{3}i$$
$$\text{Res}@z_2 = \frac{\log z_2}{(z_2-z_1)(z_2-z_3)} = \frac{\ln 1+\pi i/3}{\left[\frac{1+\sqrt{3}i}{2}+1\right]\left[\frac{1+\sqrt{3}i}{2}-\frac{1-\sqrt{3}i}{2}\right]} = \frac{2\pi}{9}\frac{1}{\sqrt{3}+i}$$
$$\text{Res}@z_3 = \frac{\log z_3}{(z_3-z_1)(z_3-z_2)} = \text{etc} = \frac{10\pi}{9}\frac{1}{i-\sqrt{3}}$$
so $$2\pi i\left(\frac{\pi}{3}i + \frac{2\pi}{9}\frac{1}{\sqrt{3}+i} + \frac{10\pi}{9}\frac{1}{i-\sqrt{3}}\right) = \int_\epsilon^R \frac{\cancel{\ln x}}{x^3+1}dx + \int_{C_R} + \int_R^\epsilon \frac{\cancel{\ln x}+i2\pi}{x^3+1}dx + \int_{C_\epsilon} \quad ✱$$
Bounds: $$\left|\int_{C_R}\right| \le \frac{\max|\ln R+i\theta|}{(R-1)^3}2\pi R = \sqrt{(\ln R)^2+4\pi^2}\,\frac{2\pi R}{(R-1)^3} \sim 2\pi\frac{\ln R}{R^2} \to 0 \text{ as } R\to\infty$$
$$\left|\int_{C_\epsilon}\right| \le \frac{\max|\ln\epsilon+i\theta|}{(1-\epsilon)^3}2\pi\epsilon = \sqrt{(\ln\epsilon)^2+4\pi^2}\,\frac{2\pi\epsilon}{(1-\epsilon)^3} \sim 2\pi\epsilon|\ln\epsilon| \to 0 \text{ as } \epsilon\to 0$$
so, letting $R\to\infty$ and $\epsilon\to 0$ in ✱ gives
$$-\frac{4\pi^2}{3\sqrt{3}}i = -2\pi i\int_0^\infty \frac{dx}{x^3+1} \quad \text{so} \quad \int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$$

(b) Using Maple, $\mathcal{I} = \frac{\pi}{108} - \frac{1}{180} - \frac{1}{54}\tan^{-1}(\frac{1}{3})$

(c) Using Maple, $\mathcal{I} = 2\pi/(3\sqrt{3})$

(d) Set $t = (1-x)/x$ to send $\int_0^1$ to $\int_0^\infty$.

$\mathcal{I} = \int_0^1 \frac{dx}{x^2+1} = \int_0^\infty \frac{dt}{t^2+2t+2}$, so consider $\mathcal{J} = \oint_C \frac{\log z}{z^2+2z+2}\,dz$



$$2\pi i(\text{Res} @ \underset{\nwarrow z_+}{-1+i}) + 2\pi i(\text{Res} @ \underset{\nwarrow z_-}{-1-i})$$

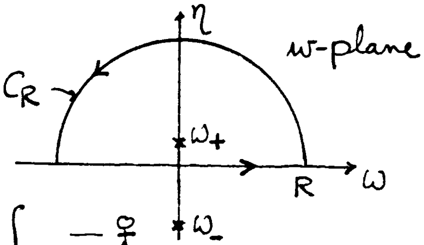
$$= \int_\epsilon^R \frac{\ln x + i0}{x^2+2x+2}\,dx + \int_{C_R} + \int_R^\epsilon \frac{\ln x + i2\pi}{x^2+2x+2}\,dx + \int_{C_\epsilon}$$

We can (but won't) show that $\int_{C_R} \to 0$ as $R \to \infty$ and that $\int_{C_\epsilon} \to 0$ as $\epsilon \to 0$ so the latter becomes

$$\cancel{2\pi} i\left(\frac{\log z_+}{z_+ - z_-} + \frac{\log z_-}{z_- - z_+}\right) = \cancel{2\pi} i \int_\infty^0 \frac{dx}{x^2+2x+2}$$

so $\mathcal{I} = -\left(\frac{\cancel{\ln\sqrt{2}} + i3\pi/4}{2i} + \frac{\cancel{\ln\sqrt{2}} + i5\pi/4}{-2i}\right) = \frac{\pi}{4}$

10. (a) $f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega x}}{\omega^2 + i\omega + 2}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega x}}{(\omega-\omega_+)(\omega-\omega_-)}\,d\omega$ where $\omega_\pm = (-1 \pm \sqrt{3})i/2$



Consider $\mathcal{J} = \frac{1}{2\pi}\oint_C \frac{e^{ixw}}{(w-\omega_+)(w-\omega_-)}\,dw$

$$= 2\pi i(\text{Res} @ \omega_+) = 2\pi i \frac{1}{2\pi}\frac{e^{ix\omega_+}}{\omega_+ - \omega_-}$$

$$= \frac{e^{-(\sqrt{3}-1)x/2}}{\sqrt{3}} \quad \text{also} = \frac{1}{2\pi}\int_{-R}^R \frac{e^{i\omega x}\,d\omega}{\omega^2+i\omega+2} + \int_{C_R} \quad \text{— ♀}$$

Now, $\left|\int_{C_R}\right| \le \frac{\max|e^{ix(\omega+i\eta)}|}{\min|w-\omega_+|\min|w-\omega_-|\,2\pi}\pi R = \frac{\pi R}{(R-|\omega_+|)\sqrt{R^2+|\omega_-|^2}}\frac{1}{2\pi} \sim \frac{1}{2R}$

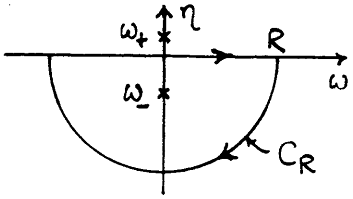
$\to 0$ as $R \to \infty$

so letting $R \to \infty$ in ♀ gives

$$\frac{e^{-(\sqrt{3}-1)x/2}}{\sqrt{3}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega}_{f(x)} + 0$$

so $f(x) = e^{-(\sqrt{3}-1)x/2}/\sqrt{3}$.

However, understand that this result has only for $x > 0$, for if $x > 0$ then $\max|\exp[ix(\omega+i\eta)]| = \max|\exp(ix\omega)\exp(-x\eta)|$ $= \max|\exp(ix\omega)|\max|\exp(-x\eta)| = 1\max e^{-x\eta} = 1$, provided that $x > 0$. If $x < 0$ we need to close C on the bottom instead:



This time $$J = \frac{1}{2\pi}\oint_C \frac{e^{ix\omega}}{(\omega-\omega_+)(\omega-\omega_-)}\,d\omega$$
$$= -2\pi i(\text{Res } @\, \omega_-) = -2\pi i \frac{1}{2\pi}\frac{e^{ix\omega_-}}{\omega_- - \omega_+}$$
$$= \frac{e^{(\sqrt{3}+1)x/2}}{\sqrt{3}}, \text{ also } = \frac{1}{2\pi}\int_{-R}^{R}\frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega + \int_{C_R} \quad \text{———} \mp$$

Now, $\left|\int_{C_R}\right| \le \dfrac{\max|e^{ix(\omega+i\eta)}|}{\min|\omega-\omega_+|\,\min|\omega-\omega_-|\,2\pi}\pi R$. This time $x<0$ but so is η, so $\max|e^{ix(\omega+i\eta)}| = \max|e^{ix\omega}|\,\max|e^{-x\eta}| = \max e^{-x\eta} = 1$, and
$$\left|\int_{C_R}\right| \le \frac{\pi R}{\sqrt{R^2+|\omega_+|^2}\,(R-|\omega_-|)\,2\pi} \sim \frac{1}{2R} \to 0 \text{ as } R\to\infty$$
so letting $R\to\infty$ in $\mp$ gives
$$\frac{e^{(\sqrt{3}+1)x/2}}{\sqrt{3}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{\omega^2+i\omega+2}\,d\omega}_{f(x)} + 0$$

so $f(x) = e^{(\sqrt{3}+1)x/2}/\sqrt{3}$ for $x<0$. Summarizing,
$$f(x) = \begin{cases} e^{(\sqrt{3}+1)x/2}/\sqrt{3}, & x<0 \\ e^{-(\sqrt{3}-1)x/2}/\sqrt{3}, & x>0 \end{cases}$$

Though not asked to do this let us check this with the Fourier transform table in Appendix D. Using partial fractions,
$$\frac{1}{\omega^2+i\omega+2} = \frac{1}{(\omega-\omega_+)(\omega-\omega_-)} = \frac{1}{\omega_+-\omega_-}\left(\frac{1}{\omega-\omega_+}-\frac{1}{\omega-\omega_-}\right) = \frac{1}{\sqrt{3}i}\left(\frac{1}{\omega-\omega_+}-\frac{1}{\omega-\omega_-}\right)$$
Using entries 2 and 3 we need to obtain the forms $\frac{1}{a+i\omega}$ or $\frac{1}{a-i\omega}$, where $\operatorname{Re} a>0$. Well,
$$\frac{1}{\omega^2+i\omega+2} = \frac{1}{\sqrt{3}}\left(\frac{1}{\left(\frac{\sqrt{3}-1}{2}\right)+i\omega} - \frac{1}{\left(\frac{-1-\sqrt{3}}{2}\right)+i\omega}\right)$$
$$= \frac{1}{\sqrt{3}}\left(\frac{1}{\left(\frac{\sqrt{3}-1}{2}\right)+i\omega} + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)-i\omega}\right)$$
so Entries 2 and 3, respectively, give the inverse as
$$f(x) = \frac{1}{\sqrt{3}}\left(H(x)e^{-(\sqrt{3}-1)x/2} + H(-x)e^{(\sqrt{3}+1)x/2}\right),$$
$$= \begin{cases} e^{-(\sqrt{3}-1)x/2}/\sqrt{3}, & x>0 \\ e^{(\sqrt{3}+1)x/2}/\sqrt{3}, & x<0, \end{cases}$$
as given above. ✓

(b) Same idea as in (a), but this time both roots $w = i, 2i$ are in the upper half plane so for $x > 0$ (closing the contour above $\curvearrowleft$) we get

$$f(x) = 2\pi i(\text{Res}@i + \text{Res}@2i) = 2\pi i \frac{1}{2\pi}\left(\frac{e^{-x}}{-i} + \frac{e^{-2x}}{i}\right) = -e^{-x} + e^{-2x}$$

and for $x < 0$ we get (closing the contour below)

$$f(x) = 0.$$

(c) Same idea as in (a). This time both roots $w = -i, -2i$ are in the lower half plane so for $x > 0$ (closing the contour above) we get

$$f(x) = 0$$

and for $x < 0$ (closing the contour below) we get

$$f(x) = -2\pi i(\text{Res}@-i + \text{Res}@-2i) = -2\pi i \frac{1}{2\pi}\left(\frac{e^{x}}{i} + \frac{e^{2x}}{-i}\right) = -e^{x} + e^{2x}$$

(d) Same idea as in (a). This time the only root $w = 2/i = -2i$ is in the lower half plane so for $x > 0$ (closing above) we get

$$f(x) = 0$$

and for $x < 0$ (closing below) we get what? The integrand is

$$\frac{1}{2\pi}\frac{e^{iwx}}{(2-iw)^2} = -\frac{1}{2\pi}\frac{e^{iwx}}{(w+2i)^2}$$

so for $x < 0$ we have

$$f(x) = -2\pi i(\text{Res}@-2i) = -2\pi i \frac{d}{dw}\left(-\frac{1}{2\pi}(w\cancel{+2i})^2 \frac{e^{iwx}}{(w\cancel{+2i})^2}\right)\Big|_{w\to 2i}$$
$$= -xe^{-2x}$$

(g) Same idea as in (a). This time the integrand has

$$\frac{1}{2\pi}\frac{e^{iwx}}{(w^2+1)^2} = \frac{1}{2\pi}\frac{e^{iwx}}{(w-i)^2(w+i)^2}$$

has 2nd order poles at $w = i, -i$. For $x > 0$ we close the contour above and get

$$f(x) = 2\pi i(\text{Res}@i) = 2\pi i \frac{d}{dw}\left(\frac{(w\cancel{-i})^2}{2\pi}\frac{e^{iwx}}{(w\cancel{-i})^2(w+i)^2}\right)\Big|_{w=i} = \frac{1+x}{4}e^{-x}$$

and for $x < 0$ we close the contour below and get

$$f(x) = -2\pi i(\text{Res}@-i) = -2\pi i \frac{d}{dw}\left(\frac{(w\cancel{+i})^2}{2\pi}\frac{e^{iwx}}{(w-i)^2(w\cancel{+i})^2}\right)\Big|_{w=-i} = \frac{1-x}{4}e^{x},$$

so

$$f(x) = \frac{1+|x|}{4}e^{-|x|}$$

NOTE: If we use entry 4 of Appendix D we have $f(x) = \frac{1}{2}e^{-|x|} * \frac{1}{2}e^{-|x|}$ $= \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-\xi|}e^{-|\xi|}\,d\xi$, which does give the same result.

11. (a)

$$\int_{-1}^{3}\frac{dx}{x} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\int_{-1}^{0-\epsilon_1}\frac{dx}{x}+\int_{0+\epsilon_2}^{3}\frac{dx}{x}\right\}$$

$$= \lim_{\epsilon_1\to 0}\ln|x|\Big|_{-1}^{-\epsilon_1} + \lim_{\epsilon_2\to 0}\ln|x|\Big|_{\epsilon_2}^{3}$$

$$= \lim_{\epsilon_1\to 0}\ln\epsilon_1 + \ln 3 - \lim_{\epsilon_2\to 0}\epsilon_2$$

does not exist because $\lim_{\epsilon_1\to}\ln\epsilon_1 = -``\infty"$ does not exist, and similarly for $\lim_{\epsilon_2\to 0}\epsilon_2$. However,

$$\fint_{-1}^{3}\frac{dx}{x} = \lim_{\epsilon\to 0}\left\{\int_{-1}^{0-\epsilon}\frac{dx}{x}+\int_{0+\epsilon}^{3}\frac{dx}{x}\right\} = \lim_{\epsilon\to 0}\left(\cancel{\ln\epsilon}+\ln 3-\cancel{\ln\epsilon}\right)$$

$= \lim_{\epsilon\to 0}\ln 3$ does exist, and it $= \ln 3$.

(b)

$$\int_{1}^{4}\frac{dx}{x(x-2)} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left\{\int_{1}^{2-\epsilon_1}\frac{dx}{x(x-2)}+\int_{2+\epsilon_2}^{4}\frac{dx}{x(x-2)}\right\}$$

Note: By partial fractions $\frac{1}{x(x-2)} = -\frac{1}{2x}+\frac{1}{2}\frac{1}{x-2}$

$$= \lim_{\epsilon_1\to 0}\left(-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|x-2|\right)\Big|_{1}^{2-\epsilon_1}$$

$$+ \lim_{\epsilon_2\to 0}\left(-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|x-2|\right)\Big|_{2+\epsilon_2}^{4}$$

$$= \lim_{\epsilon_1\to 0}\left(-\frac{1}{2}\ln|2-\epsilon_1|+\frac{1}{2}\ln\epsilon_1\right) - \left(-\frac{1}{2}\cancel{\ln 1}+\frac{1}{2}\cancel{\ln 1}\right)$$

$$+ \left(-\frac{1}{2}\ln 4+\frac{1}{2}\ln 2\right) - \lim_{\epsilon_2\to 0}\left(-\frac{1}{2}\ln|2+\epsilon_2|+\frac{1}{2}\ln\epsilon_2\right)$$

$$= -\cancel{\tfrac{1}{2}\ln 2}+\frac{1}{2}\lim_{\epsilon_1\to 0}\ln\epsilon_1-\frac{1}{2}\ln 4+\frac{1}{2}\ln 2+\cancel{\tfrac{1}{2}\ln 2}-\frac{1}{2}\lim_{\epsilon_2\to 0}\ln\epsilon_2 \qquad *$$

does not exist because each of the limits in * fails to exist. However, if $\epsilon_1 = \epsilon_2$ then the two limit terms in * cancel and

$$\fint_{1}^{4}\frac{dx}{x(x-2)} \text{ does exist and } = -\frac{1}{2}\ln 4+\frac{1}{2}\ln 2$$

$$= -\ln 2+\frac{1}{2}\ln 2 = -\frac{1}{2}\ln 2$$

12. So that the $\int_{C_R}$ integral $\to 0$ as $R\to\infty$, consider

$$J = \oint_C \frac{e^{iz}}{z}\,dz$$

rather than $\oint_C \frac{\sin z}{z}\,dz$. However, whereas $\sin z/z$ is analytic everywhere, the $e^{iz}/z \sim 1/z$ as $z\to 0$; that is, it has a singularity (1st order pole) right on the path of integration. Thus, modify the path C by "indenting"

C at the origin with a semicircle C_ϵ, as shown in the exercise. Inside the indented contour e^{iz}/z is analytic so (by Cauchy's theorem—or the residue theorem)

$$\oint = 0 = \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right)\frac{e^{iz}}{z}dz + \int_{C_\epsilon} + \int_{C_R},$$

which holds for each R (no matter how large) and for each ϵ (no matter how small). Thus it holds in the limit as $R\to\infty$ and as $\epsilon\to 0$, so

$$0 = \fint_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}dx + \lim_{R\to\infty}\int_{C_R} + \lim_{\epsilon\to 0}\int_{C_\epsilon}. \qquad ☆$$

Now, on C_R we have $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}||e^{-y}| = e^{-y} \le 1$ and $|z| = R$, so

$$\left|\int_{C_R}\frac{e^{iz}}{z}dz\right| \le \frac{1}{R}\pi R = \pi,$$

which is not sharp enough to determine whether $\int_{C_\epsilon}\to 0$ or not. Thus, use the sharper bound

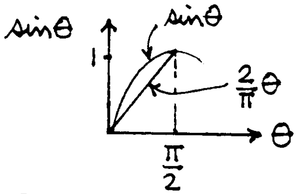
$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz|$$

given in Exercise 7 of Sec. 23.2:

$$\left|\int_{C_R}\frac{e^{iz}}{z}dz\right| \le \int_0^{\pi}\frac{e^{-y}}{R}Rd\theta = \int_0^{\pi}e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2}e^{-R\sin\theta}d\theta$$ (since $\sin\theta$ is symmetric about $\theta = \frac{\pi}{2}$)

The latter is a hard integral, but since all we need is a bound, use the fact that

$$\sin\theta \ge \frac{2}{\pi}\theta$$



so

$$\left|\int_{C_R}\right| \le 2\int_0^{\pi/2}e^{-\frac{2R}{\pi}\theta}d\theta = \frac{\pi}{R}(1-e^{-R}) \to 0 \text{ as } R\to\infty.$$

Good. Now consider the C_ϵ integral. Following the hint, write the Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2} + \cdots \qquad \text{in } 0<|z|<\infty.$$

Since the latter converges in $0<|z|<\infty$, then the part

$$g(z) \equiv i - \frac{z}{2} + \cdots \qquad ♀$$

does too. Since ♀ converges at $z=1$, say, it must converge (by theorem 24.2.1) inside $|z|<1$ and hence (by Theorem 24.2.8) be analytic there. Since $g(z)$ is analytic there it is bounded on C_ϵ by m, say. Thus,

$$\left|\int_{C_\epsilon} g(z)dz\right| \le (m)(\pi\epsilon) \to 0 \text{ as } \epsilon\to 0.$$

Further,

$$\int_{C_\epsilon}\frac{1}{z}dz = \int_{C_\epsilon}\frac{1}{\epsilon e^{i\theta}}d(\epsilon e^{i\theta}) = \int_{\pi}^{0}\frac{\epsilon i e^{i\theta}}{\epsilon e^{i\theta}}d\theta = -\pi i,$$

so $\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z} dz$ is $= -\pi i$

Hence, ★ becomes

$$0 = \fint_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx + 0 - \pi i$$

or $$\fint_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi i \,.$$

Equating real and imaginary parts give

$$\fint_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0 \qquad ✡$$

and $$\fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi,$$

of which the latter is actually

$$\fint_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \quad \text{since } \frac{\sin x}{x} \text{ is nonsingular at } x=0$$

$$= 2\int_0^{\infty} \frac{\sin x}{x} dx$$

so $$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \,.$$

The former, ✡, is a "bonus" result, but not very interesting since it follows from the antisymmetry of the integrand $\cos x / x$